Research Article

Boundary Control Problem for Heat Convection Equations with Slip Boundary Condition

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We analyze an optimal boundary control problem for heat convection equations in a three-dimensional domain, with mixed boundary conditions. We prove the existence of optimal solutions, by considering boundary controls for the velocity vector and the temperature. The analyzed optimal control problem includes the minimization of a Lebesgue norm between the velocity and some desired field, as well as the temperature and some desired temperature. By using the Lagrange multipliers theorem we derive an optimality system. We also give a second-order sufficient condition.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain with Lipschitz boundary Γ . We deal with the existence of weak solutions of a boundary value problem describing the motion of a viscous heat conduction fluid in Ω , with a Navier slip condition on the boundary for the velocity vector. This model is given by the following system of partial differential equations:

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p + \beta \mathbf{G} T = \mathbf{f} \quad \text{in } \Omega, \quad (1)$$

$$-\kappa \Delta T + \mathbf{u} \cdot \nabla T = g \quad \text{in } \Omega, \quad (2)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega. \quad (3)$$

Here, $\mathbf{u}(x)$ is the velocity field, $p(x)$ represents the hydrostatic pressure, and $T(x)$ is the temperature of the fluid at point $x \in \Omega$; the constant $\nu > 0$ denotes the kinematic viscosity; β is the volume extension coefficient; $\mathbf{G}(x)$ is the free fall acceleration vector; the field $\mathbf{f}(x)$ represents external source of linear momentum; the constant $\kappa > 0$ is the temperature conduction and $g(x)$ is the volume density of heat sources. Without loss of generality, we assume that density of the fluid is equal to one; thus if the variables in (1)-(2) are nondimensionalized, then the coefficient of viscosity ν is

simply the inverse of the Reynolds number Re (see, for example, [1] p. 166 and [2]).

We prove the existence of weak solutions for system (1)–(3). Moreover, we study an optimal boundary control problem. To do this, we consider the following boundary conditions:

$$\mathbf{u}_{\mathbf{g}_1} = \begin{cases} \mathbf{u}_0 & \text{on } \Gamma_1, \\ \mathbf{g}_1 & \text{on } \Gamma_2, \end{cases}$$

$$\mathbf{u} \cdot \mathbf{n} = 0,$$

$$[D(\mathbf{u}) \mathbf{n} + \alpha \mathbf{u}]_{\text{tang}} = 0 \quad \text{on } \Gamma_3, \quad (4)$$

$$T = g_2 \quad \text{on } \Gamma_4,$$

$$\kappa \left(\frac{\partial T}{\partial \mathbf{n}} + \gamma T \right) = \varphi \quad \text{on } \Gamma_5.$$

Here the boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma_4 \cup \Gamma_5$, where $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \bar{\Gamma}_1 \cap \bar{\Gamma}_3 = \bar{\Gamma}_2 \cap \bar{\Gamma}_3 = \bar{\Gamma}_4 \cap \bar{\Gamma}_5 = \emptyset$, and $\mathbf{n}$ denotes the outward normal vector on Γ . The function $\mathbf{u}_0$ is defined on Γ_1 ; $\gamma \geq 0$ and φ are functions given on Γ_5 , and the functions $\mathbf{g}_1, g_2$ describe the Dirichlet boundary control for velocity $\mathbf{u}$ on Γ_2 and for temperature T on Γ_4 , respectively. The controls

$\mathbf{g}_1, \mathbf{g}_2$ lie in the closed convex sets $\mathcal{U}_1 \subset \mathbf{H}^{1/2}(\Gamma_2)$ and $\mathcal{U}_2 \subset \mathbf{H}^{1/2}(\Gamma_4)$, respectively. The term $[D(\mathbf{u})\mathbf{n} + \alpha\mathbf{u}]_{\text{tang}} := D(\mathbf{u})\mathbf{n} + \alpha\mathbf{u} - [(D(\mathbf{u})\mathbf{n} + \alpha\mathbf{u}) \cdot \mathbf{n}]\mathbf{n}$ represents the tangential component of the vector $D(\mathbf{u})\mathbf{n} + \alpha\mathbf{u}$, where $D(\mathbf{u}) := (1/2)(\nabla\mathbf{u} + \nabla^T\mathbf{u})$ is the deformation tensor, and the real number $\alpha \geq 0$ is the friction coefficient which measures the tendency of the fluid to slip on Γ_3 . When $\alpha = 0$ and the boundary is flat, the fluid slips along the Γ_3 without friction and there is no boundary layers. When $\alpha \rightarrow +\infty$, the friction is so intense that the fluid is almost at rest near the boundary; the condition $[D(\mathbf{u})\mathbf{n} + \alpha\mathbf{u}]_{\text{tang}} = \mathbf{0}, \mathbf{u} \cdot \mathbf{n} = 0$ on Γ_3 converges to the Dirichlet condition. The case when α is a scalar function has been studied in [3], for the 2D nonstationary Navier-Stokes equations. The condition $[D(\mathbf{u})\mathbf{n} + \alpha\mathbf{u}]_{\text{tang}} = \mathbf{0}, \mathbf{u} \cdot \mathbf{n} = 0$ on Γ_3 is a Navier friction boundary condition. The Navier boundary condition was proposed by Navier [4], who claimed that the tangential component of the viscous stress at the boundary should be proportional to the tangential velocity. Navier boundary condition was also derived by Maxwell [5] from the kinetic theory of gases and rigorously justified as a homogenization of the no-slip condition on a rough boundary [6].

Before we describe in detail our results, let us briefly comment on previous related work. The heat convection equations, or also known as Boussinesq model, have been extensively analyzed, and, among the many articles on these equations, we just mention [7–12]. In these papers are studied the existence of very weak, weak, and strong solutions and uniqueness with Dirichlet boundary conditions on the velocity and temperature or Dirichlet boundary condition on the velocity and mixed boundary conditions on the temperature. The tools used are Galerkin method, linearization, and fixed point results. The results obtained are similar to those known for the classical Navier-Stokes equations.

When Navier friction boundary condition is considered, the authors do not know results of existence for the classical Boussinesq model, except work due to Rionero and Mulone [13], where it is considered a slightly more general model on thermodiffusive mixture.

In the context of the classic Navier-Stokes equations there is a good amount of work with Navier friction boundary condition, among which we can mention [3, 14–19], and references therein, where the existence, regularity, and uniqueness of weak and strong solutions are studied. In the context of steady-state microstructure fluids, also called micropolar fluids, we can mention works [20, 21]. In [20] a three-dimensional domain is considered where the density of the fluid is constant, while in [21] the study is performed on a two-dimensional domain with variable density.

For optimal control problems for the classical Boussinesq stationary model, see [22–27], and the references therein. In such works, standard results have been obtained, such as the existence of an optimal solution and stability as well as first-order necessary optimality conditions, from which the authors derive an optimality system. However, as far as we know, for this model there are no known studies of sufficient second-order conditions for the existence of a local extreme, except for the recent work [28], in which the authors study a problem of control of limits associated with the

stationary system of Rayleigh-Bénard-Marangoni (similar to system (1)–(3)), for which they obtain results of existence, uniqueness, and regularity of the model; in addition they provide the existence of optimal solutions, derive optimality conditions, obtain a second-order sufficient condition, and establish a result of uniqueness for the optimal solution.

The outline of this paper is as follows. In Section 2, we introduce the spaces of functions appropriate for the development of the article. In Section 3, we prove the existence and uniqueness of weak solutions to model (1)–(4), by using the Galerkin method. In Section 4, we establish the optimal control problem and also we prove the existence of optimal solutions, and, using the Lagrange multipliers method, we obtain the first-order optimality conditions, from which we derive an optimality system. In Section 5, we obtain a second-order sufficient optimality condition.

2. Preliminaries

Throughout this paper we will use the classical Lebesgue space $L^p(\Omega)$, $1 \leq p \leq \infty$, with norm denoted by $\|\cdot\|_p$. In particular, the norm and inner product in $L^2(\Omega)$ will be represented by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. The norm of $L^p(\Gamma)$ will be denoted by $\|\cdot\|_{L^p(\Gamma)}$. Also we use the Sobolev space $H^1(\Omega) := W^{1,2}(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega)\}$, for a multi-index α , with $|\alpha| = 1$. Its norm will be denoted by $\|\cdot\|_{H^1}$. Spaces of $\mathbb{R}^3$ valued functions, as well as their elements, are represented by bold face letters; thus, for example, we have $\mathbf{H}^1(\Omega)$, $\mathbf{L}^2(\Omega)$, and so on. We will use the Hilbert space $H_0 = \{T \in H^1(\Omega) : T = 0 \text{ on } \Gamma_4\}$ with the inner product $(u, v)_{H_0} := (\nabla u, \nabla v)$ and norm $\|u\|_{H_0} := \|\nabla u\|$. We also consider the following solenoidal Banach spaces $\mathbf{H}_\sigma^1 = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \text{div } \mathbf{u} = 0 \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_3\}$, endowed with the usual norm of $\mathbf{H}^1(\Omega)$; the space $\tilde{\mathbf{H}}_\sigma^1 = \{\mathbf{u} \in \mathbf{H}_\sigma^1 : \mathbf{u} = \mathbf{0} \text{ on } \Gamma \setminus \Gamma_3\}$, which is a Hilbert space with the inner product $(\mathbf{u}, \mathbf{v})_{\tilde{\mathbf{H}}_\sigma^1} := (D(\mathbf{u}), D(\mathbf{v}))$ and norm $\|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1} := \|D(\mathbf{u})\|$.

If X is general Banach space, its topological dual will be denoted by X' and the duality product by $\langle \cdot, \cdot \rangle_{X'}$ or simply by $\langle \cdot, \cdot \rangle$ unless this leads to ambiguity. The space H' denotes the dual of H_0 , the space $\mathbf{H}'_\sigma$ denotes the dual of $\mathbf{H}_\sigma^1$, and the space $\tilde{\mathbf{H}}'_\sigma$ denotes the dual of $\tilde{\mathbf{H}}_\sigma^1$. For simplicity, we consider the spaces $\mathbb{X} = \tilde{\mathbf{H}}_\sigma^1 \times H_0$, $\mathbb{X}' = \tilde{\mathbf{H}}'_\sigma \times H'$. If Γ_k is a connected subset of Γ , we consider the trace space by $\mathbf{H}^{1/2}(\Gamma_k) = \{\mathbf{u}|_{\Gamma_k} : \mathbf{u} \in \mathbf{H}^1(\Omega)\}$ (the restriction of the elements of $\mathbf{H}^1(\Omega)$ to Γ_k) and

$$\begin{aligned} \mathbf{H}_{00}^{1/2}(\Gamma_k) &= \{\mathbf{v} \in \mathbf{L}^2(\Gamma_k) : \text{there exists } \hat{\mathbf{v}} \\ &\in \mathbf{H}^{1/2}(\Gamma), \hat{\mathbf{v}}|_{\Gamma \setminus \Gamma_k} = \mathbf{0}, \hat{\mathbf{v}}|_{\Gamma_k} = \mathbf{v}\}, \end{aligned} \quad (5)$$

$$\tilde{\mathbf{H}}_{00}^{1/2}(\Gamma_k) = \left\{ \mathbf{v} \in \mathbf{H}_{00}^{1/2}(\Gamma_k) : \int_{\Gamma_k} \mathbf{v} \cdot \mathbf{n} d\Gamma = 0 \right\}.$$

The space $\mathbf{H}_{00}^{1/2}(\Gamma_k)$ is a closed subspace of $\mathbf{H}^{1/2}(\Gamma_k)$; moreover, it is continuously embedded in $\mathbf{L}^2(\Gamma_k)$ (see [29, 30]). For the space $\mathbf{H}^{1/2}(\Gamma_k)$, $\mathbf{H}^{-1/2}(\Gamma_k)$ denote its dual space and $\langle \cdot, \cdot \rangle_{\Gamma_k}$ denote the duality. The letter C will denote diverse positive

constants which may change from line to line or even within the same line.

In order to establish the weak formulation of problem (1)–(4) we consider the following result.

Lemma 1 (see [19]). *Let $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and $\mathbf{v} \in \mathbf{H}^1(\Omega)$ be divergence-free vector fields tangent to the boundary Γ . Then,*

$$\begin{aligned} - \int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} \, dx &= 2 \int_{\Omega} D(\mathbf{u}) : D(\mathbf{v}) \, dx \\ &- 2 \int_{\Gamma} [D(\mathbf{u}) \mathbf{n}]_{\text{tang}} \cdot \mathbf{v} \, d\Gamma. \end{aligned} \quad (6)$$

Then, motivated by the formula of integration by parts and Lemma 1, we establish the following definition of weak solution of system (1)–(4).

Definition 2. Let $(\mathbf{f}, g) \in \mathbb{X}'$, $\mathbf{G} \in \mathbf{L}^\infty(\Omega)$, $\mathbf{u}_0 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_1)$, $\mathbf{g}_1 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_2)$, $g_2 \in H_{00}^{1/2}(\Gamma_4)$, $\varphi \in L^2(\Gamma_5)$, and $\gamma \in L^\infty(\Gamma_5)$. A weak solution of (1)–(4) is a pair $(\mathbf{u}, T) \in \mathbf{H}_\sigma^1 \times H^1(\Omega)$ satisfying

$$2\nu(D(\mathbf{u}), D(\mathbf{v})) + 2\alpha\nu \int_{\Gamma_3} \mathbf{u} \cdot \mathbf{v} \, d\Gamma + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) \quad (7)$$

$$+ \beta(\mathbf{G}T, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{\widetilde{\mathbf{H}}_\sigma'},$$

$$\kappa(\nabla T, \nabla z) + (\mathbf{u} \cdot \nabla T, z) + \kappa \int_{\Gamma_5} \gamma T \cdot z \, d\Gamma = \langle g, z \rangle_{H'} \quad (8)$$

$$+ \langle \varphi, z \rangle_{\Gamma_5},$$

$$\mathbf{u} = \mathbf{u}_{\mathbf{g}_1} \quad \text{on } \Gamma \setminus \Gamma_3, \quad (9)$$

$$T = g_2 \quad \text{on } \Gamma_4,$$

for all $(\mathbf{v}, z) \in \mathbb{X}$.

