Research Article

On the Asymptotic Hyperstability of Linear Time-Invariant Continuous-Time Systems under a Class of Controllers Satisfying Discrete-Time Popov’s Inequality

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This paper is concerned with the property of asymptotic hyperstability of a continuous-time linear system under a class of continuous-time nonlinear and perhaps time-varying feedback controllers belonging to a certain class with two main characteristics; namely, (a) it satisfies discrete-type Popov’s inequality at sampling instants and (b) the control law within the intersample period is generated based on its value at sampling instants being modulated by two design weighting auxiliary functions. The closed-loop continuous-time system is proved to be asymptotically hyperstable, under some explicit conditions on such weighting functions, provided that the discrete feed-forward transfer function is strictly positive real.

1. Introduction

Continuous-time and discrete-time positive systems have been studied in detail in recent years [1–10]. If the state and output possess the positivity property under nonnegative initial conditions and controls, the positivity is said to be internal or, simply, the system is positive. If the output possesses such a property, the system is said to be externally positive. Thus, positive systems are intrinsically interesting to describe some problems like Markov chains, queuing problems, certain distillation columns and biological, and other physical compartmental problems where populations or concentrations cannot be negative [2, 3]. It is well known that time-invariant dynamic linear systems which are externally positive, while having positive real or strictly positive real transfer matrices, are, in addition, hyperstable or asymptotically hyperstable, i.e., globally Lyapunov stable for any nonlinear and/or time-varying feedback device satisfying Popov’s type inequality for all time, [11, 12]. The converse assertion is not generically true in the sense that a hyperstable linear system, then characterized by a positive real transfer function, is not necessarily an externally positive one. In particular, hyperstable linear systems can have a positive instantaneous input-output power and a positive input-output energy measure for all time. Thus, they have identical signs of the input and output for all time instants while they are not externally positive if those signals are not everywhere nonnegative. Thus, external positivity of a SISO system is not implied by the positive realness of its transfer function. The property of asymptotic hyperstability generalizes that of absolute stability [13–15] which generalizes the most basic concept of stability of dynamic systems. See, for instance [2, 3, 11, 13–26], and references therein. It is well known that closed-loop hyperstability is, by nature, a powerful version of closed-loop stability since it refers to the stability of a hyperstable linear feed-forward plant (in the sense of the positive realness of the associated transfer matrix) under a wide class of feedback controllers applied. The above important properties make very attractive potential research issues for kind of more complex dynamic systems with applied projection including those lying in the class of continuous/digital hybrid systems. On the other hand, the class of hybrid systems consisting of continuous-time and discrete-time (or digital) systems are of an increasing
interest since many existing industrial installations combine both kinds of systems. An elementary well-known case is when a discrete-time controller is used for a continuous-time plant. Another case is related to teleoperation systems where certain variables evolve in a discrete-time or digital fashion. A background literature and related relevant results are given in [1, 7, 11, 17, 18, 27–30] and some of the references therein.

Some conditions have been given in the background literature to guarantee the positive realness of discrete transfer functions from related conditions on their continuous-time counterparts which are then maintained under discretization. See, for instance [31]. The results are useful to guarantee the hyperstability of discrete-time systems under the hyperstability of the continuous-time ones when the class of feedback discrete-time controllers satisfies Popov’s type inequality. However, it turns out that positive realness of a continuous-time system does not guarantee the positive realness of its discretized transfer function by a zero-order hold for any sampling period and, conversely, the strong positive realness of a discretized transfer function does not guarantee in some limited cases the strong positive realness of its continuous-time counterpart, [31–33]. This paper is devoted to a certain kind of inverse problem stated in the following terms. Given is an asymptotically hyperstable discrete-time system of sampling period $T$ such that its feed-forward transfer function is strictly positive real and the discrete-timesystemofsamplingperiod

\begin{equation}
\Phi(T)
\end{equation}

such that its feed-forward transfer function is strictly positive real and the nonlinear time-varying feedback controller belongs to a class of $\Phi_d(T)$ satisfying certain discrete Popov’s type inequality. The elementary question that arises is how the asymptotic hyperstability of the continuous-time system is guaranteed for a certain modified class of controllers such that its member at sampling instants is in the class $\Phi_d(T)?$ To answer this question, an intersample controller based on the class $\Phi_d(T)$ at sampling instants is designed which corrects the control law along the intersample period via the concourse of two modulating continuous-time functions which have to satisfy certain reasonable constraints. The paper is organized as follows. Firstly, a notation and terminology subsection is allocated in the subsequent section. Section 3 presents the linear and time-invariant system of the feed-forward part of the closed-loop system together with its analytic output expressions some auxiliary results of positivity and boundedness for all the sampling instants of an input-output energy measure. The continuous-time control input is proposed to be generated within each intersample period from the last sampling instant defining the starting point of the current intersample period and some design auxiliary functions which modulate the control signal along such an intersample period. Some of the formulas presented are derived in detail in the appendix. On the other hand, some auxiliary preparatory lemmas concerning the input-output energy measure are also formulated which are of usefulness for the main result. Section 4 is devoted to obtain the main asymptotic hyperstability result which is proved based on some given and previously proved auxiliary preparatory lemmas. Basically, it is proved that the continuous-time closed-loop system is asymptotically hyperstable provided that (a) the discretized feed-forward transfer function is strictly positive real, (b) the discretized system at sampling instants is asymptotically hyperstable for all nonlinear and time-varying feedback controllers which satisfy certain Popov’s type inequality at sampling instants, and (c) some extra additional conditions on smallness and boundedness are fulfilled by the abovementioned auxiliary modulated functions which define the controls along the intersample periods. Such conditions make any continuous-time controller belonging to the appropriate class to satisfy continuous-type Popov’s inequality for a class of controllers provided that the former one at sampling instants is fulfilled by the discretized controllers. Some examples are discussed in Section 5. Finally, conclusions end the paper.

2. Notation

(i) $R = R_{0+} \cup \{0\}$, $Z = Z_{0+} \cup \{0\}$, $R_{0+} = \{r \in R : r \geq 0\}$, and $Z_{0+} = \{z \in Z : z \geq 0\}.$

(ii) $\mathbb{C} = [\infty, +\infty]$ is the closure of the real field, that is, set of real numbers together with the $\pm$ infinity points.

(iii) The continuous and discrete-time arguments $t \in R_{0+}$, and $k \in Z_{0+}$, are denoted with parenthesis and brackets, that is, $(t)$ and $[k]$, respectively.

(iv) If $T > 0$ is the sampling period then $u(t) = u[k] + \hat{u}(t)$; $\forall t \in [kT, (k+1)T)$; $\forall t \in [kT, (k+1)T)$; $\forall k \in Z_{0+}$ is the continuous-time input, where the piecewise-continuous intersample incremental input is $\hat{u}(t) = u(t) - u[k]$ with $u[k] = u(kT)$ and $\hat{u}[k] = \hat{u}(kT) = 0$; $\forall t \in [kT, (k+1)T)$; $\forall k \in Z_{0+}$. The auxiliary purely discrete input is $u_d[k]$, such that $u_d[k] = u[k]$; $\forall k \in Z_{0+}$, if $\hat{u}[t] = 0$; $\forall t \in R_{0+}$.

(v) $y_f(t)$; $\forall t \in R_{0+}$, and $y_h[k] = y_h(kT)$; $\forall k \in Z_{0+}$, are the forced and homogeneous (unforced) output of the system, respectively.

(vi) $y_{df}[k] = y_{df}(kT)$ and $y_{dh}[k] = y_{dh}(kT)$; $\forall k \in Z_{0+}$ are the forced and homogeneous output of the system, respectively, when the input is piecewise-constant with constant value in-between each two consecutive sampling instants, i.e., when the input is generated from a zero- order-hold (ZOH) device.

(vii) The Fourier transform of any real vector function $f(t)$ is denoted by $\tilde{f}(\omega)$ if it exists; $\forall t, \omega \in R$, where $i = \sqrt{-1}$ is the imaginary complex unit.

(viii) For the purpose of formally using Fourier transforms for the subsequent developments, we define continuous-time and discrete-time truncated signals from their untruncated counterparts on the respective continuous-time and discrete-time intervals $[0, t]$ and $[0, kT]$ as follows:

\begin{equation}
\nu_j(t) = \begin{cases} 
\nu(t) & \text{if } 0 \leq \tau \leq t \\
0 & \text{if } \tau > t \\
0 & \text{if } \tau < 0;
\end{cases}
\end{equation}

$\forall t, \tau \in R_{0+}.$
for all \( |z| \geq 1 \) transfer functions, \([13, 31]\).