3. Existence of a Weak Solution to Problem (1)–(4)

In order to prove the existence of a solution to system (7)–(9), we reduce the problem to an auxiliary problem with homogeneous conditions for the fields velocity on $\Gamma \setminus \Gamma_3$ and the temperature on Γ_4 . For this, we introduce the following results.

Lemma 3. Assume that $\mathbf{u}_0 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_1)$, $\mathbf{g}_1 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_2)$. For any $\varepsilon > 0$ there exists $\mathbf{u}^\varepsilon \in \mathbf{H}_\sigma^1$ with $\mathbf{u}^\varepsilon = \mathbf{g}_1$ on Γ_2 , $\mathbf{u}^\varepsilon = \mathbf{u}_0$ on Γ_1 , and $\mathbf{u}^\varepsilon = \mathbf{0}$ on Γ_3 such that

$$\begin{aligned} |(\mathbf{v} \cdot \nabla \mathbf{u}^\varepsilon, \mathbf{v})| &\leq \varepsilon \|\mathbf{v}\|_{\widetilde{\mathbf{H}}_\sigma^1}^2 \quad \forall \mathbf{v} \in \widetilde{\mathbf{H}}_\sigma^1, \\ \|\mathbf{u}^\varepsilon\|_{\mathbf{H}_\sigma^1} &\leq C \|\mathbf{u}_{\mathbf{g}_1}\|_{\mathbf{H}^{1/2}(\Gamma \setminus \Gamma_3)}, \end{aligned} \quad (10)$$

where the constant $C > 0$ depends only on Ω . Moreover, if $g_2 \in H_{00}^{1/2}(\Gamma_4)$, then there exists $T^\delta \in H^1(\Omega)$ such that

$$T^\delta = g_2 \quad \text{on } \Gamma_4,$$

$$\frac{\partial T^\delta}{\partial n} + \gamma T^\delta = 0 \quad \text{on } \Gamma_5, \quad (11)$$

$$\|T^\delta\|_4 \leq \delta,$$

$$\|T^\delta\|_{H^1} \leq C \|g_2\|_{H^{1/2}(\Gamma_4)},$$

where the constant C depends only on Ω .

Proof. The existence of $\mathbf{u}^\varepsilon \in \mathbf{H}_\sigma^1$, satisfying the relations given in (10), follows from [21], Lemma 3 (see, also [20], Lemma 3.2). The existence of $T^\delta \in H^1(\Omega)$ satisfying (11) follows from [31]. $\square$

Lemma 4 (see [32]). *Let X be a finite dimensional Hilbert space with inner product $(\cdot, \cdot)_X$ and norm $\|\cdot\|_X$, and let P be a continuous mapping from X into itself such that*

$$(P(\phi), \phi)_X > 0 \quad \text{for } \|\phi\|_X = K > 0. \quad (12)$$

Then, there exists $\phi \in X$, $\|\phi\|_X \leq K$, such that $P(\phi) = 0$.

Rewriting the unknowns $(\mathbf{u}, T) \in \mathbf{H}_\sigma^1 \times H^1(\Omega)$ in the form $\mathbf{u} = \mathbf{u}^\varepsilon + \widehat{\mathbf{u}}$ and $T = T^\delta + \widehat{T}$, where $(\widehat{\mathbf{u}}, \widehat{T}) \in \mathbb{X}$ are new unknown functions, from (7)–(9) we can obtain the following problem. Find $(\widehat{\mathbf{u}}, \widehat{T}) \in \mathbb{X}$ such that

$$\begin{aligned} 2\nu(D(\widehat{\mathbf{u}}), D(\mathbf{v})) + 2\alpha\nu \int_{\Gamma_3} \widehat{\mathbf{u}} \cdot \mathbf{v} \, d\Gamma + (\widehat{\mathbf{u}} \cdot \nabla \widehat{\mathbf{u}}, \mathbf{v}) \\ + (\widehat{\mathbf{u}} \cdot \nabla \mathbf{u}^\varepsilon, \mathbf{v}) + (\mathbf{u}^\varepsilon \cdot \nabla \widehat{\mathbf{u}}, \mathbf{v}) \\ + \beta(\mathbf{G}\widehat{T}, \mathbf{v}) = \langle \widehat{\mathbf{f}}, \mathbf{v} \rangle_{\widetilde{\mathbf{H}}_\sigma'} \quad \forall \mathbf{v} \in \widetilde{\mathbf{H}}_\sigma^1, \end{aligned} \quad (13)$$

$$\kappa(\nabla \widehat{T}, \nabla z) + (\widehat{\mathbf{u}} \cdot \nabla \widehat{T}, z) + (\widehat{\mathbf{u}} \cdot \nabla T^\delta, z) + (\mathbf{u}^\varepsilon \cdot \nabla \widehat{T}, z)$$

$$+ \kappa \int_{\Gamma_5} \gamma \widehat{T} \cdot z \, d\Gamma = \langle \widehat{g}, z \rangle_{H'} \quad \forall z \in H_0,$$

where

$$\begin{aligned} \langle \widehat{\mathbf{f}}, \mathbf{v} \rangle_{\widetilde{\mathbf{H}}_\sigma'} &:= -2\nu(D(\mathbf{u}^\varepsilon), D(\mathbf{v})) \\ &- (\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \beta \mathbf{G}T^\delta, \mathbf{v}) + \langle \mathbf{f}, \mathbf{v} \rangle_{\widetilde{\mathbf{H}}_\sigma'}, \\ \langle \widehat{g}, z \rangle_{H'} &:= -\kappa(\nabla T^\delta, \nabla z) - (\mathbf{u}^\varepsilon \cdot \nabla T^\delta, z) \\ &- \kappa \int_{\Gamma_5} \gamma T^\delta \cdot z \, d\Gamma + \langle g, z \rangle_{H'} \\ &+ \langle \varphi, z \rangle_{\Gamma_5}. \end{aligned} \quad (14)$$

Thus, we have the following result.

Theorem 5. Let $(\mathbf{f}, g) \in \mathbb{X}'$, $\mathbf{G} \in \mathbf{L}^\infty(\Omega)$, $\mathbf{u}_0 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_1)$, $\mathbf{g}_1 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_2)$, $g_2 \in H_{00}^{1/2}(\Gamma_4)$, $\varphi \in L^2(\Gamma_5)$, and $\gamma \in L^\infty(\Gamma_5)$.

Given these conditions, if ν is large enough and $\|\gamma\|_{L^\infty(\Gamma_3)}$ is small enough such that

$$\min \{2\nu, \kappa - \kappa C \|\gamma\|_{L^\infty(\Gamma_3)}\} > C(\beta \|\mathbf{G}\|_\infty + \delta), \quad (15)$$

where C is a positive constant depending on Ω such that $\|\nabla \mathbf{u}\| \leq C\|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}$ and δ is given in (11), then, there exists $(\hat{\mathbf{u}}, \hat{T}) \in \mathbb{X}$, which is a solution of system (13).

Moreover, the following estimate is satisfied:

$$\|(\hat{\mathbf{u}}, \hat{T})\|_{\mathbb{X}} \leq \frac{1}{\delta} \|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'}, \quad (16)$$

where $\tilde{\delta} := \min\{2\nu - \varepsilon, \kappa - \kappa C \|\gamma\|_{L^\infty(\Gamma_3)}\} - C(\beta \|\mathbf{G}\|_\infty + \delta) > 0$.

Proof. The existence of a solution will be proved by using the Galerkin method. For this purpose, we consider an approximate solution of (13) and then we pass to the limit. Since $\tilde{\mathbf{H}}_\sigma^1$ and H_0 are separable spaces, there exist Hilbert spaces bases $\{\mathbf{v}_i\}_{i=1}^\infty$ and $\{z_i\}_{i=1}^\infty$ of $\tilde{\mathbf{H}}_\sigma^1$ and H_0 , respectively. For each $k \in \mathbb{N}$, we consider $\tilde{\mathbf{H}}_k$ the space spanned by $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and H_k the space spanned by $\{z_1, \dots, z_k\}$; let $\mathbb{X}_k = \tilde{\mathbf{H}}_k \times H_k \subset \mathbb{X}$. The scalar product on $\mathbb{X}_k$ is the scalar product $(\cdot, \cdot)_{\mathbb{X}_k}$ induced by $\mathbb{X}$.

Thus, for each $k \in \mathbb{N}$, fixed, we define an approximate solution $(\mathbf{u}_k, T_k) \in \mathbb{X}_k$ of system (13), given by

$$\begin{aligned} \mathbf{u}_k &:= \sum_{i=1}^k \xi_{i,k} \mathbf{v}_i(x), \\ T_k(x) &:= \sum_{i=1}^k \theta_{i,k} T_i(x), \end{aligned} \quad (17)$$

where $\xi_{i,k}, \theta_{i,k} \in \mathbb{R}$, and satisfying

$$\begin{aligned} &2\nu (D(\mathbf{u}_k), D(\mathbf{v}_i)) + 2\alpha\nu \int_{\Gamma_3} \mathbf{u}_k \cdot \mathbf{v}_i d\Gamma + (\mathbf{u}_k \cdot \nabla \mathbf{u}_k, \mathbf{v}_i) \\ &+ (\mathbf{u}_k \cdot \nabla \mathbf{u}^\varepsilon, \mathbf{v}_i) + (\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}_k, \mathbf{v}_i) \\ &+ \beta (\mathbf{G}T_k, \mathbf{v}_i) = \langle \hat{\mathbf{f}}, \mathbf{v}_i \rangle_{\tilde{\mathbf{H}}_\sigma'}, \end{aligned} \quad (18)$$

$$\begin{aligned} &\kappa (\nabla T_k, \nabla z_i) + (\mathbf{u}_k \cdot \nabla T_k, z_i) + (\mathbf{u}_k \cdot \nabla T^\delta, z_i) \\ &+ (\mathbf{u}^\varepsilon \cdot \nabla T_k, z_i) + \kappa \int_{\Gamma_5} \gamma T_k \cdot z_i d\Gamma = \langle \hat{g}, z_i \rangle_{H'} \end{aligned}$$

for all $(\mathbf{v}_i, z_i) \in \mathbb{X}_k$.

In order to prove the existence of a solution for system (18) we apply the result of Lemma 4. For this purpose, we consider the continuous mapping $P : \mathbb{X}_k \rightarrow \mathbb{X}_k$, which for $\mathbf{w}_k = (\mathbf{u}_k, T_k) \in \mathbb{X}_k$ is given by

$$\begin{aligned} (P(\mathbf{w}_k), \mathbf{w})_{\mathbb{X}_k} &= 2\nu (D(\mathbf{u}_k), D(\mathbf{v})) \\ &+ 2\alpha\nu \int_{\Gamma_3} \mathbf{u}_k \cdot \mathbf{v} d\Gamma + (\mathbf{u}_k \cdot \nabla \mathbf{u}_k, \mathbf{v}) \\ &+ (\mathbf{u}_k \cdot \nabla \mathbf{u}^\varepsilon, \mathbf{v}) + (\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}_k, \mathbf{v}) \end{aligned}$$

$$\begin{aligned} &+ \beta (\mathbf{G}T_k, \mathbf{v}) - \langle \hat{\mathbf{f}}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}_\sigma'} \\ &+ \kappa (\nabla T_k, \nabla z) + (\mathbf{u}_k \cdot \nabla T_k, z) \\ &+ (\mathbf{u}_k \cdot \nabla T^\delta, z) + (\mathbf{u}^\varepsilon \cdot \nabla T_k, z) \\ &+ \kappa \int_{\Gamma_5} \gamma T_k \cdot z d\Gamma - \langle \hat{g}, z \rangle_{H'}, \end{aligned} \quad (19)$$

for all $\mathbf{w} = (\mathbf{v}, z) \in \mathbb{X}_k$. By replacing $\mathbf{w}$ by $(\mathbf{u}_k, T_k)$ in (19) and using Lemma 3, we obtain

$$\begin{aligned} (P(\mathbf{w}_k), \mathbf{w}_k)_{\mathbb{X}_k} &\geq 2\nu \|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma^1}^2 - \varepsilon \|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + \kappa \|T_k\|_{H_0}^2 \\ &+ \beta (\mathbf{G}T_k, \mathbf{u}_k) + (\mathbf{u}_k \cdot \nabla T^\delta, T_k) \\ &+ \kappa \int_{\Gamma_5} \gamma T_k \cdot T_k d\Gamma - \langle \hat{\mathbf{f}}, \mathbf{u}_k \rangle_{\tilde{\mathbf{H}}_\sigma'} \\ &- \langle \hat{g}, T_k \rangle_{H'}. \end{aligned} \quad (20)$$

Now, we will bind the right-hand side of (20). Since $\|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + \|T_k\|_{H_0}^2 = \|\mathbf{w}_k\|_{\mathbb{X}_k}^2$, then by using the Hölder and Young inequalities we have

$$\begin{aligned} &|\langle \hat{\mathbf{f}}, \mathbf{u}_k \rangle_{\tilde{\mathbf{H}}_\sigma'} + \langle \hat{g}, T_k \rangle_{H'}| \\ &\leq \|\mathbf{f}\|_{\mathbf{H}_\sigma'} \|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma^1} + \|\hat{g}\|_{H'} \|T_k\|_{H'} \\ &\leq \|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'} \|\mathbf{w}_k\|_{\mathbb{X}_k}. \end{aligned} \quad (21)$$

Applying the Hölder and Young inequalities, and taking into account the inequalities $\|\mathbf{u}\|_4 \leq C\|\nabla \mathbf{u}\|$, $\|\nabla \mathbf{u}\| \leq C\|D(\mathbf{u})\|$, and $\|T^\delta\|_4 \leq \delta$, we obtain

$$\begin{aligned} |\beta (\mathbf{G}T_k, \mathbf{u}_k)| &\leq \beta \|\mathbf{G}\|_\infty \|T_k\| \|\mathbf{u}_k\| \\ &\leq \beta C \|\mathbf{G}\|_\infty \|T_k\|_{H_0} \|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma^1} \\ &\leq \beta C \|\mathbf{G}\|_\infty (\|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + \|T_k\|_{H_0}^2), \\ \left| \kappa \int_{\Gamma_5} \gamma T_k \cdot T_k d\Gamma \right| &\leq \kappa \|\gamma\|_{L^\infty(\Gamma_5)} \|T_k\|_{L^2(\Gamma_5)}^2 \\ &\leq \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} \|T_k\|_{H_0}^2, \\ |(\mathbf{u}_k \cdot \nabla T^\delta, T_k)| &= |(\mathbf{u}_k \cdot \nabla T_k, T^\delta)| \\ &\leq \|\mathbf{u}_k\|_4 \|\nabla T_k\| \|T^\delta\|_4 \\ &\leq \delta C \|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma'} \|T_k\|_{H_0} \\ &\leq \delta C (\|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma'}^2 + \|T_k\|_{H_0}^2). \end{aligned} \quad (22)$$