(1)

\[
v_{[k]}[j] = \begin{cases} 
  v[j] & \text{if } 0 \leq j \leq k \\
  0 & \text{if } j > k \\
  0 & \text{if } j < 0;
\end{cases} \\
\forall j, k \in \mathbb{Z}_0^+.
\]

(ix) Let \( s \) and \( z = e^{-Ts} \) the Laplace and \( z \)-transforms (for sampling period \( T \)) complex arguments. The set \([SPR]\) is the set of strictly positive real continuous \( g(s) \) (respectively, discrete) transfer functions \( \hat{g}(z) \); i.e., with poles in \( \text{Re } s < 0 \) (respectively, \( |z| < 1 \)) which satisfy \( \text{Re } \hat{g}(s) > 0 \) for all \( s \in \mathbb{C} \) (respectively, \( \text{Re } \hat{g}(z) > 0 \) for all \( |z| \geq 1 \)). These conditions imply that \( \text{Re } \hat{g}(i\omega) > 0, \forall \omega \in \mathbb{R}, \) \([13, 31, 34]\). The strictly positive continuous and discrete real sets are not distinguished at the level of notation since they are easy to identify them according to context.

(x) The so-called set \([SSPR]\) is the set of (continuous or discrete depending on context) strongly positive real transfer functions which are those in \([SPR]\) whose real part is strictly positive also as \( |s| \to \infty \). It can be pointed out that Szegö-Kalman-Popov Lemma (also so-called Positive Real Lemma) which relates positive realness of transfer functions to associated state-space realization properties states that discrete transfer function in \([SPR]\) are of relative degree (namely, pole-zero excess) equal to zero so that they are also in \([SSPR]\) and both sets are equivalent. This equivalence concern between the discrete sets \([SPR]\) and \([SSPR]\) does not apply to continuous transfer functions, \([13, 31]\).

3. The Continuous-Time and Discrete-Time Linear Time-Invariant Systems and Some Auxiliary Results

Consider an \( n \)-th order linear and time-invariant dynamic system in state-space description:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + bu(t); \quad x(0) = x_0 \in \mathbb{R}^n, \\
y(t) &= c^T x(t) + du(t),
\end{align*}
\]

(2)

where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}, \) and \( y(t) \in \mathbb{R} \) are the state, piecewise-continuous control input, and output and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1}, C \in \mathbb{R}^{1 \times n}, \) and \( D \in \mathbb{R} \) are the matrix of dynamics, control vector, output vector, and input-output interconnection gain. The auxiliary purely discrete output and the continuous-time and sampled output are calculated from (2) via the impulse responses \( g(t) \) and \( g_d[k] \) which become

\[
\begin{align*}
g(t) &= c^T e^{A(t-\tau)} b + d \delta(t-\tau); \quad \forall \tau, t \geq 0 \in \mathbb{R}_0^+; \\
g_d[k-j] &= \int_{jT}^{(k-j)T} g(kT-\tau) \, d\tau.
\end{align*}
\]

\[
\begin{align*}
y[k] &= \left( \sum_{j=-\infty}^{\infty} g_d[k-j] \right) u[k] \\
&\quad + \int_{(k-1)T}^{kt} g((kT-\tau)) \, d\tau \quad \forall k \in \mathbb{Z}_0^+.
\end{align*}
\]

(6)

and \( g(t-\tau) = 0; \forall \tau, t \in \mathbb{R}_0^+ \), where \( \delta(t-\tau) \) is the Dirac distribution, \( u(-\tau) = 0; \forall \tau \in \mathbb{R}_0^+, \) and \( g_d[k-j] = 0; \forall j > k, \) \( k \in \mathbb{Z}_0^+ \) with \( g_d[0] = d \). The initial input and output conditions are \( u[0] = y[0] = 0 \) for any integer \( j < 0 \) and \( y(t) = u(-t) = \bar{u}(-t) = 0; \forall t \in \mathbb{R}_0^+ \) in causal dynamic systems subject to initial conditions \( x(0) = x_0 \). Let the control input be

\[
\begin{align*}
u(t) &= u[k] + \bar{u}(t) = u_d[k] + \bar{u}_d(t); \\
&\quad \forall k \in \mathbb{Z}_0^+.
\end{align*}
\]

(4)

Then, the auxiliary purely discrete output becomes under piecewise-constant input with eventual finite jumps at sampling instants:

\[
\begin{align*}
y_d[k] &= \left( \sum_{j=-\infty}^{\infty} g_d[k-j] \right) u_d[k] \\
&\quad + \int_{(k-1)T}^{kt} g((kT-\tau)) \, d\tau \quad \forall k \in \mathbb{Z}_0^+.
\end{align*}
\]

(5)

The continuous-time output becomes

\[
\begin{align*}
y(t) &= \left( \sum_{j=-\infty}^{\infty} g_d[k-j] \right) u[k] \\
&\quad + \int_{(k-1)T}^{kt} g((kT-\tau)) \, d\tau \quad \forall k \in \mathbb{Z}_0^+.
\end{align*}
\]

(7)

while the sampled real output becomes

\[
\begin{align*}
y[k] &= y(kT) \\
&\quad + \int_{(k-1)T}^{kt} g((kT-\tau)) \bar{u}_d(kT) \, d\tau \quad \forall k \in \mathbb{Z}_0^+.
\end{align*}
\]

Expressions (5)–(7) are derived in the appendix.
Remark 1. Note that the difference between the sampled continuous-time sequence \( \{ y[k] \} \) and the purely discrete output sequence \( \{ \tilde{y}_d[k] \} \) is that the first one is the exact value of the output at sampling instants including the effects of the intersample input ripple while the second one is the discretized output at sampling instants in the presence of a zero-order hold. As a result, note that \( y[k] = \tilde{y}_d[k] \) for any given \( k \in \mathbb{Z}^+ \) if and only if
\[
\int_{kT}^{(k+1)T} g(t-\tau) \tilde{u}(\tau) \, d\tau = 0.
\]
In particular, \( y[k] = \tilde{y}_d[k] \) if \( \tilde{u}(t) = 0; \forall t \in \mathbb{R}^+ \). Causality implies that (6)-(7) can be rewritten as
\[
y(t) = \left( \sum_{j=0}^{\infty} g_d[k-j] \right) u[k] + \left( \int_{kT}^{t} g(t-\tau) \, d\tau \right) u[k]
\]
\[
+ \int_{kT}^{\infty} g(kT-\tau) \tilde{u}_d(\tau) \, d\tau + y_h(t);
\]
\[
\forall t \in [kT, (k+1)T); \forall k \in \mathbb{Z}^+.
\]

The following result is direct.

Lemma 2. Any given piecewise-continuous control input can be decomposed into purely discrete-time control plus incremental intersample period ones as follows:
\[
u(t) = u[k] + \bar{u}(t) = u_d[k] + \bar{u}_d(t)
\]
\[
= (1 + \lambda (t)) u[k] + \sigma(t);
\]
\[
\forall t \in [kT, (k+1)T); \forall k \in \mathbb{Z}^+.
\]
through piecewise-continuous functions \( \lambda, \sigma : \cup_{k \in \mathbb{Z}^+} [kT, (k+1)T) \rightarrow \mathbb{R} \) defined by
\[
\lambda(t) = \begin{cases} 
\bar{u}(t) & \text{if } u[k] \neq 0 \\
0 & \text{if } u[k] = 0,
\end{cases}
\]
\[
\sigma(t) = \begin{cases} 
0 & \text{if } u[k] \neq 0 \\
\bar{u}(t) & \text{if } u[k] = 0.
\end{cases}
\]

The purely auxiliary input-output and true ones energy measures on the discrete-time interval \([0, kT]\) as well as the continuous-time ones on \([0, t]; \forall t \in [kT, (k+1)T) \) and \( \forall k \in \mathbb{Z}^+ \), are, respectively, from (5)-(7) and (9)-(10), given by
\[
E_d[k] = \sum_{n=0}^{N} T y_d[n] u_d[n] = \sum_{n=0}^{N} \sum_{\ell=0}^{N} T \left( \int_{\ell T}^{n T} g(nT-\tau) \, d\tau \right) u_d[\ell] u_d[n]
\]
\[
= \sum_{n=0}^{N-1} \sum_{\ell=0}^{n} T g_d[n] \left[ n - \ell \right] u_d[n] u_d[n-\ell] + T g_d[0] u_d^2[n]; \forall k \in \mathbb{Z}^+.
\]
\[
E[k] = \int_{0}^{kT} y(t) u(t) \, d\tau
\]
\[
= \sum_{n=0}^{N-1} \sum_{\ell=0}^{n} \int_{\ell T}^{(n+1)T} T \left( \int_{\ell T}^{(n+1)T} g(t-\theta) \, d\theta \right) u_d[n] u_d[\ell] \, d\tau
\]
\[
+ T g_d[0] u_d^2[n]; \forall k \in \mathbb{Z}^+.
\]
\[
E(t) = \int_{0}^{t} y(t) u(t) \, d\tau = E[k] + \int_{kT}^{t} \int_{0}^{T} g(t-\theta) \left[ (1 + \lambda (\theta)) u[k] + \sigma(\theta) \right] \left[ (1 + \lambda (\theta)) u[k] + \sigma(\theta) \right] \, d\theta \, d\tau
\]
\[
= E[k] + \int_{kT}^{t} g(t-\theta) \left[ (1 + \lambda (\theta)) u[k] + \sigma(\theta) \right] \left[ (1 + \lambda (\theta)) u[k] + \sigma(\theta) \right] \, d\theta \, d\tau;
\]
\[
\forall t \in [kT, (k+1)T); \forall k \in \mathbb{Z}^+.
\]

Remark 3. Since the control input is piecewise-continuous, the truncated input \( u_d(\tau) \) for \( \tau \in [0, t] \) has Fourier transforms for any \( t \in \mathbb{R}_+ \) so that the corresponding truncated output \( \tilde{y}_d(\tau) \) for \( \tau \in [0, t] \) and also the energy measure have also Fourier transforms for any \( t \in \mathbb{R}_+ \). This follows from the fact that these signals are calculated via truncated functions which
Mathematical Problems in Engineering

are then zero at infinity so that they are square-integrable and their sequences of sampled values of sampling period $T$ are square-summable as a result.

The subsequent two simple auxiliary lemmas will be then used in the next section to establish the main result.

**Lemma 4.** Assume that there exist constants $\gamma_{d0}, (\gamma_d > \gamma_{d0}) \in \mathbb{R}$ such that $\gamma_{d0} \leq E_d[k] \leq \gamma_d; \forall k \in \mathbb{Z}_0$. Then, there exist $\gamma_0 (\gamma > \gamma_0) \in \mathbb{R}$ such that

\[
\gamma_0 \leq E(t) \leq \gamma; \forall t \in [(k_0 + 1)T, \infty)
\]

where

\[
E(t) = E[k] + \tilde{E}(kT,t) = E_d[k] + \tilde{E}_d(kT,t)
\]

\[
\forall t \in (kT,(k+1)T)
\]

\[
\tilde{E}_d(kT,T) = \sum_{j=0}^{T} \int_{0}^{T} g(t - \theta) \left[(1 + \lambda (\theta)) u[k] + \sigma (\theta)\right] \left[(1 + \lambda (\theta)) u[k] + \sigma (\theta)\right] d\theta dt
\]

\[
\forall t \in (kT,(k+1)T), \forall k \in \mathbb{Z}_0.
\]

**Lemma 5.** Assume that $\phi(x,t)$ is subadditive for all $x \in \mathbb{R}$, i.e., $\phi(x + y, t) \leq \phi(x, t) + \phi(y, t); \forall x, y \in \mathbb{R}, \forall t \in \mathbb{R}_0$. Then, $\phi(y(t), t) \leq \phi(y[k], k) + \phi(y(t) - y[k], t,T)$, and as a result \( \bar{u}(t) \geq \phi(y(t) - y[k], t,T) \) and $\phi(y[k], k) \geq -\phi(-y[k], k); \forall t \in [kT,(k+1)T), \forall k \in \mathbb{Z}_0$.

If $\phi(x,t)$ is superadditive for all $x \in \mathbb{R}$, i.e., $\phi(x + y, t) \geq \phi(x, t) + \phi(y, t); \forall x, y \in \mathbb{R}, \forall t \in \mathbb{R}_0$, then, $\phi(y(t), t) \geq \phi(y[k], k) + \phi(y(t) - y[k], t,T)$, and as a result $\bar{u}(t) \leq \phi(y(t) - y[k], t,T)$ and $\phi(y[k], k) \leq -\phi(-y[k]); \forall t \in [kT,(k+1)T), \forall k \in \mathbb{Z}_0$.

**Proof.** From the first identity in (9), if $\phi(x,t)$ is subadditive then

\[
-\phi(y(t)) = u(t) = u[k] + \bar{u}(t)
\]

\[
\geq -\phi(y[k]) + (y(t) - y[k])
\]

\[
\geq \phi(y[k]) - \phi(y(t) - y[k])
\]

\[
\forall t \in (kT,(k+1)T), \forall k \in \mathbb{Z}_0.
\]

so that

\[
\phi(y(t)) \leq \phi(y[k]) + \phi(y(t) - y[k])
\]

\[
\forall t \in [kT,(k+1)T), \forall k \in \mathbb{Z}_0.
\]

If $\phi(x,t)$ is superadditive then

\[
-\phi(y(t)) \leq -\phi(y[k]) + (y(t) - y[k])
\]

\[
\leq -\phi(y[k]) - \phi(y(t) - y[k])
\]

\[
\forall t \in [kT,(k+1)T), \forall k \in \mathbb{Z}_0.
\]
so that
\[
\varphi(y(t)) \geq \varphi(y[k]) + \varphi(y(t) - y[k]),
\]
\[
\varphi(y[k]) \leq -\varphi(-y[k]); \quad \forall t \in [kT, (k+1)T), \; \forall k \in \mathbb{Z}_{0+}.
\]

(20)

4. The Main Result

Definition 6 (Popov’s inequality). A continuous-time feedback nonlinear and eventually time-varying continuous-time controller \(u(t) = -\varphi(y(t), t)\) and \(\forall t \in \mathbb{R}_{0+}\), where \(\varphi : \mathbb{R}_{0+} \times \mathbb{R}_{0+} \rightarrow \mathbb{R}_{0+}\) is in the class \(\{\Phi\}\), denoted as \(\varphi \in \{\Phi\}\), is said to satisfy Popov’s type integral inequality if, for some \(\gamma_0 \in \mathbb{R}_{+}\), one has
\[
\int_0^t y(\tau) \varphi(y(\tau), \tau) d\tau \geq -\gamma_0; \quad \forall t \in \mathbb{R}_{0+}
\]

(21)

Definition 7 (discrete-time Popov’s inequality). A discrete-time feedback nonlinear and eventually time-varying discrete-time controller \(u(t) = u[k] = u(kT) = -\varphi(y(kT), kT), \; \forall t \in [kT, (k+1)T), \; \forall k \in \mathbb{Z}_{0+}\), where \(\varphi : \mathbb{R}_{0+} \times \mathbb{Z}_{0+} \rightarrow \mathbb{R}_{0+}\) is in the class \(\{\Phi_d(T)\}\) of sampling period \(T \in \mathbb{R}_{+}\), denoted as \(\varphi \in \Phi_d(T)\), is said to satisfy a Popov’s type discrete-time inequality if, for some \(\gamma_d \in \mathbb{R}_{+}\), one has
\[
\sum_{j=0}^k y[j] \varphi[k] \geq -\gamma_d T; \quad \forall k \in \mathbb{Z}_{0+}
\]

(22)

I. D. Landau refers to controllers satisfying a Popov’s type inequality (21) (or, respectively, (22)) as hyperstable controllers of class \(\{\Phi\}\) (or, respectively, of class \(\{\Phi_d(T)\}\)), [13], bearing in mind that if any controller of such a class is coupled to a linear time-invariant forward system of transfer function being continuous-time (or, respectively, discrete-time) strictly positive real then the overall closed-loop system is asymptotically hyperstable, namely, globally asymptotically stable for any arbitrary controller belonging to the respective class. It can be pointed out that the above comments referred to discrete-time systems and, in particular Definition 7, are also applicable to digital systems, in the same way as they are applicable to discrete-time systems, i.e., those which are fully described in the discrete domain without having specific links to a discretization process on a certain continuous-time system.