By replacing (21) and (22) in (20) and taking into account (11), we have

$$\begin{aligned}
& (P(\mathbf{w}_k), \mathbf{w}_k)_{\mathbb{X}_k} \\
& \geq (2\nu - \varepsilon) \|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + (\kappa - \kappa C \|\gamma\|_{L^\infty(\Gamma_3)}) \|T_k\|_{H_0}^2 \\
& \quad - \beta C \|\mathbf{G}\|_\infty \left(\|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + \|T_k\|_{H_0}^2 \right) \\
& \quad - \delta C \left(\|\mathbf{u}_k\|_{\tilde{\mathbf{H}}_\sigma}^2 + \|T_k\|_{H_0}^2 \right) - \|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'} \|\mathbf{w}_k\|_{\mathbb{X}_k} \quad (23) \\
& \geq \min \{2\nu - \varepsilon, \kappa - \kappa C \|\gamma\|_{L^\infty(\Gamma_3)}\} \|\mathbf{w}_k\|_{\mathbb{X}_k}^2 \\
& \quad - (\beta C \|\mathbf{G}\|_\infty + \delta C) \|\mathbf{w}_k\|_{\mathbb{X}_k}^2 - \|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'} \|\mathbf{w}_k\|_{\mathbb{X}_k} \\
& \geq \|\mathbf{w}_k\|_{\mathbb{X}_k} \left[\tilde{\delta} \|\mathbf{w}_k\|_{\mathbb{X}_k} - \|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'} \right],
\end{aligned}$$

where, by hypothesis,

$$\begin{aligned}
\tilde{\delta} &:= \min \{2\nu - \varepsilon, \kappa - \kappa C \|\gamma\|_{L^\infty(\Gamma_3)}\} \\
&\quad - (\beta C \|\mathbf{G}\|_\infty + \delta C) > 0, \quad (24)
\end{aligned}$$

for ε small enough. Then, for $\|\mathbf{w}_k\|_{\mathbb{X}_k} = K$ with $K > (1/\tilde{\delta})\|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'}$, from (23) it follows that

$$(P(\mathbf{w}_k), \mathbf{w}_k)_{\mathbb{X}_k} \geq K(\tilde{\delta}K - \|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'}) > 0. \quad (25)$$

Thus, by Lemma 4, there exists $(\mathbf{u}_k, T_k) \in \mathbb{X}_k$ such that $\|(\mathbf{u}_k, T_k)\|_{\mathbb{X}_k} \geq K$ and

$$(P(\mathbf{u}_k, T_k), (\mathbf{v}, z))_{\mathbb{X}_k} = 0 \quad \forall (\mathbf{v}, z) \in \mathbb{X}_k. \quad (26)$$

Moreover, since $(\mathbf{v}_i, 0), (0, z_i) \in \mathbb{X}_k$, from (19) and (26), we conclude that $(\mathbf{u}_k, T_k)$ is a solution to (18).

We now pass to the limit. Since $\mathbf{w}_k = (\mathbf{u}_k, T_k) \in \mathbb{X}_k$, then (26) implies $(P(\mathbf{w}_k), \mathbf{w}_k)_{\mathbb{X}_k} = 0$ and, from (23), we conclude that

$$\|(\mathbf{u}_k, T_k)\|_{\mathbb{X}_k} \leq \frac{1}{\tilde{\delta}} \|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'}. \quad (27)$$

Thus, from (27), we conclude that the sequence $\{(\mathbf{u}_k, T_k)\}_{k \geq 1}$ is bounded in $\mathbb{X}$; then there exists a subsequence, still denoted by $\{(\mathbf{u}_k, T_k)\}_{k \geq 1}$, and some element $(\hat{\mathbf{u}}, \hat{T}) \in \mathbb{X}$, such that when $k \rightarrow \infty$,

$$\begin{aligned}
\mathbf{u}_k &\rightharpoonup \hat{\mathbf{u}} \text{ weakly in } \tilde{\mathbf{H}}_\sigma^1 \text{ and strongly in } L^2(\Omega), \\
T_k &\rightharpoonup \hat{T} \text{ weakly in } H_0 \text{ and strongly in } L^2(\Omega). \quad (28)
\end{aligned}$$

The convergences (28) allow us to pass to the limit in (18) and to prove that $(\hat{\mathbf{u}}, \hat{T})$ is a solution to system (13). Moreover, from (27), we have

$$\|(\hat{\mathbf{u}}, \hat{T})\|_{\mathbb{X}} \leq \liminf_{k \rightarrow \infty} \|(\mathbf{u}_k, T_k)\|_{\mathbb{X}_k} \leq \frac{1}{\tilde{\delta}} \|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'}, \quad (29)$$

which implies (16). $\square$

Theorem 6 (existence of weak solutions). *Under the conditions of Theorem 5, there exists a solution $(\mathbf{u}, T) \in \mathbf{H}_\sigma^1 \times H^1(\Omega)$ for system (7)–(9). Moreover, the following estimate holds:*

$$\|\mathbf{u}\|_{\mathbf{H}_\sigma^1} + \|T\|_{H^1} \leq \tilde{C}_1 \Theta, \quad (30)$$

where $\tilde{C}_1$ is a positive constant which depends only on $\Omega, \nu, \kappa, \beta, \|\mathbf{G}\|_\infty$, and $\|\gamma\|_{L^\infty(\Gamma_3)}$ and $\Theta = \|\mathbf{u}_{\mathbf{g}_1}\|_{\mathbf{H}^{1/2}(\Gamma \setminus \Gamma_3)} + \|g_2\|_{H^{1/2}(\Gamma_4)} + \|\mathbf{u}_{\mathbf{g}_1}\|_{\mathbf{H}^{1/2}(\Gamma \setminus \Gamma_3)}^2 + \|g_2\|_{H^{1/2}(\Gamma_4)}^2 + \|\varphi\|_{L^2(\Gamma_3)} + \|(\mathbf{f}, g)\|_{\mathbb{X}'}$.

Proof. As a consequence of Theorem 5, we have the fact that there exists a solution $(\mathbf{u}, T) \in \mathbf{H}_\sigma^1 \times H^1(\Omega)$ for system (7)–(9), where $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}^\varepsilon$ and $T = \hat{T} + T^\delta$, with $(\hat{\mathbf{u}}, \hat{T}) \in \mathbb{X}$ being a solution of (13) given in Theorem 5. Thus, from (16), we have

$$\begin{aligned}
\|\mathbf{u}\|_{\mathbf{H}_\sigma^1} + \|T\|_{H^1} &\leq \|\hat{\mathbf{u}}\|_{\tilde{\mathbf{H}}_\sigma^1} + \|\hat{T}\|_{H_0} + \|\mathbf{u}^\varepsilon\|_{\mathbf{H}_\sigma^1} + \|T^\delta\|_{H^1} \\
&\leq \sqrt{2} \|(\hat{\mathbf{u}}, \hat{T})\|_{\mathbb{X}} + \sqrt{2} \|(\mathbf{u}^\varepsilon, T^\delta)\|_{\mathbf{H}_\sigma^1 \times H^1} \\
&\leq \frac{\sqrt{2}}{\delta} \|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'} \\
&\quad + \sqrt{2} \|(\mathbf{u}^\varepsilon, T^\delta)\|_{\mathbf{H}_\sigma^1 \times H^1}. \quad (31)
\end{aligned}$$

From (14) we can obtain

$$\begin{aligned}
\|(\hat{\mathbf{f}}, \hat{g})\|_{\mathbb{X}'} &\leq C_1 \left(\|(\mathbf{u}^\varepsilon, T^\delta)\|_{\mathbf{H}_\sigma^1 \times H^1} + \|(\mathbf{u}^\varepsilon, T^\delta)\|_{\mathbf{H}_\sigma^1 \times H^1}^2 \right. \\
&\quad \left. + \|\varphi\|_{L^2(\Gamma_3)} + \|(\mathbf{f}, g)\|_{\mathbb{X}'} \right), \quad (32)
\end{aligned}$$

where C_1 is a positive constant which depends on $\beta, \|\mathbf{G}\|_\infty, \|\gamma\|_{L^\infty(\Gamma_3)}, \nu$, and κ . Thus, from (31), we have

$$\begin{aligned}
\|\mathbf{u}\|_{\mathbf{H}_\sigma^1} + \|T\|_{H^1} &\leq C_1 \frac{C}{\delta} \left(\|(\mathbf{u}^\varepsilon, T^\delta)\|_{\mathbf{H}_\sigma^1 \times H^1} + \|(\mathbf{u}^\varepsilon, T^\delta)\|_{\mathbf{H}_\sigma^1 \times H^1}^2 \right) \\
&\quad + C_1 \frac{C}{\delta} \left(\|\varphi\|_{L^2(\Gamma_3)} + \|(\mathbf{f}, g)\|_{\mathbb{X}'} \right). \quad (33)
\end{aligned}$$

Since $\|(\mathbf{u}^\varepsilon, T^\delta)\|_{\mathbf{H}_\sigma^1 \times H^1} \leq \|\mathbf{u}^\varepsilon\|_{\mathbf{H}_\sigma^1} + \|T^\delta\|_{H^1}$, then, by Lemma 3, we have

$$\begin{aligned}
\|(\mathbf{u}^\varepsilon, T^\delta)\|_{\mathbf{H}_\sigma^1 \times H^1} &\leq C \left(\|\mathbf{u}_{\mathbf{g}_1}\|_{\mathbf{H}^{1/2}(\Gamma \setminus \Gamma_3)} + \|g_2\|_{H^{1/2}(\Gamma_4)} \right), \\
\|(\mathbf{u}^\varepsilon, T^\delta)\|_{\mathbf{H}_\sigma^1 \times H^1}^2 &\leq C \left(\|\mathbf{u}_{\mathbf{g}_1}\|_{\mathbf{H}^{1/2}(\Gamma \setminus \Gamma_3)}^2 + \|g_2\|_{H^{1/2}(\Gamma_4)}^2 \right). \quad (34)
\end{aligned}$$

Therefore, the above inequalities together with (33) imply (30). $\square$

Remark 7. Since $\text{Re} = 1/\nu$, then condition (15) shows us that the existence of solution for the system (7)–(9) is guaranteed when the Reynolds number is small (the flow is laminar).

Theorem 8 (uniqueness of solutions). *Under conditions of Theorem 6, if ν, κ are large enough and $\|\gamma\|_{L^\infty(\Gamma_5)}$ is small enough such that*

$$\min \{2\nu - \beta C \|\mathbf{G}\|_\infty, \kappa - \beta C \|\mathbf{G}\|_\infty - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)}\} > C\tilde{C}_1\Theta, \quad (35)$$

where C is a positive constant that depends on the domain Ω and $\tilde{C}_1, \Theta$ are given in (30), then the solution $(\mathbf{u}, T) \in \mathbf{H}_\sigma^1 \times H^1(\Omega)$ of system (7)–(9) provided by Theorem 6 is unique.

Proof. Let $(\mathbf{u}_1, T_2), (\mathbf{u}_2, T_2) \in \mathbf{H}_\sigma^1 \times H^1(\Omega)$ be two solutions of system (7)–(9). Then, subtracting the corresponding equations and denoting $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $T = T_1 - T_2$, we obtain $(\mathbf{u}, T) \in \mathbb{X}$ and the following system:

$$2\nu(D(\mathbf{u}), D(\mathbf{v})) + 2\alpha\nu \int_{\Gamma_3} \mathbf{u} \cdot \mathbf{v} d\Gamma + (\mathbf{u}_1 \cdot \nabla \mathbf{u}, \mathbf{v}) + (\mathbf{u} \cdot \nabla \mathbf{u}_2, \mathbf{v}) + \beta(GT, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \tilde{\mathbf{H}}_\sigma^1, \quad (36)$$

$$\begin{aligned} \kappa(\nabla T, \nabla z) + (\mathbf{u}_1 \cdot \nabla T, z) + (\mathbf{u} \cdot \nabla T_2, z) \\ + \kappa \int_{\Gamma_5} \gamma T \cdot z d\Gamma = 0 \quad \forall z \in H_0. \end{aligned} \quad (37)$$

Now, by replacing $\mathbf{v} = \mathbf{u}$ in (36) and $z = T$ in (37) and adding the results, we have

$$\begin{aligned} 2\nu \|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + \kappa \|T\|_{H_0}^2 &\leq |(\mathbf{u} \cdot \nabla \mathbf{u}_2, \mathbf{u})| + \beta |(GT, \mathbf{u})| \\ &\quad + |(\mathbf{u} \cdot \nabla T_2, T)| \\ &\quad + \kappa \left| \int_{\Gamma_5} \gamma T \cdot T d\Gamma \right|. \end{aligned} \quad (38)$$

By using the Hölder and Young inequalities, we obtain

$$\begin{aligned} |(\mathbf{u} \cdot \nabla \mathbf{u}_2, \mathbf{u})| &\leq \|\mathbf{u}\|_3 \|\nabla \mathbf{u}_2\| \|\mathbf{u}\|_6 \\ &\leq C \|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}^2 \|\mathbf{u}_2\|_{\mathbf{H}_\sigma^1}, \\ \beta |(GT, \mathbf{u})| &\leq \beta \|\mathbf{G}\|_\infty \|T\| \|\mathbf{u}\| \\ &\leq \beta C \|\mathbf{G}\|_\infty \|T\|_{H_0} \|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1} \\ &\leq \beta C \|\mathbf{G}\|_\infty \left(\|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + \|T\|_{H_0}^2 \right), \\ |(\mathbf{u} \cdot \nabla T_2, T)| &\leq \|\mathbf{u}\|_3 \|\nabla T_2\| \|T\|_6 \\ &\leq C \|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1} \|T_2\|_{H^1} \|T\|_{H_0} \\ &\leq C \|T_2\|_{H^1} \left(\|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + \|T\|_{H_0}^2 \right), \\ \kappa \left| \int_{\Gamma_5} \gamma T \cdot T d\Gamma \right| &\leq \kappa \|\gamma\|_{L^\infty(\Gamma_5)} \|T\|_{L^2(\Gamma_5)}^2 \\ &\leq \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} \|T\|_{H_0}^2. \end{aligned} \quad (39)$$