It can be pointed out that Definition 7 can be also established for a digital system not being related to the time discretization of a continuous-time system in terms of Popov’s inequality of the following form:
\[
\sum_{j=0}^k y[k] \varphi[k] \geq -\gamma_d; \quad \forall k \in \mathbb{Z}_{0+}
\]

(23)

for some constant \(\gamma_d \in \mathbb{R}_+\) for any digital controller of class \(\{\Phi_d\}\). In this context the sampling period either can be nonrelevant—then nonmade notational explicit in the class \(\{\Phi_d\}\)—or can even have nonsense since the inequality is applied to a certain discrete sequence.

The following result obtains useful expressions for the discretized input-output energy measure under some conditions concerning the subadditivity constraints on the controllers of class \(\{\Phi_d(T)\}\). The result is also a preparatory one for the next Lemma 9 which addresses the positivity and boundedness of the input-output energy measure at sampling instants.

Lemma 8. The following properties are fulfilled:

(i) The subsequent formulas hold:

\[
y[k] = \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} (1 + \lambda_k[j] + \xi_k[j]) g((k-j)T) \right) d\tau \right] u[j] + \int_0^{\infty} \left[ g(kT-\tau) \sigma(kT)(\tau) d\tau \right] + y_h[k];
\]

(24)

\[
E[k] = \sum_{\ell=0}^{k} y[\ell] u[\ell]
\]

\[
= \sum_{\ell=0}^{k} \left( \sum_{j=0}^{\ell} \left( \int_{jT}^{(j+1)T} (1 + \lambda_{\ell}[j] + \xi_{\ell}[j]) g((\ell-j)T) \right) d\tau + \int_0^{\infty} \left[ g(\ellT-\tau) \sigma(\ellT)(\tau) d\tau \right] u[j] + y_h[\ell] \right) u[\ell]
\]

\[
= \sum_{k=0}^{\infty} y[k] u[k]
\]

\[
= \sum_{\ell=0}^{\infty} \left( \sum_{j=0}^{\ell} \left( \int_{jT}^{(j+1)T} (1 + \lambda_{\ell}[j] + \xi_{\ell}[j]) g((\ell-j)T) \right) d\tau + \int_0^{\infty} \left[ g(\ellT-\tau) \sigma(\ellT)(\tau) d\tau \right] u[j] + y_h[\ell] \right) u[\ell];
\]

(25)

\[
\forall k \in \mathbb{Z}_{0+},
\]

\[
\forall k \in \mathbb{Z}_{0+},
\]
where
\[
\xi_{[k]}[j] = \int_{jT}^{(j+1)T} \lambda (jT + \tau) g ((k-j)T - \tau) d\tau - \lambda [j] g ((k-j)T) \quad \forall j \leq k, k \in \mathbb{Z}_0^+.
\]

(ii) Assume that \( \varphi \in \{ \Phi_d(T) \} \) and that \( \varphi(x, kT) \) is subadditive for all \( x \in \mathbb{R} \), i.e., \( \varphi(x + y, kT) \leq \varphi(x, kT) + \varphi(y, kT) \); \( \forall x, y \in \mathbb{R}, \forall k \in \mathbb{Z}_0^+ \). Then, the subsequent relations are true if the sequence \{\varepsilon[k]\} is defined by \( \varepsilon[k] = \lambda[k] + \xi[k] \); \( \forall k \in \mathbb{Z}_0^+ \):

\[
y_d[k] + \int_0^{\infty} [g(\ell T - \tau)] \sigma_{(kT)}(\tau) d\tau
\leq \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} \varepsilon_{[k]} [j] g ((k-j)T) d\tau \right) \right] \\
\cdot \varphi_{[k]} (y_d[j], j) + \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} \varepsilon_{[k]} [j] g ((k-j)T) d\tau \right) \right] \\
\cdot \varphi_{[k]} (-y_d[j], j) \quad \forall k \in \mathbb{Z}_0^+.
\]

Proof. One has

\[
y[k] = \int_0^{kT} g(kT - \tau) u(\tau) d\tau + y_h[k] = \sum_{j=0}^{k} \left( \int_0^{kT} g [(k-j)T] d\tau \right) u[j] + \int_0^{kT} g(t - \tau) \bar{u}(\tau) d\tau + y_h[k]
\]

\[
= \sum_{j=0}^{k} \left( \int_{jT}^{(j+1)T} g [(k-j)T] d\tau \right) u[j] + \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} [g(kT - \tau)] (\lambda (\tau) u[j] + \sigma(\tau)) d\tau + y_h[k]
\]

\[
= \sum_{j=0}^{k-1} \left( \int_{jT}^{(j+1)T} g [(k-j)T] d\tau \right) u[j] + g_d[0] u[k] + \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} [g(kT - \tau)] (\lambda (\tau) u[j] + \sigma(\tau)) d\tau + y_h[k]
\]

\[
= \left[ \sum_{j=0}^{k-1} \left( \int_{jT}^{(j+1)T} g [(k-j)T] + \lambda (jT) g ((k-j)T - \tau) d\tau \right) \right] u[j] + g_d[0] u[k]
\]

\[
\quad + \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} [g(kT - \tau)] \sigma(\tau) d\tau + y_h[k]
\]

\[
= \left[ \sum_{j=0}^{k-1} \left( \int_{jT}^{(j+1)T} (1 + \lambda (jT)) g [(k-j)T] d\tau \right) \right] u[j]
\]

\[
+ \sum_{j=0}^{k-1} \left( \int_{jT}^{(j+1)T} (\lambda (jT + \tau) g ((k-j)T - \tau)) - \lambda (jT) g ((k-j)T) d\tau \right) \right] u[j] + g_d[0] u[k]
\]

\[
+ \int_0^{kT} [g(kT - \tau)] \sigma(\tau) d\tau + y_h[k] \quad \forall k \in \mathbb{Z}_0^+.
\]
Then, since $\lambda_{[k]}[k] = \sigma_{[k]}[k] = 0; \forall k \in \mathbb{Z}_0^+$ then $\xi_{[k]}[k] = 0; \forall k \in \mathbb{Z}_0^+$; it follows that

$$y[k] = \left[ \sum_{j=0}^{k} \left( \int_{jT}^{(j+1)T} (1 + \lambda_{[k]}[j] + \xi_{[k]}[j]) g \left[ (k-j)T \right] \right) dr \right] \cdot u[j] + \int_0^{dT} \left[ \int_{\ell T}^{(\ell T + \tau)} g(\ell T - \tau) \sigma_{(kT)}(\tau) d\tau \right] dy_h[k]; \forall k \in \mathbb{Z}_0^+;$$

which yields (24) and leads to (25). Now, define the sequence $\{\epsilon_{[k]}[j]\}$ by $\epsilon_{[k]}[j] = \lambda_{[k]} + \xi_{[k]}[j] = \int_{jT}^{(j+1)T} (\lambda(jT + \tau)g((k-j)T - \tau) - \lambda[j]g((k-j)T))d\tau; \forall k \in \mathbb{Z}_0^+$

and one gets (27), since $y_{dh}[k] = y_{dh}[k]; \forall k \in \mathbb{Z}_0^+$, from the following relation:

$$y_{dh}[k] + \int_0^{\infty} \left[ \int_{\ell T}^{(\ell T + \tau)} g(\ell T - \tau) \sigma_{(kT)}(\tau) d\tau \right] dy_h[k] \leq \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} g \left[ (k-j)T \right] \right) d\tau \right] \cdot \varphi_{[k]}(y[j], j) - \varphi_{[k]}(-y_d[j], j)) \leq - \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} \epsilon_{[k]}[j] g \left[ (k-j)T \right] \right) d\tau \right] u_{[k]}[k] \leq \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} g \left[ (k-j)T \right] \right) d\tau \right] \cdot u_{[k]}[k]$$

$$\cdot \varphi_{[k]}(-y_d[j], j); \forall k \in \mathbb{Z}_0^+;$$

and then

$$\varphi_{[k]}(y[j], j) \leq - \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} g \left[ (k-j)T \right] \right) d\tau \right] \cdot \varphi_{[k]}(y[j], j) \leq - \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} \epsilon_{[k]}[j] g \left[ (k-j)T \right] \right) d\tau \right] \varphi_{[k]}(y[j], j) \leq - \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} \epsilon_{[k]}[j] g \left[ (k-j)T \right] \right) d\tau \right] \cdot u_{[k]}[k] + \left[ \sum_{j=0}^{\infty} \left( \int_{jT}^{(j+1)T} g \left[ (k-j)T \right] \right) d\tau \right] \cdot (\varphi_{[k]}(y_d[j], j) + \varphi_{[k]}(-y_d[j], j)); \forall k \in \mathbb{Z}_0^+;$$