By replacing (39) in (38), we can obtain

$$\begin{aligned} (2\nu - \beta C \|\mathbf{G}\|_\infty) \|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}^2 \\ + (\kappa - \beta C \|\mathbf{G}\|_\infty - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)}) \|T\|_{H_0}^2 \\ \leq C \left(\|\mathbf{u}_2\|_{\mathbf{H}_\sigma^1} + \|T_2\|_{H^1} \right) \|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + C \|T_2\|_{H^1} \|T\|_{H_0}^2. \end{aligned} \quad (40)$$

From (30) we have $\|\mathbf{u}_2\|_{\mathbf{H}_\sigma^1} + \|T_2\|_{H^1} \leq \tilde{C}_1\Theta$; then (40) implies

$$\begin{aligned} (2\nu - \beta C \|\mathbf{G}\|_\infty) \|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}^2 \\ + (\kappa - \beta C \|\mathbf{G}\|_\infty - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)}) \|T\|_{H_0}^2 \\ \leq C\tilde{C}_1\Theta \left(\|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}^2 + \|T\|_{H_0}^2 \right), \end{aligned} \quad (41)$$

and thus

$$\begin{aligned} (2\nu - \beta C \|\mathbf{G}\|_\infty - C\tilde{C}_1\Theta) \|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1}^2 \\ + (\kappa - \beta C \|\mathbf{G}\|_\infty - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} - C\tilde{C}_1\Theta) \|T\|_{H_0}^2 \\ \leq 0. \end{aligned} \quad (42)$$

Therefore, if $\min\{2\nu - \beta C \|\mathbf{G}\|_\infty, \kappa - \beta C \|\mathbf{G}\|_\infty - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)}\} > C\tilde{C}_1\Theta$, from (42), we have $\|\mathbf{u}\|_{\tilde{\mathbf{H}}_\sigma^1} = \|T\|_{H_0} = 0$, that is, $\mathbf{u}_1 = \mathbf{u}_2$ and $T_1 = T_2$, and the theorem is proved. $\square$

Remark 9. The condition (35) assures us of the uniqueness of solution of system (7)–(9) when the viscosity ν and the temperature conduction coefficient κ are large enough, which implies that the Rayleigh number is small. That is, the uniqueness of weak solution is obtained when the heat transfer is produced primarily by conduction.

4. The Optimal Control Problem

In this section the statement of the boundary control problem to study is established. We suppose that $\mathcal{U}_1 \subset \tilde{\mathbf{H}}_{00}^{1/2}(\Gamma_2)$ and $\mathcal{U}_2 \subset H_{00}^{1/2}(\Gamma_4)$ are nonempty sets. We consider data $(\mathbf{f}, g) \in \mathbb{X}'$, $\mathbf{G} \in \mathbf{L}^\infty(\Omega)$, $\mathbf{u}_0 \in \tilde{\mathbf{H}}_{00}^{1/2}(\Gamma_1)$, $\varphi \in L^2(\Gamma_5)$, $\gamma \in L^\infty(\Gamma_5)$, and the functions $\mathbf{g}_1 \in \mathcal{U}_1$, $g_2 \in \mathcal{U}_2$, describing the Dirichlet boundary controls for $\mathbf{u}$ on Γ_2 and T on Γ_4 , respectively. For simplicity, here and hereinbelow, we will use the product space $\mathbb{H} = \mathbf{H}_\sigma^1 \times H^1(\Omega)$, and we consider the following objective functional $J : \mathbb{H} \times \mathcal{U}_1 \times \mathcal{U}_2 \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} J(\mathbf{u}, T, \mathbf{g}_1, g_2) &= \hat{J}(\mathbf{u}, T) + \frac{b_1}{2} \|\mathbf{g}_1\|_{\mathbf{H}^{1/2}(\Gamma_2)}^2 \\ &\quad + \frac{b_2}{2} \|g_2\|_{H^{1/2}(\Gamma_4)}^2, \end{aligned} \quad (43)$$

where $\hat{J} : \mathbf{H}_\sigma^1 \times H^1(\Omega) \rightarrow \mathbb{R}$ is a weakly lower semicontinuous functional, and the nonnegative constants b_1, b_2 measure the cost of the controls.

The following are examples of the weakly lower semicontinuous functional in (43) most interesting from a physical viewpoint:

$$\begin{aligned} J_1(\mathbf{u}) &= \frac{1}{2} \int_{\Omega} |\mathbf{u} - \mathbf{u}_d|^2 d\Omega, \\ J_2(T) &= \frac{1}{2} \int_{\Omega} |T - T_d|^2 d\Omega, \\ J_3(\mathbf{u}) &= \frac{\nu}{2} \int_{\Omega} |\nabla \mathbf{u} + \nabla^T \mathbf{u}|^2 d\Omega, \end{aligned} \quad (44)$$

Find $(\mathbf{u}, T, \mathbf{g}_1, g_2) \in \mathbb{H}$

$$\text{such that } J(\mathbf{u}, T, \mathbf{g}_1, g_2) = \frac{1}{2} \|\mathbf{u} - \mathbf{u}_d\|^2 + \frac{1}{2} \|T - T_d\|^2 + \frac{b_1}{2} \|\mathbf{g}_1\|_{\mathbf{H}^{1/2}(\Gamma_2)}^2 + \frac{b_2}{2} \|g_2\|_{H^{1/2}(\Gamma_4)}^2 \quad (45)$$

reaches its minimum over the weak solutions of system (1)–(4).

The set of admissible solutions of optimal control problem (45) is defined by

$$\begin{aligned} \mathcal{S}_{\text{ad}} &= \{\mathbf{s} = (\mathbf{u}, T, \mathbf{g}_1, g_2) \in \mathbb{H} \times \mathcal{U}_{\text{ad}} \text{ such that } J(\mathbf{s}) \\ &< \infty \text{ and } \mathbf{s} \text{ satisfies (7)–(9)}\}, \end{aligned} \quad (46)$$

where $\mathcal{U}_{\text{ad}} := \mathcal{U}_1 \times \mathcal{U}_2$. A similar control problem has been studied in [20, 21] for micropolar fluids. In [20] a three-dimensional domain was considered with density constant; while in [21] the problem was studied in a two-dimensional domain, where the fluid density was considered as an extra variable in the system.

4.1. Existence of Optimal Solution. In this subsection we will prove the existence of an optimal solution for problem (45). For this purpose we introduce the following result. The proof follows a technique that is standard for optimal control problems. We will repeat it for convenience of the reader.

Theorem 10. *Under the hypotheses of Theorem 6, and also assuming that the conditions are satisfied:*

(i) $b_i \geq 0$, for $i = 1, 2$,

$$\mathcal{U}_1 \subset \tilde{\mathbf{H}}_{00}^{1/2}(\Gamma_2),$$

$$\mathcal{U}_2 \subset H_{00}^{1/2}(\Gamma_4)$$

are bounded, closed and convex set;

(ii) $b_i > 0$, for $i = 1, 2$,

$$\mathcal{U}_1 \subset \tilde{\mathbf{H}}_{00}^{1/2}(\Gamma_2),$$

$$\mathcal{U}_2 \subset H_{00}^{1/2}(\Gamma_4)$$

are closed and convex set;

where $\mathbf{u}_d \in \mathbf{L}^2(\Omega)$ and $T_d \in L^2(\Omega)$ are the desired states. The functionals J_1 and J_2 describe, respectively, the deviation of the velocity of flow from a given velocity and the deviation of the temperature from a given temperature. The functional J_3 describes the resistance in a fluid due to viscous friction (see [22]).

For simplicity we will do the study with $\hat{J} = J_1 + J_2$; also the study can be done considering $\hat{J}$ as the sum of two or more functionals given in (44). Thus, we define the following constrained minimization problem related to system (7)–(9):

then problem (45) has at least one optimal solution; that is, there exist $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{T}, \tilde{\mathbf{g}}_1, \tilde{g}_2) \in \mathcal{S}_{\text{ad}}$ such that

$$J(\tilde{\mathbf{u}}, \tilde{T}, \tilde{\mathbf{g}}_1, \tilde{g}_2) = \min_{\mathbf{s} \in \mathcal{S}_{\text{ad}}} J(\mathbf{u}, T, \mathbf{g}_1, g_2). \quad (48)$$

Proof. From Theorem 6 we have that the set of admissible solutions $\mathcal{S}_{\text{ad}} \neq \emptyset$, and since the functional J is bounded below, there exists a minimizing sequence $\{\mathbf{s}^m = (\mathbf{u}^m, T^m, \mathbf{g}_1^m, g_2^m)\}_{m \geq 1} \in \mathcal{S}_{\text{ad}}$, $m \in \mathbb{N}$, such that $\lim_{m \rightarrow \infty} J(\mathbf{s}^m) = \inf_{\mathbf{s} \in \mathcal{S}_{\text{ad}}} J(\mathbf{s})$. Also, by definition of set $\mathcal{S}_{\text{ad}}$, we have the fact that $\mathbf{s}^m$ satisfies system (7)–(9); that is,

$$\begin{aligned} 2\nu(D(\mathbf{u}^m), D(\mathbf{v})) + 2\alpha\nu \int_{\Gamma_3} \mathbf{u}^m \cdot \mathbf{v} d\Gamma \\ + (\mathbf{u}^m \cdot \nabla \mathbf{u}^m, \mathbf{v}) + \beta(GT^m, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}_0^t}, \\ \kappa(\nabla T^m, \nabla z) + (\mathbf{u}^m \cdot \nabla T^m, z) \\ + \kappa \int_{\Gamma_5} \gamma T^m \cdot z d\Gamma = \langle g, z \rangle_{H^t} + \langle \varphi, z \rangle_{\Gamma_5}, \\ \mathbf{u}^m = \mathbf{u}_{\mathbf{g}_1^m}^m \quad \text{on } \Gamma \setminus \Gamma_3, \\ T^m = g_2^m \quad \text{on } \Gamma_4, \end{aligned} \quad (49)$$

for all $(\mathbf{v}, z) \in \mathbb{X}$. Now, if one of the conditions (i) or (ii) in (47) is satisfied, then there exists a constant C , independent of m , such that $\|\mathbf{g}_1^m\|_{\mathbf{H}^{1/2}(\Gamma_2)}^2 + \|g_2^m\|_{H^{1/2}(\Gamma_4)}^2 \leq C$; thus, from (30), we conclude that $\|\mathbf{u}^m\|_{\mathbf{H}_0^1} + \|T^m\|_{H^1} \leq C$. Therefore, since $\mathcal{U}_{\text{ad}}$ is a closed convex subset of $\tilde{\mathbf{H}}_{00}^{1/2}(\Gamma_2) \times H_{00}^{1/2}(\Gamma_4)$, there exists

$\bar{\mathbf{s}} = (\bar{\mathbf{u}}, \bar{T}, \bar{\mathbf{g}}_1, \bar{g}_2) \in \mathbb{H} \times \mathcal{U}_{\text{ad}}$ such that, for some subsequence of $\{\mathbf{s}^m\}_{m \geq 1}$, still denoted by $\{\mathbf{s}^m\}_{m \geq 1}$, when $m \rightarrow \infty$, we have

$$\begin{aligned} \mathbf{u}^m &\rightharpoonup \bar{\mathbf{u}} \text{ weakly in } \mathbf{H}_\sigma^1, \\ T^m &\rightharpoonup \bar{T} \text{ weakly in } H^1(\Omega), \\ \mathbf{g}_1^m &\rightharpoonup \bar{\mathbf{g}}_1 \text{ weakly in } \mathcal{U}_1, \\ g_2^m &\rightharpoonup \bar{g}_2 \text{ weakly in } \mathcal{U}_2; \end{aligned} \quad (50)$$

by standard Sobolev embeddings, we obtain that

$$\begin{aligned} \mathbf{u}^m &\rightarrow \bar{\mathbf{u}} \text{ strongly in } \mathbf{L}^2(\Omega), \\ T^m &\rightarrow \bar{T} \text{ strongly in } L^2(\Omega), \\ \mathbf{g}_1^m &\rightarrow \bar{\mathbf{g}}_1 \text{ strongly in } \mathbf{L}^2(\Gamma_2), \\ g_2^m &\rightarrow \bar{g}_2 \text{ strongly in } L^2(\Gamma_4). \end{aligned} \quad (51)$$

Moreover, since $\mathbf{u}^m = \mathbf{u}_0$ on Γ_1 , $\mathbf{u}^m = \mathbf{g}_1^m$ on Γ_2 , and $T^m = g_2^m$ on Γ_4 , then from (51) it follows that $\bar{\mathbf{u}} = \mathbf{u}_{\bar{\mathbf{g}}_1}$ on $\Gamma \setminus \Gamma_3$ and $\bar{T} = \bar{g}_2$ on Γ_4 ; thus, $\bar{\mathbf{s}}$ satisfies the boundary conditions given in (9). A standard procedure permits passing the limit in (51), as m goes to ∞ , and proving that $\bar{\mathbf{s}}$ is solution of system (7)–(9). Consequently we have $\bar{\mathbf{s}} \in \mathcal{S}_{\text{ad}}$ and

$$\lim_{m \rightarrow \infty} J(\mathbf{s}^m) = \inf_{\mathbf{s} \in \mathcal{S}_{\text{ad}}} J(\mathbf{s}) \leq J(\bar{\mathbf{s}}). \quad (52)$$

Also, since the functional J is weakly lower semicontinuous on $\mathcal{S}_{\text{ad}}$, we have that $J(\bar{\mathbf{s}}) \leq \lim_{m \rightarrow \infty} \inf J(\mathbf{s}^m)$. Therefore, from (52) and last inequality we conclude $J(\bar{\mathbf{u}}, \bar{T}, \bar{\mathbf{g}}_1, \bar{g}_2) = J(\bar{\mathbf{s}}) = \min_{\mathbf{s} \in \mathcal{S}_{\text{ad}}} J(\mathbf{s})$, which implies the existence of an optimal solution to the control problem (45). $\square$

4.2. First-Order Necessary Optimality Conditions and an Optimality System. In order to obtain first-order optimality conditions, and thus derive an optimality system, we start by considering the following Banach space:

$$\mathbb{Y} := \mathbb{X}' \times \tilde{\mathbf{H}}_{00}^{1/2}(\Gamma \setminus \Gamma_3) \times H_{00}^{1/2}(\Gamma_4) \quad (53)$$

and the vector operator $\mathbb{F} = (\mathbf{F}_1, F_2, \mathbf{F}_3, F_4) : \mathbb{H} \times \mathcal{U}_{\text{ad}} \rightarrow \mathbb{Y}$, where $\mathbf{F}_1 : \mathbb{H} \times \mathcal{U}_{\text{ad}} \rightarrow \tilde{\mathbf{H}}_\sigma'$, $F_2 : \mathbb{H} \times \mathcal{U}_{\text{ad}} \rightarrow H'$, $\mathbf{F}_3 : \mathbb{H} \times \mathcal{U}_{\text{ad}} \rightarrow \tilde{\mathbf{H}}_{00}^{1/2}(\Gamma \setminus \Gamma_3)$, and $F_4 : \mathbb{H} \times \mathcal{U}_{\text{ad}} \rightarrow H_{00}^{1/2}(\Gamma_4)$, at each point $\mathbf{s} = (\mathbf{u}, T, \mathbf{g}_1, g_2) \in \mathbb{H} \times \mathcal{U}_{\text{ad}}$, are defined by

$$\begin{aligned} \langle \mathbf{F}_1(\mathbf{s}), \mathbf{v} \rangle_{\tilde{\mathbf{H}}_\sigma'} &:= 2\nu(D(\mathbf{u}), D(\mathbf{v})) + 2\alpha\nu \int_{\Gamma_3} \mathbf{u} \cdot \mathbf{v} \, d\Gamma \\ &+ (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) + \beta(GT, \mathbf{v}) \\ &- \langle \mathbf{f}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}_\sigma'}, \end{aligned}$$

$$\begin{aligned} \langle F_2(\mathbf{s}), z \rangle_{H'} &:= \kappa(\nabla T, \nabla z) + (\mathbf{u} \cdot \nabla T, z) \\ &+ \kappa \int_{\Gamma_3} \gamma T \cdot z \, d\Gamma \\ &- (\langle g, z \rangle_{H'} + \langle \varphi, z \rangle_{\Gamma_5}), \end{aligned}$$

$$\mathbf{F}_3(\mathbf{s}) := \mathbf{u}|_{\Gamma \setminus \Gamma_3} - \mathbf{u}_{\mathbf{g}_1},$$

$$F_4(\mathbf{s}) := T|_{\Gamma_4} - g_2,$$

(54)

for all $(\mathbf{v}, z) \in \mathbb{X}$. Thus, the optimal control problem (45) is rewritten as follows:

$$\begin{aligned} \text{Find } \mathbf{s} &= ((\mathbf{u}, T), (\mathbf{g}_1, g_2)) \\ &\in \mathbb{H} \times \mathcal{U}_{\text{ad}} \end{aligned}$$

such that the functional $J(\mathbf{u}, T, \mathbf{g}_1, g_2)$

$$\begin{aligned} &= \frac{1}{2} \|\mathbf{u} - \mathbf{u}_d\|^2 \\ &+ \frac{1}{2} \|T - T_d\|^2 \\ &+ \frac{b_1}{2} \|\mathbf{g}_1\|_{\mathbf{H}^{1/2}(\Gamma_2)}^2 \\ &+ \frac{b_2}{2} \|g_2\|_{H^{1/2}(\Gamma_4)}^2 \end{aligned} \quad (55)$$

is minimized subject to $\mathbb{F}(\mathbf{s}) = \mathbf{0}$;

$$\text{that is, } \langle \mathbf{F}_1(\mathbf{s}), \mathbf{v} \rangle_{\tilde{\mathbf{H}}_\sigma'} = 0,$$

$$\langle F_2(\mathbf{s}), z \rangle_{H'} = 0,$$

$$\mathbf{F}_3(\mathbf{s}) = \mathbf{0},$$

$$F_4(\mathbf{s}) = 0,$$

for all $(\mathbf{v}, z) \in \mathbb{X}$.

Recall that problem (45) is a problem with constraint; one method to obtain necessary conditions of optimality is Lagrange method, which consists of transforming the problem (45) into an unrestricted one, using an auxiliary function (Lagrange functional or Lagrangian) and additional variables, called Lagrange multipliers. The Lagrangian associated with optimal control problem (45) is given by

$$\begin{aligned} \mathcal{L}(\mathbf{s}, \boldsymbol{\phi}) &= J(\mathbf{s}) - \langle \mathbf{F}_1(\mathbf{s}), \boldsymbol{\lambda} \rangle_{\tilde{\mathbf{H}}_\sigma'} - \langle F_2(\mathbf{s}), \eta \rangle_{H'} \\ &- \langle \boldsymbol{\chi}, \mathbf{F}_3(\mathbf{s}) \rangle_{\Gamma \setminus \Gamma_3} - \langle \vartheta, F_4(\mathbf{s}) \rangle_{\Gamma_4}, \end{aligned} \quad (56)$$

where $\boldsymbol{\phi} = ((\boldsymbol{\lambda}, \eta), \boldsymbol{\chi}, \vartheta) \in \mathbb{X} \times \tilde{\mathbf{H}}_{00}^{-1/2}(\Gamma \setminus \Gamma_3) \times H_{00}^{-1/2}(\Gamma_4)$.

Concerning differentiation of the objective functional J and the constraint operator $\mathbb{F}$, we have the following results.

Lemma 11. *The functional J is Fréchet differentiable with respect to the point $\mathbf{s} = ((\mathbf{u}, T), (\mathbf{g}_1, g_2)) \in \mathbb{H} \times \mathcal{U}_{\text{ad}}$. Moreover, at an arbitrary point $\bar{\mathbf{s}} = ((\bar{\mathbf{u}}, \bar{T}), (\bar{\mathbf{g}}_1, \bar{g}_2)) \in \mathbb{H} \times \mathcal{U}_{\text{ad}}$, the*

Fréchet derivative of J with respect to $\mathbf{s}$ is a linear and bounded functional $J_s(\bar{\mathbf{s}}) : \mathbb{H} \times \mathcal{U}_{ad} \rightarrow \mathbb{R}$ such that, at each point $\mathbf{t} = ((\boldsymbol{\omega}, \psi), (\boldsymbol{q}, \tau)) \in \mathbb{H} \times \mathcal{U}_{ad}$, it is defined by

$$J_s(\bar{\mathbf{s}}) \mathbf{t} = (\tilde{\mathbf{u}} - \mathbf{u}_d, \boldsymbol{\omega}) + (\tilde{T} - T_d, \psi) + b_1 \int_{\Gamma_2} \tilde{\mathbf{g}}_1 \cdot \boldsymbol{q} d\Gamma + b_2 \int_{\Gamma_4} \tilde{g}_2 \cdot \tau d\Gamma. \quad (57)$$

Lemma 12. The operator $\mathbb{F}$ is Fréchet differentiable with respect to the point $\mathbf{s} = ((\mathbf{u}, T), (\mathbf{g}_1, g_2)) \in \mathbb{H} \times \mathcal{U}_{ad}$. Moreover, at an arbitrary point $\bar{\mathbf{s}} = ((\tilde{\mathbf{u}}, \tilde{T}), (\tilde{\mathbf{g}}_1, \tilde{g}_2)) \in \mathbb{H} \times \mathcal{U}_{ad}$, the Fréchet derivative of $\mathbb{F}$ with respect to $\mathbf{s}$ is a linear and bounded operator $\mathbb{F}_s(\bar{\mathbf{s}}) : \mathbb{H} \times \mathcal{U}_{ad} \rightarrow \mathbb{R}$ such that, at each point $\mathbf{t} = ((\boldsymbol{\omega}, \psi), (\boldsymbol{q}, \tau)) \in \mathbb{H} \times \mathcal{U}_{ad}$, it is defined by

$$\begin{aligned} \langle \mathbb{F}_{1s}(\bar{\mathbf{s}}) \mathbf{t}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}'_\sigma} &= 2\nu(D(\boldsymbol{\omega}), D(\mathbf{v})) + 2\alpha\nu \int_{\Gamma_3} \boldsymbol{\omega} \cdot \mathbf{v} d\Gamma \\ &\quad + (\tilde{\mathbf{u}} \cdot \nabla \boldsymbol{\omega}, \mathbf{v}) + (\boldsymbol{\omega} \cdot \nabla \tilde{\mathbf{u}}, \mathbf{v}) \\ &\quad + \beta(\mathbf{G}\psi, \mathbf{v}), \\ \langle \mathbb{F}_{2s}(\bar{\mathbf{s}}) \mathbf{t}, z \rangle_{H'} &= \kappa(\nabla \psi, \nabla z) + \kappa \int_{\Gamma_5} \gamma \psi \cdot z d\Gamma \\ &\quad + (\tilde{\mathbf{u}} \cdot \nabla \psi, z) + (\boldsymbol{\omega} \cdot \nabla \tilde{T}, z), \end{aligned} \quad (58)$$

$$\mathbb{F}_{3s}(\bar{\mathbf{s}}) \mathbf{t} = \boldsymbol{\omega}|_{\Gamma \setminus \Gamma_3} - \mathcal{B}\boldsymbol{q},$$

$$\mathbb{F}_{4s}(\bar{\mathbf{s}}) \mathbf{t} = \psi|_{\Gamma_4} - \tau,$$

for all $(\mathbf{v}, z) \in \mathbb{X}$. The operator $\mathcal{B} \in \mathcal{L}(\tilde{\mathbf{H}}_{00}^{1/2}(\Gamma_2), \tilde{\mathbf{H}}_{00}^{1/2}(\Gamma \setminus \Gamma_3))$ is defined by

$$\mathcal{B}\boldsymbol{q} := \begin{cases} \mathbf{0} & \text{on } \Gamma_1, \\ \boldsymbol{q} & \text{on } \Gamma_2. \end{cases} \quad (59)$$

The following is an adaptation to the definition of regular point, given by [33, p. 50] in abstract, for optimal control problem (45).

Definition 13. Let $\bar{\mathbf{s}} = ((\tilde{\mathbf{u}}, \tilde{T}), (\tilde{\mathbf{g}}_1, \tilde{g}_2)) \in \mathbb{H} \times \mathcal{U}_{ad}$ be an admissible point for the control problem (55). We say that $\bar{\mathbf{s}}$ is a regular point of the operator $\mathbb{F}$, defined by (54), if

$$\mathbb{F}_s(\bar{\mathbf{s}})(\mathbb{H} \times \mathcal{C}(\tilde{\mathbf{g}}_1) \times \mathcal{C}(\tilde{g}_2)) = \mathbb{Y}, \quad (60)$$

where

$$\begin{aligned} \mathcal{C}(\tilde{\mathbf{g}}_1) \times \mathcal{C}(\tilde{g}_2) &= \{(\theta_1(\mathbf{g}_1 - \tilde{\mathbf{g}}_1), \theta_2(g_2 - \tilde{g}_2)) : \theta_1 \\ &\geq 0, \theta_2 \geq 0, (\mathbf{g}_1, g_2) \in \mathcal{U}_{ad}\}. \end{aligned} \quad (61)$$

In the following lemma, we give a condition to assure that $\bar{\mathbf{s}} \in \mathcal{S}_{ad}$ is a regular point for optimal control problem (45). Thereafter the existence of Lagrange multipliers is shown.

Lemma 14. Let $\bar{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{T}, \tilde{\mathbf{g}}_1, \tilde{g}_2) \in \mathcal{S}_{ad}$ be a feasible solution for the optimal problem (45). If ν, κ are sufficiently large and $\|\gamma\|_{L^\infty(\Gamma_5)}$ is small enough, such that

$$\begin{aligned} \min \{ &2\nu - \beta C \|\mathbf{G}\|_\infty - C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}}, \kappa - \beta C \|\mathbf{G}\|_\infty \\ &- \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} - C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}} \} > 0, \end{aligned} \quad (62)$$

where the constant $C > 0$ depends on the domain Ω , then $\bar{\mathbf{s}}$ is a regular point.