Finally, (28) follows from (25) and (27).}

The subsequent result establishes lower and upper-bounds for the discretized input-output energy measures based on the formulas obtained in Lemma 8 and strict positive realness of the discrete-time transfer function of the feed-forward linear and time-invariant system. The result invokes the boundedness and the smallness of the parameters which define the calculation of the intersample control input from their preceding values at sampling instants. Note that the control deviation in the intersample periods related to the sampling time instants has to be sufficiently moderate enough so that the hyperstability property is kept from the discrete-time system to the continuous-time one. In particular, the positivity and boundedness of the energy measure at the sampling instants is guaranteed if $\sup_{t \in \mathbb{R}_+} |\sigma(t)|$ is small enough and $\{\Phi_d(T)\}$ is subadditive.
Lemma 9. Assume that $\tilde{g}_d \in \{\text{SPR}\}$ and that $\varphi \in \{\Phi_d(T)\}$. Then, the following properties hold:

(i) The input-output energy measure of the auxiliary discretized system satisfies

$$0 < \gamma_{da} \leq \frac{T}{2\pi \omega} \min_{\omega \in R_{k+}} \text{Re} \tilde{g}_d \left( e^{-i\omega T} \right) \left( \sum_{k=0}^{\infty} u_d^2[k] \right)$$

$$\leq E_d[k] = \sum_{j=0}^{k} T \varphi(y_d[k], j) u_d[k] \leq \gamma_{da}; \quad \forall k > k_0 \in Z_+$$

(34)

for some $\gamma_{da} > \gamma_{da} \in R_+$ and for any nonidentically zero controls and zero initial conditions, i.e., for any forced response.

(ii) Assume, in addition, that the class $\{\Phi_d(T)\}$ consists of subadditive functions. Assume also that $\sup_{\epsilon \in Z_+} |\varphi(\epsilon)|$ is sufficiently small, $\inf_{\epsilon \in Z_+} |\varphi(\epsilon)| > -1$, and $\sup_{\epsilon \in Z_+} |\varphi(\epsilon)| < 1$, where $\varphi(\epsilon) = \lambda(\epsilon) + \xi(\epsilon)$; $\forall \epsilon \in Z_+$, and that, furthermore,

$$\sup_{j, \xi \in Z_+} |\varphi(\epsilon)|$$

$$\leq \gamma_{da}; \quad \forall \epsilon \in \{\Phi_d(T)\}$$

< $\infty$; \quad $\forall \epsilon \in \{\Phi_d(T)\}$

Then, the input-output energy measure of the auxiliary discretized system satisfies

$$0 < \gamma_{ad} \leq \frac{T}{2\pi \omega} \min_{\omega \in R_{k+}} \text{Re} \tilde{g}_d \left( e^{-i\omega T} \right) \left( \sum_{k=0}^{\infty} u_d^2[k] \right)$$

$$\leq E_d[k] = \sum_{j=0}^{k} T \varphi(y_d[k], j) u_d[k] \leq \gamma_{ad}; \quad \forall k > k_0 \in Z_+$$

(36)

for some $\gamma_{ad} > \gamma_{ad} \in R_+$ and for any nonidentically zero controls and zero initial conditions.

Proof. The upper-bounding constraint of (34) follows since

$$E_d[k] = \sum_{j=0}^{k} y_d[k] \varphi(y_d[k], j) T$$

$$\leq \sum_{j=0}^{k} y_d[k] (kT) \varphi(y_d[k], kT) \geq \frac{\gamma_{da}}{T}, \quad \forall k \in Z_+$$

(37)

for some $\gamma_{da} \in R_+$, from the first assumption of the theorem. On the other hand, the initial conditions can be neglected since the assumption $\tilde{g}_d \in \{\text{SPR}\}$, implying that the discrete-time transfer function is (strictly) stable which makes their values irrelevant for purposes of stability analysis. Then, the discrete frequency response being applicable to piecewise-constant inputs in-between any two consecutive sampling instants is given by

$$\tilde{g}_d \left( e^{-i\omega T} \right) = \sum_{n=0}^{\infty} g_d[n] e^{-i\omega T} = \sum_{n=-\infty}^{\infty} g_d[n] e^{-i\omega T}$$

(38)

since the subsequence $\{g[\cdot n]\} = 0$; $\forall n \in Z_+$, and the transfer frequency response of a sampling and zero-order-hold operator $Z$ of period $T$ defined by $Z(T, \omega)(t) = v(t) = v[kT] = v[k]$ for any given $v : R_{k+} \rightarrow R$ and all $t \in [kT, (k+1)T)$, $k \in Z_+$, is given by $Z(T, \omega) = (1 - e^{-i\omega T})/\omega$ [31, 35]. It turns out that $|Z(T, \omega)| = |(1 - e^{-i\omega T})/\omega| < 1$; $\forall \omega(\neq 0) \in R$ and $|Z(T, \omega)| = 1$. Thus, the auxiliary discretized system satisfies the equivalent stability relation in the frequency domain to the discrete-time relation (11). It turns out that Fourier transforms exist in the impulse response and truncated control and auxiliary functions of (11)–(13) since the truncated functions are square-integrable in $l^2 R = [-\infty, \infty]$. By using the discrete Parseval’s theorem under zero initial conditions one has

$$E_d[k]$$

$$= \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} g_d[n] e^{-i\omega T} a_d[k, (\omega)] a_d[k, (-\omega)] d\omega$$

$$= \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} s(kT) e^{-i\omega T} a_d[k, (\omega)]^2 d\omega$$

(39)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 1 - e^{-i\omega} \right] a_d[k, (\omega)]^2 d\omega; \quad \forall k \in Z_+$$

The lower-bounding constraint of (37) follows by using (39) and the discrete Parseval’s theorem for the equivalence between the input-output energy measures from the frequency domain to the discrete-time domain

$$E_d[k] \geq \frac{T}{2\pi} \min_{\omega \in R_{k+}} \text{Re} \tilde{g}_d \left( e^{-i\omega T} \right) \left( \sum_{k=0}^{\infty} u_d^2[k] \right)$$

$$\geq \frac{T}{2\pi} \min_{\omega \in R_{k+}} \text{Re} \tilde{g}_d \left( e^{-i\omega T} \right) \left( \sum_{k=0}^{\infty} u_d^2[k] \right) \geq \gamma_{ad}; \quad \forall k \in Z_+$$

(40)

from the assumption $\tilde{g}_d \in \{\text{SPR}\}$; i.e., $\min_{\omega \in R_{k+}} \text{Re} \tilde{g}_d \left( e^{-i\omega T} \right) = \gamma_{da} > 0$; $\forall k \in Z_+$, $k \geq k_0 \in Z_+$ since

(a) strictly positive real discrete transfer functions have

zero relative degree (i.e., an identical number of poles and zeros) from the Discrete Positive Real Lemma (Szegö-Kalman-Popov Lemma) [13, 31, 34], so that the real parts of their frequency hodographs in the argument $e^{-i\omega T}$ are positively lower-bounded;
(b) \[ \text{Im} \hat{g}_d(e^{-\omega T}) = -\text{Im} \hat{g}_d(e^{\omega T}); \quad \forall \omega \in \mathbb{R}_0, \]
so that their integrals in the argument \( \omega \) on frequency intervals \([-k\pi, k\pi]\) for any \( k \in \mathbb{Z}_1 \) are null.

Property (i) has been proved. To prove Property (ii), note from (28) [Lemma 8(ii)] that
\[
0 < E[k] \leq \left( 1 + \sup_{k \in \mathbb{Z}_0} |e[k]| \right) \gamma_{da} \\
+ \sum_{j=0}^{\infty} \left[ \sum_{\ell=0}^{\ell} \left( \sum_{j=0}^{j+1} g(k-\ell T) \right) \right] \cdot \left( q(e) (y_d[j], j) + q(e) (-y_d[j], j) \right) - \gamma_{dh}[k] \\
- \int_0^\infty \left[ g(\ell T - \tau) \right] \sigma(kT)(\tau) d\tau < \gamma_{da} = K \gamma_{da},
\]
for some \( K \in \mathbb{R}_+ \), provided that \( \sup_{k \in \mathbb{Z}_0} |e[k]| < -1 \), and \( \sup_{k \in \mathbb{Z}_0} |e[k]| < \infty \), where \( e[k] = \lambda[k] + \xi[k] \); \( \forall k \in \mathbb{Z}_0 \), since \( g(t) \) is bounded and converges exponentially to zero since \( \hat{g}_d \in \{\text{SPR}\} \) and then convergent (that is, strictly stable in the discrete context), \( \{y_{dh}[k]\} \rightarrow 0 \), and it is a bounded sequence. \( \square \)

The next result, which is a preparatory result to then establish the main asymptotic hyperstability result, addresses the relevant property that the input-output measure \( E(t) \) of the continuous-time system is positively lower-bounded and finitely upper-bounded for all time under the conditions of Lemmas 4 and 9.

**Lemma 10.** If \( \bar{\lambda} \) and \( \bar{\sigma} \) are sufficiently small then
\[
\gamma_0 - \gamma_d \leq E(t) - E_d[k] \\
\leq \frac{1 + \bar{\lambda}^2}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) |\hat{u}[i]\| \omega \| \omega \| d\omega \\
+ \bar{\sigma} \left( 2\bar{\lambda} \sup_{0 \leq j \leq k} u_{j+1}^2 + \bar{\sigma} \right) \int_0^{kT} |g(kT - \tau)| d\tau \leq \gamma_0 \quad (42)
\]
provided that \( \gamma_0 \leq E_d[k] \leq \gamma_d; \quad \forall k \in \mathbb{Z}_+ \)

Proof. One gets the following from direct calculations by using the continuous-time control laws (9)-(10) and (16) if \( \bar{\lambda} = \sup_{t \in \mathbb{R}_+} |\bar{\lambda}(t)| \) and \( \bar{\sigma} = \sup_{t \in \mathbb{R}_+} |\sigma(t)| \):

\[
- \frac{\bar{\lambda}^2}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) |\hat{u}[i]\| \omega \| \omega \| d\omega - \bar{\sigma} \left( 2\bar{\lambda} \sup_{0 \leq j \leq k} u_{j+1}^2 + \bar{\sigma} \right) \int_0^{kT} |g(kT - \tau)| d\tau \leq E(t) - E_d(kT, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{y}(i) \omega \]
\[
\cdot \hat{u}(-\omega) d\omega - E_d(kT, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) |\hat{u}(\omega)|^2 d\omega - E_d(kT, t) \\
= \sum_{j=0}^{ \infty} \sum_{\ell=0}^{j+1} \int_{\ell T}^{(j+1)T} \int_{jT}^{(j+1)T} g(\tau - \theta) \left[ \lambda_{jT}(\theta) u_{[j]} \right] \sigma(\theta) \] \[
\cdot \left[ \lambda_{(j+1)T}(\tau) u_{[j+1]} \right] + \sigma(\tau) \right) \hat{u}_{d,j}(\omega) \right) d\tau + \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) \hat{u}(\omega) \]
\[
\cdot \hat{u}_{d,j}(\omega) \right) d\omega - \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} g_d[n] e^{-i\omega T} \hat{u}_{d,j}(\omega) \hat{u}_{d,j}(\omega) \hat{u}(\omega) \right) d\omega \leq \frac{1 + \bar{\lambda}^2}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) |\hat{u}[i]\| \omega \| \omega \| d\omega - \frac{T}{2\pi} \]
\[
\cdot \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} g_d[n] e^{-i\omega T} \hat{u}_{d,j}(\omega) \hat{u}_{d,j}(\omega) \hat{u}(\omega) \right) d\omega + \bar{\sigma} \left( 2\bar{\lambda} \sup_{0 \leq j \leq k} u_{j+1}^2 + \bar{\sigma} \right) \int_0^{kT} |g(kT - \tau)| d\tau \quad \forall t \in (kT, (k + 1) T), \quad \forall k \in \mathbb{Z}_r, \]
Theorem 11. Consider the dynamic system (2) subject to a control:

\[ u(t) = (1 + \lambda(t)) u[k] + \sigma(t); \quad \forall t \in \mathbb{R}_0^+ \]

\[ u[k] = \varphi[y[k], k]; \quad \forall k \in \mathbb{Z}_0^+, \text{ where } \varphi[k] \in \{ \Phi_d(T) \}. \tag{46} \]

Assume that

(i) \( \varphi(y[k], kT) \) is in the class \( \Phi_d(T) \) (that is, \( \varphi(y[k], kT) \in \{ \Phi_d(T) \} \) defined by the discrete Popov's type sum inequality \( \Sigma_{\ell=0}^{\infty} y_{d}[\ell] u[\ell] \geq -\gamma_d T \) for some \( \gamma_d > 0 \) \( \forall k \in \mathbb{Z}_0^+ \); and some \( k_0 \in \mathbb{Z}_0^+ \);

(ii) \( \varphi(x, kT) \) is subadditive for all \( x \in \mathbb{R}; \forall k \in \mathbb{Z}_0^+ \);

(iii) \( \lambda(t) \) and \( \sigma(t) \) are sufficiently small;

(iv) \( \inf_{k \in \mathbb{Z}_0^+} e[k] > -1 \) and \( \sup_{k \in \mathbb{Z}_0^+} |e[k]| < \infty \), where \( e[k] = \lambda[k] + \xi[k]; \forall k \in \mathbb{Z}_0^+ \) with

\[ \xi[k][j] = \int_{jT}^{(j+1)T} (\lambda(jT + \tau) g((k-j)T - \tau) \]  

\[ - \lambda[j] g((k-j)T)) d\tau; \quad \forall k \in \mathbb{Z}_0^+. \tag{47} \]

Then, the following properties hold:

(i) The discrete closed-loop system is asymptotically hyperstable.

(ii) \( u(t) \to 0 \) as \( t \to \infty \) and the control and output are bounded for all time.

\[ y_0 - y_d \leq \frac{\lambda^2}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega) |\tilde{u}(\omega)|^2 d\omega \]

\[ -\tilde{\sigma} \left( \frac{2\lambda}{\gamma} \sup_{0 \leq j < k} u[j] + \tilde{\sigma} \right) \int_0^{kT} |g(kT - \tau)| d\tau \]

\[ \leq E(t) - E_d[k] \]

\[ \leq \frac{1 + \lambda^2}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega) |\tilde{u}(\omega)|^2 d\omega \]

\[ + \tilde{\sigma} \left( \frac{2\lambda}{\gamma} \sup_{0 \leq j < k} u[j] + \tilde{\sigma} \right) \int_0^{kT} |g(kT - \tau)| d\tau \]

\[ \leq y_0 - y_d; \quad \forall k(k > k_0) \in \mathbb{Z}_0^+; \quad \forall t \in (0, T) \]

if \( \lambda \) and \( \tilde{\sigma} \) are sufficiently small such that (43) holds leading to \( y_0 > 0 \) for any given positive \( y_d \) and \( y_T > y_d \).

The main first result follows.
Since it has a critically stable pole in the discrete domain it is positive real although it is not strictly positive real.

Example 13. Consider the discrete transfer function $\tilde{g}_d(z) = (z^2 + d_1z + d_2)/(z^2 - a_1z - a_2)$ with real numerator and denominator coefficients. Its strict positive realness requires as necessary conditions its convergence (i.e., strict stability in the discrete sense), so that its poles have to be within the complex open unit circle as well as the convergence of its inverse, so that, its zeros have also to lie inside the open unit circle and, furthermore, its relative degree (i.e., its pole-zero excess has to be zero). Note that the last condition is not needed for strict (nonstrong) positive realness of continuous transfer functions where their positive real part can converge to zero as $|s| \to \infty$. By using the homographic bilinear transformation $z = (1 + s)/(1 - s)$, which transforms the unit circle in the left-hand-side complex plane, to get the auxiliary continuous transfer function $\tilde{g}_d(s) = \tilde{g}_d(z) = (1 + s)/(1 - s)$, it is found in [13] that the denominator and numerator polynomials of $\tilde{g}_d(z)$ are subject to $a_2 \leq 1$, $a_1 + a_2 < 1$, and $a_2 > -1$. Take the particular values $a_1 = 0.5$ and $a_2 = 0.4$ leading to the values of poles $z_1 = 0.5$ and $z_2 = 0.4$ and the coefficients of the numerator polynomial are fixed to $d_1 = 1.2$, $d_2 = 0.3$. Then, we fix a discrete transfer function $\hat{g}_d(z) = (z^2 + 1.2z + 0.3)/(z^2 - 0.5z - 0.4)$ yielding $\min_{\omega \in \mathbb{R}} \Re \hat{g}(s) \geq 0.09$, so $\hat{g}_d \in \{\text{SSPR}\}$. The decomposition of $\tilde{g}_d(z)$ in simple fractions leads to $\tilde{g}_d(z) = 1 + 1.68/(z - 0.93) + 0.758/(z + 0.43)$. By using an ideal (i.e., with instantaneous sampling) sampler and zero-order-hold device of period $T$, of transfer function $1 - e^{-Ts}$, on a continuous transfer function $\tilde{g}(s)$ to be specified to generate $\hat{g}_d(z)$ one has the identity $Z((1 - e^{-Ts})\tilde{g}(s)/s) = \tilde{g}_d(z)$, where $Z(F(s))$ is an abbreviate usual informal notation to denote $F^*(s) = L((f(t))^*)$, where $L(*)$ and $Z(*)$ denote the Laplace transform of $(*)$, of argument $s$, and $Z$-transform of $(*)$, of argument $z = e^{Ts}$, and $f^*(t) = \sum_{j=0}^\infty f(t)\delta(t - jT)$ denotes the pulsed function of the sampled function $f(t)$ with sampling period $T$. As a result, we get for a sampling period $T = 0.1$ sec.: 