Proof. Given $(\mathbf{a}, b, \mathbf{c}, d) \in \mathbb{Y}$, it is sufficient to show the existence of the point $\mathbf{t} = ((\boldsymbol{\omega}, \psi), (\boldsymbol{q}, \tau)) \in \mathbb{H} \times \mathcal{U}_{ad}$, such that

$$\begin{aligned} 2\nu(D(\boldsymbol{\omega}), D(\mathbf{v})) + 2\alpha\nu \int_{\Gamma_3} \boldsymbol{\omega} \cdot \mathbf{v} d\Gamma + (\tilde{\mathbf{u}} \cdot \nabla \boldsymbol{\omega}, \mathbf{v}) \\ + (\boldsymbol{\omega} \cdot \nabla \tilde{\mathbf{u}}, \mathbf{v}) + \beta(\mathbf{G}\psi, \mathbf{v}) &= \langle \mathbf{a}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}'_\sigma} \quad \forall \mathbf{v} \in \tilde{\mathbf{H}}_\sigma^1, \\ \kappa(\nabla \psi, \nabla z) + \kappa \int_{\Gamma_5} \gamma \psi \cdot z d\Gamma + (\tilde{\mathbf{u}} \cdot \nabla \psi, z) \\ + (\boldsymbol{\omega} \cdot \nabla \tilde{T}, z) &= \langle b, z \rangle_{H'} \quad \forall z \in H_0, \\ \boldsymbol{\omega}|_{\Gamma \setminus \Gamma_3} &= \mathbf{c} + \mathcal{B}(\boldsymbol{q} - \tilde{\mathbf{g}}_1), \\ \psi|_{\Gamma_4} &= d + (\tau - \tilde{g}_2). \end{aligned} \quad (63)$$

Setting $\boldsymbol{q} = \tilde{\mathbf{g}}_1$ and $\tau = \tilde{g}_2$, we have $\boldsymbol{\omega}|_{\Gamma \setminus \Gamma_3} = \mathbf{c}$ and $\psi|_{\Gamma} = d$; thus, from Lemma 3, there exists $(\boldsymbol{\omega}^\varepsilon, \psi^\varepsilon) \in \mathbb{H}$ such that $\boldsymbol{\omega}^\varepsilon|_{\Gamma \setminus \Gamma_3} = \mathbf{c}$, $\boldsymbol{\omega}^\varepsilon|_{\Gamma_3} = \mathbf{0}$, $\psi|_{\Gamma_4} = d$, and $\psi|_{\Gamma_5} = 0$. Then, rewriting the unknown $(\boldsymbol{\omega}, \psi) \in \mathbb{H}$ in the form $\boldsymbol{\omega} = \boldsymbol{\omega}^\varepsilon + \hat{\boldsymbol{\omega}}$ and $\psi = \psi^\varepsilon + \hat{\psi}$, with $(\hat{\boldsymbol{\omega}}, \hat{\psi}) \in \mathbb{X}$ new unknowns functions, from system (63), we obtain the following system: Find $(\hat{\boldsymbol{\omega}}, \hat{\psi}) \in \mathbb{X}$ such that

$$\begin{aligned} 2\nu(D(\hat{\boldsymbol{\omega}}), D(\mathbf{v})) + 2\alpha\nu \int_{\Gamma_3} \hat{\boldsymbol{\omega}} \cdot \mathbf{v} d\Gamma + (\tilde{\mathbf{u}} \cdot \nabla \hat{\boldsymbol{\omega}}, \mathbf{v}) \\ + (\hat{\boldsymbol{\omega}} \cdot \nabla \tilde{\mathbf{u}}, \mathbf{v}) + \beta(\mathbf{G}\hat{\psi}, \mathbf{v}) &= \langle \hat{\mathbf{a}}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}'_\sigma} \quad \forall \mathbf{v} \in \tilde{\mathbf{H}}_\sigma^1, \\ \kappa(\nabla \hat{\psi}, \nabla z) + \kappa \int_{\Gamma_5} \gamma \hat{\psi} \cdot z d\Gamma + (\tilde{\mathbf{u}} \cdot \nabla \hat{\psi}, z) \\ + (\hat{\boldsymbol{\omega}} \cdot \nabla \tilde{T}, z) &= \langle \hat{b}, z \rangle_{H'} \quad \forall z \in H_0, \end{aligned} \quad (64)$$

where

$$\begin{aligned} \langle \hat{\mathbf{a}}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}'_\sigma} &:= -2\nu(D(\boldsymbol{\omega}^\varepsilon), D(\mathbf{v})) - (\tilde{\mathbf{u}} \cdot \nabla \boldsymbol{\omega}^\varepsilon, \mathbf{v}) \\ &\quad - (\boldsymbol{\omega}^\varepsilon \cdot \nabla \tilde{\mathbf{u}}, \mathbf{v}) - \beta(\mathbf{G}\psi^\varepsilon, \mathbf{v}) \\ &\quad + \langle \mathbf{a}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}'_\sigma}, \\ \langle \hat{b}, z \rangle_{H'} &:= -\kappa(\nabla \psi^\varepsilon, \nabla z) - (\tilde{\mathbf{u}} \cdot \nabla \psi^\varepsilon, z) \\ &\quad - (\boldsymbol{\omega}^\varepsilon \cdot \nabla \tilde{T}, z) + \langle b, z \rangle_{H'}. \end{aligned} \quad (65)$$

In order to prove the existence of a solution for system (64) we define the bilinear form $a : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}$ by

$$\begin{aligned} a((\widehat{\omega}, \widehat{\psi}), (v, z)) &:= 2\nu(D(\widehat{\omega}), D(v)) \\ &+ 2\alpha\nu \int_{\Gamma_3} \widehat{\omega} \cdot v \, d\Gamma + (\tilde{\mathbf{u}} \cdot \nabla \widehat{\omega}, v) \\ &+ (\widehat{\omega} \cdot \nabla \tilde{\mathbf{u}}, v) + \beta(\mathbf{G}\widehat{\psi}, v) \\ &+ \kappa(\nabla \widehat{\psi}, \nabla z) + (\tilde{\mathbf{u}} \cdot \nabla \widehat{\psi}, z) \\ &+ (\widehat{\omega} \cdot \nabla \tilde{T}, z) + \kappa \int_{\Gamma_5} \gamma \widehat{\psi} \cdot z \, d\Gamma, \end{aligned} \quad (66)$$

and the linear and continuous functional $\mathcal{F} : \mathbb{X} \rightarrow \mathbb{R}$ by

$$\mathcal{F}(v, z) := \langle \widehat{\mathbf{a}}, v \rangle_{\tilde{\mathbf{H}}_0'} + \langle \widehat{b}, z \rangle_{H^1}. \quad (67)$$

Thus, from (66)-(67), system (64) is equivalent to the following. Find $(\widehat{\omega}, \widehat{\psi}) \in \mathbb{X}$ such that

$$a((\widehat{\omega}, \widehat{\psi}), (v, z)) = \mathcal{F}(v, z) \quad \forall (v, z) \in \mathbb{X}. \quad (68)$$

We notice that the form $a(\cdot, \cdot)$ is continuous. Now, by taking $(v, z) = (\widehat{\omega}, \widehat{\psi})$ in (66), we have

$$\begin{aligned} a((\widehat{\omega}, \widehat{\psi}), (\widehat{\omega}, \widehat{\psi})) &\geq 2\nu \|\widehat{\omega}\|_{\tilde{\mathbf{H}}_0'}^2 + (\widehat{\omega} \cdot \nabla \tilde{\mathbf{u}}, \widehat{\omega}) \\ &+ \beta(\mathbf{G}\widehat{\psi}, \widehat{\omega}) + \kappa \|\widehat{\psi}\|_{H_0}^2 \\ &+ (\widehat{\omega} \cdot \nabla \tilde{T}, \widehat{\psi}) + \kappa \int_{\Gamma_5} \gamma \widehat{\psi} \cdot \widehat{\psi} \, d\Gamma. \end{aligned} \quad (69)$$

By applying the Hölder and Young inequalities to the terms in the right-hand side of (69), we have

$$\begin{aligned} &|(\widehat{\omega} \cdot \nabla \tilde{\mathbf{u}}, \widehat{\omega}) + (\widehat{\omega} \cdot \nabla \tilde{T}, \widehat{\psi})| \\ &\leq C \|\tilde{\mathbf{u}}\|_{\mathbf{H}_0^1} \|\widehat{\omega}\|_{\tilde{\mathbf{H}}_0'}^2 + C \|\tilde{T}\|_{H^1} \left(\|\widehat{\omega}\|_{\tilde{\mathbf{H}}_0'}^2 + \|\widehat{\psi}\|_{H_0}^2 \right), \\ &|\beta(\mathbf{G}\widehat{\psi}, \widehat{\omega})| \leq \beta \|\mathbf{G}\|_{\infty} \|\widehat{\psi}\| \|\widehat{\omega}\| \\ &\leq \beta C \|\mathbf{G}\|_{\infty} \left(\|\widehat{\omega}\|_{\tilde{\mathbf{H}}_0'}^2 + \|\widehat{\psi}\|_{H_0}^2 \right), \\ &\left| \kappa \int_{\Gamma_5} \gamma \widehat{\psi} \cdot \widehat{\psi} \, d\Gamma \right| \leq \kappa \|\gamma\|_{L^\infty(\Gamma_5)} \|\widehat{\psi}\|_{L^2(\Gamma_5)}^2 \\ &\leq \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} \|\widehat{\psi}\|_{H_0}^2. \end{aligned} \quad (70)$$

Then, by using the last inequalities in (69), we have

$$\begin{aligned} &a((\widehat{\omega}, \widehat{\psi}), (\widehat{\omega}, \widehat{\psi})) \\ &\geq (2\nu - C \|\tilde{\mathbf{u}}\|_{\mathbf{H}_0^1} - C \|\tilde{T}\|_{H^1} - \beta C \|\mathbf{G}\|_{\infty}) \|\widehat{\omega}\|_{\tilde{\mathbf{H}}_0'}^2 \\ &\quad + (\kappa - C \|\tilde{T}\|_{H^1} - \beta C \|\mathbf{G}\|_{\infty} - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)}) \|\widehat{\psi}\|_{H_0}^2 \\ &\geq \bar{\delta} \|(\widehat{\omega}, \widehat{\psi})\|_{\mathbb{X}}^2, \end{aligned} \quad (71)$$

where, by hypotheses, $\bar{\delta} := \min\{2\nu - \beta C \|\mathbf{G}\|_{\infty} - C \|\tilde{\mathbf{u}}\|_{\mathbf{H}_0^1} - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} - \|C(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}}\} > 0$. Thus, we obtain that the form $a(\cdot, \cdot)$ is $\mathbb{X}$ -coercive; therefore, from (68), (71), and the Lax-Milgram theorem we conclude the existence of a unique $(\widehat{\omega}, \widehat{\psi}) \in \mathbb{X}$ solution of system (64), and, consequently, we obtain that $(\omega, \psi) \in \mathbb{H}$ is solution of (63). $\square$

In the following result we prove the existence of Lagrange multipliers provided that a local optimal solution $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{T}, \tilde{\mathbf{g}}_1, \tilde{g}_2) \in \mathcal{S}_{ad}$ verifies the regular point condition provided by Lemma 14.

Theorem 15. *Let $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{T}, \tilde{\mathbf{g}}_1, \tilde{g}_2) \in \mathcal{S}_{ad}$ be a local optimal solution for the optimal control problem (45) satisfying the regular point condition given in (62). Then, there exist Lagrange multipliers $((\lambda, \eta), \chi, \vartheta) \in \mathbb{X} \times \tilde{\mathbf{H}}_{00}^{-1/2}(\Gamma \setminus \Gamma_3) \times H_{00}^{-1/2}(\Gamma_4)$ such that for all $((\omega, \psi), \varrho, \tau) \in \mathbb{H} \times \mathcal{C}(\tilde{\mathbf{g}}_1) \times \mathcal{C}(\tilde{g}_2)$ it holds that*

$$\begin{aligned} &(\tilde{\mathbf{u}} - \mathbf{u}_d, \omega) + (\tilde{T} - T_d, \psi) + b_1 \int_{\Gamma_2} \tilde{\mathbf{g}}_1 \cdot \varrho \, d\Gamma \\ &+ b_2 \int_{\Gamma_4} \tilde{g}_2 \cdot \tau \, d\Gamma - 2\nu(D(\omega), D(\lambda)) \\ &- 2\alpha\nu \int_{\Gamma_3} \omega \cdot \lambda \, d\Gamma - (\tilde{\mathbf{u}} \cdot \nabla \omega, \lambda) - (\omega \cdot \nabla \tilde{\mathbf{u}}, \lambda) \\ &- \beta(\mathbf{G}\psi, \lambda) - \kappa(\nabla \psi, \nabla \eta) - (\tilde{\mathbf{u}} \cdot \nabla \psi, \eta) \\ &- (\omega \cdot \nabla \tilde{T}, \eta) - \kappa \int_{\Gamma_5} \gamma \psi \cdot \eta \, d\Gamma \\ &- \langle \chi, \omega - \mathcal{B}\varrho \rangle_{\Gamma \setminus \Gamma_3} - \langle \vartheta, \psi - \tau \rangle_{\Gamma_4} \geq 0, \end{aligned} \quad (72)$$

where $\mathcal{C}(\tilde{\mathbf{g}}_1) \times \mathcal{C}(\tilde{g}_2)$ is defined in (61).

Proof. Since $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{T}, \tilde{\mathbf{g}}_1, \tilde{g}_2) \in \mathcal{S}_{ad}$ satisfies the regular point condition (62), then there exist Lagrange multipliers $((\lambda, \eta), \chi, \vartheta) \in \mathbb{X} \times \tilde{\mathbf{H}}_{00}^{-1/2}(\Gamma \setminus \Gamma_3) \times H_{00}^{-1/2}(\Gamma_4)$ such that

$$\begin{aligned} &J_s(\tilde{\mathbf{s}}) \mathbf{t} - \langle \mathbf{F}_{1s}(\tilde{\mathbf{s}}) \mathbf{t}, \lambda \rangle_{\tilde{\mathbf{H}}_0'} - \langle \mathbf{F}_{2s}(\tilde{\mathbf{s}}) \mathbf{t}, \eta \rangle_{H^1} \\ &- \langle \chi, \mathbf{F}_{3s}(\tilde{\mathbf{s}}) \mathbf{t} \rangle_{\Gamma \setminus \Gamma_3} - \langle \vartheta, \mathbf{F}_{4s}(\tilde{\mathbf{s}}) \mathbf{t} \rangle_{\Gamma_4} \geq 0, \end{aligned} \quad (73)$$

for all $\mathbf{t} = ((\omega, \psi), \varrho, \tau) \in \mathbb{H} \times \mathcal{C}(\tilde{\mathbf{g}}_1) \times \mathcal{C}(\tilde{g}_2)$. Thus, the proof follows from (57), (58), and (59). $\square$

4.3. Optimality System. In this subsection we derive the equations that are satisfied by the Lagrange multipliers $((\lambda, \eta), \chi, \vartheta) \in \mathbb{X} \times \tilde{\mathbf{H}}_{00}^{-1/2}(\Gamma \setminus \Gamma_3) \times H_{00}^{-1/2}(\Gamma_4)$ provided by Theorem 15.