\[
\hat{g}(s) = sZ^{-1}\left(\frac{z}{z - 1} + \frac{1.68}{z - 0.93} + \frac{0.758}{z + 0.43}\right)
\]

\[
= 1 + \frac{0.0298}{s + 0.726} + \frac{15.92}{s + 8.44}
\]

\[
= \frac{s^2 + 25.086s + 17.966172}{s^2 + 9.166s + 6.1274}
\]

Note that $\tilde{g} \in \{\text{SPR}\}$ (and also $\hat{g} \in \{\text{SSPR}\}$ since its relative degree is zero) is the continuous-time transfer function which generates $\tilde{g}_d(z)$ under an ideal sampler and zero-order-hold device of period $T = 0.1$ sec. so that $\min_{\omega \in \mathbb{R}} \Phi(\tilde{g}(s)) = \min_{\omega \in \mathbb{R}_0} \Phi(\hat{g}(\omega)) > 0$. Thus, it is guaranteed that the continuous-time energy measure is associated with the forced output $E(t) > 0$ for all $t > 0$ (i.e., it is positively lower-bounded for all time) for any piecewise-continuous control irrespectively of the particular controller in operation. In order to guarantee that any such a control that converges asymptotically to zero it is necessary to finitely upper-bound such an energy for all time. The above concerns also guarantee that $E[k] = E(kT) \geq \gamma_{d0} > 0$ for all integer $k > 0$ under any piecewise-constant control $u[k] = u(kT)$ with eventual jumps at sampling instants of any period $T > 0$. For such a control, identical conclusion arises from the discrete $\tilde{g}_d(z)$ which is also strongly strictly positive real. Assume that $u[k] = -\varphi(y[k], k)$ and $\forall k \in \mathbb{Z}_+$ with $\varphi \in \{\Phi_d\}$, where the class $\{\Phi_d\}$ is defined as the set of sequences $\{\varphi(y[k], k)\}$ which satisfy Popov's inequality $T \sum_{j=0}^k y[j] \varphi(y[j], k) \geq -\gamma_d; \forall k \in \mathbb{Z}_+$ for some finite $\gamma_d \in \mathbb{R}_+$ which is identical to

\[
E[k] = T \sum_{j=0}^k y[j] \varphi(y[j], k) \leq \gamma_d < +\infty;
\]

$\forall k \in \mathbb{Z}_+$

A valid choice to parameterize the discrete Popov's inequality is the following one:

\[
T \sum_{j=0}^k y[j] \varphi(y[j], k) = -T \left(M \sum_{j=0}^k y[j] + \sum_{j=0}^k \varepsilon[j]\right)
\]

\[
= -TM(1 - y[k+1]) - T \sum_{j=0}^k \varepsilon[j] \geq -\gamma_d, \forall k \in \mathbb{Z}_+
\]

for some real constants $M > 0$, $\nu \in (0, 1)$ and some real summable sequence $\{\varepsilon[k]\}$. Then, the class $\{\Phi_d\}$ is defined by any real sequence $\{\varphi(y[k], k)\}$ being parameterized by any real constants $M > 0$, $\nu \in (0, 1)$ and any real summable sequence $\{\varepsilon[k]\}$ such that

\[
u[k] = -\varphi(y[k], k)
\]

\[
= \begin{cases} 
M^\nu y[k] + \varepsilon[k] & \text{if } y[k] \neq 0 \\
0 & \text{if } y[k] = 0,
\end{cases}
\]

$\forall k \in \mathbb{Z}_+$ and $\gamma_d \geq \max(TM(1 - \nu) + \mathbb{E}, \mathbb{F})$ for some given $\mathbb{E} = T \max_{k \in \mathbb{Z}_+} \sum_{j=0}^k \varepsilon[j]$. The continuous-time control is generated as

\[
u(t) = (1 + \lambda(y(t), t)) u(y[k], k) + \sigma(y(t), t);
\]

$\forall t \in \mathbb{R}_+$

Theorem 11 guarantees that the continuous-time controller

\[
u(t) = (1 + \lambda(t)) u[k] + \sigma(t);
\]

\[
u[k] = -\varphi[y[k], k];
\]

$\forall t \in [kT, (k + 1)T), \forall k \in \mathbb{Z}_+$

where $\varphi[k] \in \{\Phi_d(T)\}$, satisfies Popov's type discrete inequality, which makes the feedback system globally asymptotically stable under the theorem hypothesis. Thus, the discrete-time closed-loop system is asymptotically hyperstable. In particular, under the additional assumptions in Theorem 11(iii), the closed-loop continuous-time system is asymptotically hyperstable.
One of the referees has given a comment of interest on this example related to continuous versus discrete counterpart properties which has motivated us to give a further intuitive discussion on it. Note the following. In the continuous-time case, there are two kinds of strictly positive real transfer functions as it is well known, namely, the so-called strict positive real which can be zero (or, respectively, positive) at infinity frequency and which, a realizable, has relative degree one (or, respectively zero). In the first case, the direct input-output interconnection gain \(d\) is zero in a state-space representation (\(A,b,c,d = 0\)) of the transfer function. The transfer functions of the second case are often referred to as the class of strong strict positive real transfer functions. This class is not a disjoint class of first above one of strict positive real transfer functions since it is included in it as a potential particular case. Note that the members of this strong class of strict positive real transfer functions have relative degree zero and a positive direct input-output interconnection gain \(d\). Related to this concern, the Yakubovich-Kalman-Popov lemma needs, for its own coherency, that the input-output interconnection \(d\)-gain be nonzero since; otherwise, one of the relevant constraint in the lemma fails, [13]. So, strict positive realness in the discrete domain in this context is typically strong and then the relative degree of its transfer function is zero so that there is a direct interconnection positive gain from the input to the output. Since we assume that the discrete system is strictly positive real, it has a positive input-output interconnection from the input to the output and the minimum value of the real part of its frequency response is strictly positive even as the frequency tends to infinity. On the other hand, note that a zero-order hold “converts” any strictly proper (continuous-time) transfer function of any relative degree into a strictly discrete one of unity relative order while it “converts” a continuous one of zero relative order (i.e., biproper) into a discrete one still of zero-relative order and with the same input-output interconnection gain. See [48]. Therefore, we deal in this example with a biproper strict positive real transfer function of positive input-output interconnection gain identical to that of its discrete-time counterpart got through a ZOH sampling and hold device.