Theorem 16. *Under the conditions of Theorem 15, the Lagrange multipliers $(\lambda, \eta, \chi, \vartheta)$ satisfy the following adjoint equations*

$$-\nu \Delta \lambda - \tilde{\mathbf{u}} \cdot \nabla \lambda + \nabla^T \lambda \cdot \tilde{\mathbf{u}} + \nabla^T \eta \cdot \tilde{T} = \tilde{\mathbf{u}} - \mathbf{u}_d \quad \text{in } \tilde{\mathbf{H}}_0', \quad (74)$$

$$-\kappa \Delta \eta - \tilde{\mathbf{u}} \cdot \nabla \eta + \beta \mathbf{G} \cdot \lambda = \tilde{T} - T_d \quad \text{in } H^1, \quad (75)$$

$$\operatorname{div} \boldsymbol{\lambda} = 0 \quad \text{in } \Omega, \quad (76)$$

$$\nu(\nabla \boldsymbol{\lambda}) \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma \setminus \Gamma_3, \quad (77)$$

$$\begin{aligned} \nu(\nabla \boldsymbol{\lambda}) \mathbf{n} + \alpha \boldsymbol{\lambda} &= \mathbf{0} \quad \text{on } \Gamma_3, \\ (\nabla \eta) \mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_4, \end{aligned} \quad (78)$$

$$\begin{aligned} (\nabla \eta) \mathbf{n} + \kappa(\gamma \eta) &= 0 \quad \text{on } \Gamma_5, \\ \chi &= \mathbf{0} \quad \text{on } \Gamma \setminus \Gamma_3, \\ \vartheta &= 0 \quad \text{on } \Gamma_4, \end{aligned} \quad (79)$$

and the optimality conditions

$$\begin{aligned} \langle b_1 \tilde{\mathbf{g}}_1 + \chi, \mathbf{g}_1 - \tilde{\mathbf{g}}_1 \rangle_{\Gamma_2} &\geq 0, \\ \langle b_2 \tilde{g}_2 + \vartheta, g_2 - \tilde{g}_2 \rangle_{\Gamma_4} &\geq 0. \end{aligned} \quad (80)$$

Proof. In fact, from (72), taking $(\psi, \mathbf{q}, \tau) = (0, \mathbf{0}, 0)$, we have

$$\begin{aligned} &(\tilde{\mathbf{u}} - \mathbf{u}_d, \boldsymbol{\omega}) - 2\nu(D(\boldsymbol{\omega}), D(\boldsymbol{\lambda})) - 2\alpha\nu \int_{\Gamma_3} \boldsymbol{\omega} \cdot \boldsymbol{\lambda} d\Gamma \\ &- (\tilde{\mathbf{u}} \cdot \nabla \boldsymbol{\omega}, \boldsymbol{\lambda}) - (\boldsymbol{\omega} \cdot \nabla \tilde{\mathbf{u}}, \boldsymbol{\lambda}) - (\boldsymbol{\omega} \cdot \nabla \tilde{T}, \eta) \\ &- \langle \chi, \boldsymbol{\omega} \rangle_{\Gamma \setminus \Gamma_3} = 0 \quad \forall \boldsymbol{\omega} \in \mathbf{H}_\sigma^1, \end{aligned} \quad (81)$$

and integrating by parts, for all $\boldsymbol{\omega} \in \mathbf{H}_\sigma^1$, we obtain

$$\begin{aligned} &(\tilde{\mathbf{u}} - \mathbf{u}_d, \boldsymbol{\omega}) + \nu(\Delta \boldsymbol{\lambda}, \boldsymbol{\omega}) - 2\nu \int_{\Gamma} [(\nabla \boldsymbol{\lambda}) \mathbf{n}] \cdot \boldsymbol{\omega} d\Gamma \\ &- 2\alpha\nu \int_{\Gamma_3} \boldsymbol{\lambda} \cdot \boldsymbol{\omega} d\Gamma + (\tilde{\mathbf{u}} \cdot \nabla \boldsymbol{\lambda}, \boldsymbol{\omega}) - (\nabla^T \boldsymbol{\lambda} \cdot \mathbf{u}, \boldsymbol{\omega}) \\ &- (\nabla^T \eta \cdot \tilde{T}, \boldsymbol{\omega}) - \langle \chi, \boldsymbol{\omega} \rangle_{\Gamma \setminus \Gamma_3} = 0. \end{aligned} \quad (82)$$

Now, taking $(\boldsymbol{\omega}, \mathbf{q}, \tau) = (\mathbf{0}, \mathbf{0}, 0)$ in (72), we have

$$\begin{aligned} &(\tilde{T} - T_d, \psi) - \beta(\mathbf{G}\psi, \boldsymbol{\lambda}) - \kappa(\nabla \psi, \nabla \eta) - (\tilde{\mathbf{u}} \cdot \nabla \psi, \eta) \\ &- \kappa \int_{\Gamma_5} \gamma \psi \cdot \eta d\Gamma - \langle \vartheta, \psi \rangle_{\Gamma_4} = 0, \end{aligned} \quad (83)$$

for all $\psi \in H^1(\Omega)$, and integrating by parts, for all $\psi \in H^1(\Omega)$, we obtain

$$\begin{aligned} &(\tilde{T} - T_d, \psi) + \kappa(\Delta \eta, \psi) - \kappa \int_{\Gamma} [(\nabla \eta) \mathbf{n}] \cdot \psi d\Gamma \\ &+ (\tilde{\mathbf{u}} \cdot \nabla \eta, \psi) - \beta(\mathbf{G} \cdot \boldsymbol{\lambda}, \psi) - \kappa \int_{\Gamma_5} \gamma \eta \cdot \psi d\Gamma \\ &- \langle \vartheta, \psi \rangle_{\Gamma_4} = 0. \end{aligned} \quad (84)$$

Since $(\boldsymbol{\omega}, \psi) \in \mathbb{H}$ is arbitrary, then from (82) and (84), we obtain system (74)–(79).

Now, from (72), when $(\boldsymbol{\omega}, \psi) = (\mathbf{0}, 0)$, we obtain

$$\begin{aligned} &b_1 \int_{\Gamma_2} \tilde{\mathbf{g}}_1 \cdot \mathbf{q} d\Gamma + b_2 \int_{\Gamma_4} \tilde{g}_2 \cdot \tau d\Gamma + \langle \chi, \mathcal{B}\mathbf{q} \rangle_{\Gamma \setminus \Gamma_3} \\ &+ \langle \vartheta, \tau \rangle_{\Gamma_4} \geq 0. \end{aligned} \quad (85)$$

Taking $\mathbf{q} = \mathbf{g}_1 - \tilde{\mathbf{g}}_1$, $\tau = g_2 - \tilde{g}_2$, and taking into account the definition of operator $\mathcal{B}$ given in (59), the last inequality implies

$$\langle b_1 \tilde{\mathbf{g}}_1 + \chi, \mathbf{g}_1 - \tilde{\mathbf{g}}_1 \rangle_{\Gamma_2} + \langle b_2 \tilde{g}_2 + \vartheta, g_2 - \tilde{g}_2 \rangle_{\Gamma_4} \geq 0; \quad (86)$$

thus the optimality conditions (80) follow. $\square$

Summarizing, the state equations described in (1)–(4), the adjoint equations given in (75)–(79), and the optimality conditions obtained in (80) form the optimality system of the optimal control problem (45).

Remark 17. Since the set of controls $\mathcal{U}_{ad}$ is convex, then from (80) we deduce

$$\begin{aligned} \tilde{\mathbf{g}}_1 &= \operatorname{Proj}_{\mathcal{U}_1} \left(-\frac{\chi}{b_1} \right) \quad \text{on } \Gamma_2, \\ \tilde{g}_2 &= \operatorname{Proj}_{\mathcal{U}_2} \left(-\frac{\vartheta}{b_2} \right) \quad \text{on } \Gamma_4. \end{aligned} \quad (87)$$

5. Second-Order Sufficient Condition

In this section, we will discuss sufficient conditions for $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{T}, \tilde{\mathbf{g}}_1, \tilde{g}_2) \in \mathcal{S}_{ad}$ being a local optimal solution for control problem (45). We will establish a $\mathbb{H} \times \mathcal{U}_{ad}$ -coercivity condition on the second derivative of the Lagrange functional $\mathcal{L}$ given in (56) associated with problem (45) in order to assure that an admissible point $\tilde{\mathbf{s}}$ is a local optimal solution. Due to Lemmas 11 and 12, the functional $\mathcal{L}$ is Fréchet differentiable with respect to the point $\mathbf{s} = ((\mathbf{u}, T), (\mathbf{g}_1, g_2)) \in \mathbb{H} \times \mathcal{U}_{ad}$. Then, we can obtain the fact that the Lagrange multiplier $\tilde{\boldsymbol{\phi}} = ((\boldsymbol{\lambda}, \eta), \chi, \vartheta) \in \mathbb{X} \times \tilde{\mathbf{H}}_{00}^{-1/2}(\Gamma \setminus \Gamma_3) \times H_{00}^{-1/2}(\Gamma_4)$, given by Theorem 15, for $\mathbf{x} = (\mathbf{u}, T) \in \mathbb{H}$, satisfies $\mathcal{L}_{\mathbf{x}}(\tilde{\mathbf{s}}, \tilde{\boldsymbol{\phi}})(\boldsymbol{\omega}, \psi) = 0$; that is,

$$\begin{aligned} &(\tilde{\mathbf{u}} - \mathbf{u}_d, \boldsymbol{\omega}) + (\tilde{T} - T_d, \psi) - 2\nu(D(\boldsymbol{\omega}), D(\boldsymbol{\lambda})) \\ &- 2\alpha\nu \int_{\Gamma_3} \boldsymbol{\omega} \cdot \boldsymbol{\lambda} d\Gamma - (\tilde{\mathbf{u}} \cdot \nabla \boldsymbol{\omega}, \boldsymbol{\lambda}) - (\boldsymbol{\omega} \cdot \nabla \tilde{\mathbf{u}}, \boldsymbol{\lambda}) \\ &- \beta(\mathbf{G}\psi, \boldsymbol{\lambda}) - \kappa(\nabla \psi, \nabla \eta) - (\tilde{\mathbf{u}} \cdot \nabla \psi, \eta) \\ &- (\boldsymbol{\omega} \cdot \nabla \tilde{T}, \eta) - \kappa \int_{\Gamma_5} \gamma \psi \cdot \eta d\Gamma - \langle \chi, \boldsymbol{\omega} \rangle_{\Gamma \setminus \Gamma_3} \\ &- \langle \vartheta, \psi \rangle_{\Gamma_4} = 0, \end{aligned} \quad (88)$$

for all $(\boldsymbol{\omega}, \psi) \in \mathbb{H}$; this equation is called the Lagrange-Euler equation. In this section we will follow the ideas of [20, 30] (see also [28]).

Lemma 18. *Let $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{T}, \tilde{\mathbf{g}}_1, \tilde{g}_2) \in \mathcal{S}_{ad}$ be an admissible point for the optimal control problem (45). If ν, κ are sufficiently large and $\|\gamma\|_{L^\infty(\Gamma_5)}$ is small enough, such that*

$$\begin{aligned} \hat{\delta} &:= \min \left\{ \nu - \beta C \|\mathbf{G}\|_\infty - C \left\| (\tilde{\mathbf{u}}, \tilde{T}) \right\|_{\mathbb{H}}, \frac{\kappa}{2} \right. \\ &\quad \left. - \beta C \|\mathbf{G}\|_\infty - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} - C \left\| (\tilde{\mathbf{u}}, \tilde{T}) \right\|_{\mathbb{H}} \right\} > 0, \end{aligned} \quad (89)$$

then the Lagrange multiplier $(\lambda, \eta) \in \mathbb{X}$ satisfies

$$\|(\lambda, \eta)\|_{\mathbb{X}}^2 \leq \frac{1}{\widehat{\delta}} \widetilde{C} \mathcal{N}(\tilde{\mathbf{u}}, \tilde{T}), \quad (90)$$

where $\mathcal{N} := \|\tilde{\mathbf{u}} - \mathbf{u}_d\|^2 + \|\tilde{T} - T_d\|^2$ and $\widetilde{C} := C \max\{1/4\nu, 1/2\kappa\}$, with C being a positive constant depending only on Ω .

Proof. Setting $(\omega, \psi) = (\lambda, \tau) \in \mathbb{X}$ in (88), we have

$$\begin{aligned} 2\nu \|\lambda\|_{\widetilde{\mathbf{H}}_0^1}^2 + \kappa \|\eta\|_{H_0}^2 &\leq (\tilde{\mathbf{u}} - \mathbf{u}_d, \lambda) + (\tilde{T} - T_d, \eta) \\ &\quad - (\lambda \cdot \nabla \tilde{\mathbf{u}}, \lambda) - \beta(\mathbf{G}\eta, \lambda) \\ &\quad - (\lambda \cdot \nabla \tilde{T}, \eta) - \kappa \int_{\Gamma_5} \gamma \eta \cdot \eta \, d\Gamma. \end{aligned} \quad (91)$$

Applying the Hölder and Young inequalities in terms of the right side of (91), we obtain

$$\begin{aligned} |(\tilde{\mathbf{u}} - \mathbf{u}_d, \lambda)| &\leq \|\tilde{\mathbf{u}} - \mathbf{u}_d\| \|\lambda\| \\ &\leq \frac{C}{4\nu} \|\tilde{\mathbf{u}} - \mathbf{u}_d\|^2 + \nu \|\lambda\|_{\widetilde{\mathbf{H}}_0^1}^2, \\ |(\tilde{T} - T_d, \eta)| &\leq \|\tilde{T} - T_d\| \|\eta\| \\ &\leq \frac{C}{2\kappa} \|\tilde{T} - T_d\|^2 + \frac{\kappa}{2} \|\eta\|_{H_0}^2, \\ |\beta(\mathbf{G}\eta, \lambda)| &\leq \beta \|\mathbf{G}\|_{\infty} \|\eta\| \|\lambda\| \\ &\leq \beta C \|\mathbf{G}\|_{\infty} (\|\lambda\|_{\widetilde{\mathbf{H}}_0^1}^2 + \|\eta\|_{H_0}^2), \\ |(\lambda \cdot \nabla \tilde{\mathbf{u}}, \lambda) + (\lambda \cdot \nabla \tilde{T}, \eta)| \\ &\leq C \|\tilde{\mathbf{u}}\|_{\widetilde{\mathbf{H}}_0^1} \|\lambda\|_{\widetilde{\mathbf{H}}_0^1}^2 + C \|\tilde{T}\|_{H^1} (\|\lambda\|_{\widetilde{\mathbf{H}}_0^1}^2 + \|\eta\|_{H_0}^2) \\ &\leq C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}} \|\lambda\|_{\widetilde{\mathbf{H}}_0^1}^2 + C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}} \|\eta\|_{H_0}^2, \\ \left| \kappa \int_{\Gamma_5} \gamma \eta \cdot \eta \, d\Gamma \right| &\leq \kappa \|\gamma\|_{L^\infty(\Gamma_5)} \|\eta\|_{L^2(\Gamma_5)}^2 \\ &\leq \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} \|\eta\|_{H_0}^2. \end{aligned} \quad (92)$$