Example 14. Consider that a controller of the form \(u(t) = -\varphi(y(t),t);\forall t \in R_0\), is used for the continuous-time feedforward transfer function (48) so that \(\varphi \in \{0\}\) and \(\{0\}\) is class in Example 13 integral an inequality type Popov's inequality (21) for all \((t \geq t_0) \in R_0\) and some finite \(t_0 \in R_0\). That is, the control function belongs to a class satisfying continuous-time integral Popov's inequality. Then, \(u(t) \to 0\) as \(t \to \infty\) and \(u(t)\) and \(y(t)\) are almost everywhere bounded for all time since \(\tilde{g} \in \{SSPR\}\), then being (strictly) stable with input-output energy measure being positively lower-bounded and also finitely upper-bounded for all \(t \in R_0\) from Popov's-type integral inequality. Potential useful controllers are, for some given controller gain \(k_u \in R_0\),

\[
\begin{align*}
\phi(t) &= \begin{cases} 
0 & \text{for } t \in [0, 1) \\
\frac{k_u}{t^2 y(t)} & \text{for } t \geq 1
\end{cases} \\
\phi(t) &= \begin{cases} 
0 & \text{for } t \in [0, 2) \\
\frac{k_u}{t \ln^2 t} y(t) & \text{for } t \geq 2
\end{cases} \\
\phi(t) &= \begin{cases} 
0 & \text{for } t \in [0, 1) \\
\frac{k_u}{e^{3t}} y(t) & \text{for } t \geq 1
\end{cases} \\
\phi(t) &= \begin{cases} 
k_u t^2 e^{-t} y(t) & \text{for } t \geq 0,
\end{cases}
\end{align*}
\]

where Popov's inequality on \([0, +\infty]\) of (49)–(52) is \(\int_0^\infty \varphi(y(t),t)dt = -k_{u}\), that of (57) is \(\int_0^\infty y(t)\varphi(y(t),t)dt = -2k_{u}/e\), and that of (58) is \(\int_0^\infty y(t)\varphi(y(t),t)dt = -2k_{u}/e\). Since the respective integrands are monotonically decreasing then there is some \(t_0 \in R_0\) and some \(y = y(t_0) \in R_0\) such that \(\int_0^{t_0} y(t)\varphi(y(t),t)dt \geq \gamma\) for all \((t \geq t_0) \in R_0\). The obtained closed-loop systems are in all cases asymptotically hyperstable. The piecewise-constant controls with eventual discontinuities at sampling instants generated from any discrete-time controller (51) of class \(\{\mathbb{P}_d\}\) of Example 13 lead also to asymptotic hyperstability of the discretized system since \(\tilde{g}_d \in \{SSPR\}\).

Example 15. Consider the continuous-time open-loop system (2) under a real sampler of sampling period \(T\) and data picking-up nonzero duration \(e \in (0, T)\). The state and output trajectory solutions are

\[
\begin{align*}
x(kT + \sigma) &= e^{kT} x(kT) \\
&+ \int_0^{\min(x,e)} e^{A(t - e)} b u(kT + t) dt
\end{align*}
\]

\[
y(kT + \sigma) = e^{kT} x(kT + \sigma) + du(kT + \sigma)
\]

for \(\sigma \in [0, T]; \forall k \in \mathbb{Z}_0\). In particular,

\[
x((k + 1)T) = e^{AT} x(kT) + \int_0^e e^{AT} bu(kT + t) dt
\]

\[
= e^{AT} x(kT) + \left(\int_0^e e^{AT} b du(kT + t) dt\right) u[k] + \int_0^e e^{AT} b u(kT + t) dt
\]
\[ y[k] = e^{AT} x_0 + \sum_{j=0}^{k-1} g_d(k-j) x(j) \]

where \( \rho = \int_0^\infty e^{AT} b d\tau / \int_0^\infty e^{AT} b d\tau \) and \( \bar{u}(kT + \tau) = u(kT + \tau) - u[k] \) for \( \tau \in [0, T] \), \( \forall k \in \mathbb{Z}_0^+ \). Then, if \( \Phi(T) = e^{AT} \) and \( \Gamma(T) = \int_0^T e^{AT} b d\tau \) provided that the control is continuous in the open intersample periods \((kT, kT + \varepsilon)\); \( \forall k \in \mathbb{Z}_0^+ \), one has by applying the mean value theorem

\[ y[k] = e^T e^{kAT} x_0 + \sum_{j=0}^{k-1} g_d(k-j) x(j) \]

The above equations imply that if \( \tilde{g}_d \in \{\text{SPR}\} \), then the forced output \( y_f[k] = y[k] - e^T e^{kAT} x_0 \); \( \forall k \in \mathbb{Z}_0^+ \) generates an associate input-output energy measure which is strictly positive for \( k \in \mathbb{Z}_0^+ \), since \( \rho \in (0, 1) \) for real sampling of duration \( \varepsilon \in (0, T) \). This is a necessary condition for asymptotic hyperstability under an appropriate class of controllers satisfying Popov's inequality. However, the incremental input-output energy measure generated by the incremental forced output \( e^T \sum_{j=0}^{k-1} e^{(k-j)AT} \int_0^T \lambda_{jT+\varepsilon}(\tau) e^{-At} b d\tau u[j] \) is not guaranteed to be nonnegative in general. A sufficient condition for that is that \( \lim_{\varepsilon \to 0} \sup_{\tau \in [0, \varepsilon]} \| \lambda_{jT+\varepsilon}(\tau) \| \) is sufficiently small related to the amount \((- (1 - \rho)/\varepsilon) \min_{\omega \in [0, \pi]} \Re \tilde{g}_d(e^{i\omega}) \).

### 6. Conclusions

This paper has been devoted to obtain and to prove the asymptotic hyperstability property of a linear and time-invariant continuous-time system based on some given and previously proved auxiliary preparatory lemmas. Basically, it is proved that the continuous-time closed-loop system is asymptotically hyperstable under the assumptions that the discretized feed-forward transfer function is strictly positive real and that the discretized system at sampling instants of constant sampling period is asymptotically hyperstable for all nonlinear and time-varying feedback controllers which satisfy certain Popov's type inequality at sampling instants. It is assumed that the continuous-time controller belongs to a class defined in the following way: (a) its values at sampling instants are defined by a class of controllers which satisfy discrete Popov's type inequality; (b) the intersample control is generated from its values at sampling instants with the use of two modulating auxiliary functions which are subject to some extra additional conditions on smallness and boundedness. Such conditions make the overall continuous-time controller to satisfy continuous-type Popov's inequality for all time.

### Appendix

**Output and Input-Output Energy Measure Expressions**

**Auxiliary purely discrete output**

\[ y_d[k] = y_{df}[k] + y_{dh}[k] \]

\[ = \sum_{j=0}^{k} g_d[k-j] u_d[j] + y_h[k] \]
Mathematical Problems in Engineering 15

\[
= \left( \sum_{j=0}^{k} g_d [k - j] \right) u_d [j] + y_h [k]
\]

leads to (5).

\[
= \left( \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} g (kT - \tau) d\tau \right) u_d [j] + g_d (0) u_d [k] + y_d [k]
\]

\[\text{(A.1)}\]

\[\text{Continuous-time output}\]

\[
y (t) = y_f (t) + y_h (t) = \int_{0}^{t} g (t - \tau) u (\tau) d\tau + y_h (t)
\]

\[
= \int_{0}^{t} g (t - \tau) u (\tau) d\tau + \int_{0}^{t} g (t - \tau) \bar{u} (\tau) d\tau + y_h (t)
\]

\[\text{(A.2)}\]

\[\text{Sampled real output}\]

\[
y [k] = y_f (kT) + y_h (kT)
\]

\[
= \int_{0}^{kT} g (kT - \tau) u (\tau) d\tau + y_h [k]
\]

\[\text{(A.3)}\]

\[\text{Output energy measure on the discrete-time interval } [0, kT]\]

\[
E_d [k] = \sum_{\ell=0}^{n} T \left( \sum_{\ell=0}^{n} g ((n-\ell)T - \tau) d\tau \right) u_d [\ell] u_d [n]
\]

\[\text{(A.4)}\]

\[\text{leads to (6).}\]

\[\text{Input-output energy measure on the discrete-time interval } [0, kT]\]

\[
\text{from (9)-(10) and (A.3)}
\]

\[
E [k] = \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} y (\tau) u (\tau) d\tau = \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} y (\tau)
\]

\[\text{(A.5)}\]

\[\text{leads to (11).}\]

\[\text{Continuous-time energy measure on } [0, T]\]

\[
E (t) = \int_{0}^{\infty} \left[ \left( \sum_{j=0}^{\infty} g_d [k (\tau) - j] \right) u_{[k(\tau)]} [j]
\]

\[\text{(A.6)}\]

\[\text{where } k(\tau) = \max (z \in Z_+: z T \leq \tau) \text{ and } k = k(t) = \max (z \in Z_+: z T \leq t); \forall t \in \mathbb{R}_+\text{, which yields (13).}\]
Data Availability

The underlying data to support this study are included within the references.

Conflicts of Interest

The author declares that he does not have any conflicts of interest.

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