Then, by replacing the last inequalities in (91), we have

$$\begin{aligned} &(\nu - \beta C \|\mathbf{G}\|_{\infty} - C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}}) \|\lambda\|_{\widetilde{\mathbf{H}}_0^1}^2 \\ &+ \left(\frac{\kappa}{2} - \beta C \|\mathbf{G}\|_{\infty} - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} - C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}} \right) \\ &\cdot \|\eta\|_{H_0}^2 \leq \frac{C}{4\nu} \|\tilde{\mathbf{u}} - \mathbf{u}_d\|^2 + \frac{C}{2\kappa} \|\tilde{T} - T_d\|^2; \end{aligned} \quad (93)$$

since, by hypothesis,

$$\begin{aligned} \widehat{\delta} &:= \min \left\{ \nu - \beta C \|\mathbf{G}\|_{\infty} - C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}}, \frac{\kappa}{2} \right. \\ &\quad \left. - \beta C \|\mathbf{G}\|_{\infty} - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} - C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}} \right\} > 0, \end{aligned} \quad (94)$$

then, from the last inequality, we conclude the proof. $\square$

Theorem 19. Let $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{T}, \tilde{\mathbf{g}}_1, \tilde{g}_2) \in \mathcal{S}_{ad}$ be an admissible point for the optimal control problem (45), ν, κ be sufficiently large, and $\|\gamma\|_{L^\infty(\Gamma_5)}$ be small enough, such that

$$\begin{aligned} \widehat{\delta} &= \min \left\{ \nu - \beta C \|\mathbf{G}\|_{\infty} - C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}}, \frac{\kappa}{2} - \beta C \|\mathbf{G}\|_{\infty} \right. \\ &\quad \left. - \kappa C \|\gamma\|_{L^\infty(\Gamma_5)} - C \|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}} \right\} > 0. \end{aligned} \quad (95)$$

If $(\widetilde{C}^2 C^2 / \widehat{\delta} \Pi^2) \mathcal{N}(\tilde{\mathbf{u}}, \tilde{T}) < 1$, where $\Pi := \min\{Cb_1 \widehat{\delta}^2 / (1 + \widehat{\delta})^2, Cb_2 \widehat{\delta}^2 / (1 + \widehat{\delta})^2\}$ and $\mathcal{N}(\tilde{\mathbf{u}}, \tilde{T})$, $\widetilde{C}$ are given in Lemma 18, then there exists a constant $\mathcal{K} > 0$ such that

$$\mathcal{L}_{ss}(\tilde{\mathbf{s}}, \tilde{\phi})(\mathbf{t}, \mathbf{t}) \geq \mathcal{K} \|\mathbf{t}\|_{\mathbb{H} \times \mathcal{U}_{ad}}^2, \quad (96)$$

for all $\mathbf{t} \in \ker(\mathbb{F}_s(\mathbf{s}))$, with $\tilde{\phi} = ((\lambda, \eta), \chi, \vartheta) \in \mathbb{X} \times \widetilde{\mathbf{H}}_{00}^{-1/2}(\Gamma \setminus \Gamma_3) \times H_{00}^{-1/2}(\Gamma_4)$ given by Theorem 15. Consequently, the point $\tilde{\mathbf{s}}$ is a local optimal solution for problem (45).

Proof. Let $\mathbf{t} = ((\omega, \psi), (\mathbf{q}, \tau)) \in \mathbb{H} \times \mathcal{U}_{ad}$. Then, the second derivative of Lagrangian given in (56), with respect to $\mathbf{s}$ at the point $\tilde{\mathbf{s}}$ in $(\mathbf{t}, \mathbf{t})$, is

$$\begin{aligned} \mathcal{L}_{ss}(\tilde{\mathbf{s}}, \tilde{\phi})(\mathbf{t}, \mathbf{t}) &= \|\omega\|^2 + \|\psi\|^2 + b_1 \|\mathbf{q}\|_{\mathbf{H}^{1/2}(\Gamma_2)}^2 \\ &\quad + b_2 \|\tau\|_{H^{1/2}(\Gamma_4)}^2 - 2(\omega \cdot \nabla \omega, \lambda) \\ &\quad - 2(\omega \cdot \nabla \psi, \eta). \end{aligned} \quad (97)$$

By applying the Hölder and Young inequalities, we can obtain

$$\begin{aligned} \mathcal{L}_{ss}(\tilde{\mathbf{s}}, \tilde{\phi})(\mathbf{t}, \mathbf{t}) &\geq \|\omega\|^2 + \|\psi\|^2 + b_1 \|\mathbf{q}\|_{\mathbf{H}^{1/2}(\Gamma_2)}^2 \\ &\quad + b_2 \|\tau\|_{H^{1/2}(\Gamma_4)}^2 \\ &\quad - C(\|\lambda\|_{\widetilde{\mathbf{H}}_0^1} + \|\eta\|_{H_0}) \|(\omega, \psi)\|_{\mathbb{H}}^2. \end{aligned} \quad (98)$$

On the other hand, if $\mathbf{t} \in \ker(\mathbb{F}_s(\tilde{\mathbf{s}}))$, then, from (58)-(59), we obtain

$$\begin{aligned} &2\nu(D(\omega), D(\mathbf{v})) + 2\alpha\nu \int_{\Gamma_3} \omega \cdot \mathbf{v} \, d\Gamma + (\tilde{\mathbf{u}} \cdot \nabla \omega, \mathbf{v}) \\ &\quad + (\omega \cdot \nabla \tilde{\mathbf{u}}, \mathbf{v}) + \beta(\mathbf{G}\psi, \mathbf{v}) = 0, \\ &\kappa(\nabla \psi, \nabla z) + (\tilde{\mathbf{u}} \cdot \nabla \psi, z) + (\omega \cdot \nabla \tilde{T}, z) \\ &\quad + \kappa \int_{\Gamma_5} \gamma \psi \cdot z \, d\Gamma = 0, \\ &\omega|_{\Gamma \setminus \Gamma_3} = \mathcal{B}\mathbf{q}, \\ &\psi|_{\Gamma_4} = \tau, \end{aligned} \quad (99)$$

for all $(\mathbf{v}, z) \in \mathbb{X}$. Thus, by using Lemma 3, there exists $(\omega^\varepsilon, \psi^\varepsilon) \in \mathbb{H}$ such that $\omega^\varepsilon|_{\Gamma \setminus \Gamma_3} = \mathcal{B}\mathbf{q}$, $\omega|_{\Gamma_3} = \mathbf{0}$, $\psi|_{\Gamma_4} = \tau$, and $\psi|_{\Gamma_5} = 0$. Then, rewriting $(\omega, \psi) \in \mathbb{H}$ in the form $\omega = \omega^\varepsilon + \widehat{\omega}$ and $\psi = \psi^\varepsilon + \widehat{\psi}$ with $(\widehat{\omega}, \widehat{\psi}) \in \mathbb{X}$ new unknown, and similarly, as in the proof of Lemma 14, we obtain

$$a((\widehat{\omega}, \widehat{\psi}), (\mathbf{v}, z)) = \widetilde{\mathcal{F}}(\mathbf{v}, z) \quad \forall (\mathbf{v}, z) \in \mathbb{X}, \quad (100)$$

where the form $a(\cdot, \cdot)$ is defined in (66) and $\tilde{\mathcal{F}} \in \mathbb{X}'$ is defined by $\tilde{\mathcal{F}}(\mathbf{v}, z) := \langle \tilde{\mathbf{a}}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}_0'} + \langle \tilde{\mathbf{b}}, z \rangle_{H'}$, with $\langle \tilde{\mathbf{a}}, \mathbf{v} \rangle_{\tilde{\mathbf{H}}_0'} := -2\nu(D(\omega^\varepsilon), D(\mathbf{v})) - (\tilde{\mathbf{u}} \cdot \nabla \omega^\varepsilon, \mathbf{v}) - (\omega^\varepsilon \cdot \nabla \tilde{\mathbf{u}}, \mathbf{v}) - \beta(\mathbf{G}\psi^\varepsilon, \mathbf{v})$ and $\langle \tilde{\mathbf{b}}, z \rangle_{H'} := -\kappa(\nabla \psi^\varepsilon, \nabla z) - (\tilde{\mathbf{u}} \cdot \nabla \psi^\varepsilon, z) - (\omega^\varepsilon \cdot \nabla \tilde{T}, z)$. Also, from (100), we can obtain the fact that there exists a positive constant C depending only on ν , κ and $\|(\tilde{\mathbf{u}}, \tilde{T})\|_{\mathbb{H}}$, such that $\|(\tilde{\omega}, \tilde{\psi})\|_{\mathbb{X}} \leq (C/\tilde{\delta})\|(\omega^\varepsilon, \psi^\varepsilon)\|_{\mathbb{H}}$, where $\tilde{\delta}$ is given as in (71). Thus, we have

$$\begin{aligned} \|(\omega, \psi)\|_{\mathbb{H}} &\leq \|(\tilde{\omega}, \tilde{\psi})\|_{\mathbb{X}} + \|(\omega^\varepsilon, \psi^\varepsilon)\|_{\mathbb{H}} \\ &\leq C \frac{(1 + \tilde{\delta})}{\tilde{\delta}} (\|\varrho\|_{H^{1/2}(\Gamma_2)} + \|\tau\|_{H^{1/2}(\Gamma_4)}). \end{aligned} \quad (101)$$

Using (101) in (98) and taking into account the fact that $\Pi := \min\{Cb_1\tilde{\delta}^2/(1 + \tilde{\delta})^2, Cb_2\tilde{\delta}^2/(1 + \tilde{\delta})^2\}$, we obtain

$$\begin{aligned} \mathcal{L}_{ss}(\tilde{\mathbf{s}}, \tilde{\phi})(\mathbf{t}, \mathbf{t}) &\geq \|\omega\|^2 + \|\psi\|^2 + \frac{b_1}{2} \|\varrho\|_{H^{1/2}(\Gamma_2)}^2 + \frac{b_2}{2} \|\tau\|_{H^{1/2}(\Gamma_4)}^2 \\ &\quad - C(\|\lambda\|_{\tilde{\mathbf{H}}_0'} + \|\eta\|_{H_0}) \|(\omega, \psi)\|_{\mathbb{H}}^2 + \Pi \|(\omega, \psi)\|_{\mathbb{H}}^2 \quad (102) \\ &\geq (\Pi - C(\|\lambda\|_{\tilde{\mathbf{H}}_0'} + \|\eta\|_{H_0})) \|(\omega, \psi)\|_{\mathbb{H}}^2 \\ &\quad + \frac{b_1}{2} \|\varrho\|_{H^{1/2}(\Gamma_2)}^2 + \frac{b_2}{2} \|\tau\|_{H^{1/2}(\Gamma_4)}^2. \end{aligned}$$

Notice that, from Lemma 18, we can obtain $\|\lambda\|_{\tilde{\mathbf{H}}_0'} + \|\eta\|_{H_0} \leq ((\tilde{C}/\tilde{\delta})\mathcal{N}(\tilde{\mathbf{u}}, \tilde{T}))^{1/2}$; then, from (102), we have

$$\begin{aligned} \mathcal{L}_{ss}(\tilde{\mathbf{s}}, \tilde{\phi})(\mathbf{t}, \mathbf{t}) &\geq \left[\Pi - C \left(\frac{\tilde{C}}{\tilde{\delta}} \mathcal{N}(\tilde{\mathbf{u}}, \tilde{T}) \right)^{1/2} \right] \|(\omega, \psi)\|_{\mathbb{H}}^2 \quad (103) \\ &\quad + \frac{b_1}{2} \|\varrho\|_{H^{1/2}(\Gamma_2)}^2 + \frac{b_2}{2} \|\tau\|_{H^{1/2}(\Gamma_4)}^2. \end{aligned}$$

Since $C((\tilde{C}/\tilde{\delta})\mathcal{N}(\tilde{\mathbf{u}}, \tilde{T}))^{1/2} < \Pi$, then we have $\tilde{\Pi} := \Pi - C((\tilde{C}/\tilde{\delta})\mathcal{N}(\tilde{\mathbf{u}}, \tilde{T}))^{1/2} > 0$; therefore

$$\mathcal{L}_{ss}(\tilde{\mathbf{s}}, \tilde{\phi})(\mathbf{t}, \mathbf{t}) \geq \min \left\{ \tilde{\Pi}, \frac{b_1}{2}, \frac{b_2}{2} \right\} \|\mathbf{t}\|_{\mathbb{H} \times \mathcal{U}_{ad}}^2. \quad (104)$$

Thus, taking in particular $\mathbf{t} = ((\omega, \psi), (\varrho, \tau)) \in \ker(\mathbb{F}_s(\tilde{\mathbf{s}}))$ with $(\varrho, \tau) \in \mathcal{C}(\tilde{\mathbf{g}}_1) \times \mathcal{C}(\tilde{g}_2) := \{(\theta_1(\mathbf{g}_1 - \tilde{\mathbf{g}}_1), \theta_2(g_2 - \tilde{g}_2)) : \theta_1 \geq 0, \theta_2 \geq 0, (\mathbf{g}_1, g_2) \in \mathcal{U}_{ad}\}$, we obtain that the point $\tilde{\mathbf{s}} \in \mathcal{S}_{ad}$ is a local optimal solution of control problem (45) (see [34]). $\square$

6. Conclusions

In this paper, the existence and uniqueness of weak solutions of the stationary heat convection equations with mixed boundary conditions, including the Navier slip condition, were proved. Also, an optimal boundary control problem was analyzed. Boundary controls for the velocity vector and temperature and the existence of optimal solutions were

proved. The optimal control problem analyzed includes the minimization of the L^2 -distance between the velocities and some desired fields. By using the theorem of Lagrange multipliers, an optimality system was derived. A second-order sufficient condition was also given.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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