Research Article

Stationary Gas Networks with Compressor Control and Random Loads: Optimization with Probabilistic Constraints

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We introduce a stationary model for gas flow based on simplified isothermal Euler equations in a non-cycled pipeline network. Especially the problem of the feasibility of a random load vector is analyzed. Feasibility in this context means the existence of a flow vector meeting these loads, which satisfies the physical conservation laws with box constraints for the pressure. An important aspect of the model is the support of compressor stations, which counteract the pressure loss caused by friction in the pipes. The network is assumed to have only one influx node; all other nodes are efflux nodes. With these assumptions the set of feasible loads can be characterized analytically. In addition we show the existence of optimal solutions for some optimization problems with probabilistic constraints. A numerical example based on real data completes this paper.

1. Introduction

In this paper, we present a way to model stationary gas networks with random loads. Natural gas as energy source has been popular for decades. Because of the nuclear power phase-out and aim to leave energy gained by coal behind, natural gas as energy source is more current than ever. So in this paper we study the problem of gas transport in a pipeline network mathematically. The aim of this paper is to get results in optimization problems with probabilistic constraints using the example of gas networks.

There are several studies about the mathematical problem of gas transport. A gas transport network in general can be modeled as a system of hyperbolic balance laws, like e.g., the isothermal Euler equations. In [1] one can find a great overview about existing models and their application areas. A network system is modeled as a graph with nodes and edges. It can be extended by different elements, like compressor stations, valves and resistors. In [2] the authors describe the functionality of these elements and how they can be included to the model.

We assume an active stationary state for our model that means we use compressor stations as controllable elements in

a state of time-independent flows and pressures. Our model is based on the Weymouth-Equation (see [3]), a simplification of the isothermal Euler equation (see e.g., [4, 5]). Further [3] also studies the isothermal Euler equations for real gas. We assume our gas to be ideal. The optimal control problem in pipeline networks has been studied in, e.g., [4, 6]. In our work we assume the loads to be random, so this leads to optimization problems with probabilistic constraints (see [7]).

In the next section, we introduce our gas network model, which is based on the model in [8]. This is a passive one which we extend by compressor stations. So our model contains inner control variables which makes our model an active one. These controls will be one goal of the optimization in Section 4.

In Section 3 we characterize the set of feasible loads. Instead of searching a flow and a pressure vector (based on a given load vector), which fulfills the physical balance laws, we look for these vectors fulfilling a system of inequalities, which depend on bounds for the pressure. In addition, we assume the load vector to be random. So we adapt the system to the spheric-radial decomposition, which is our central tool to handle the uncertainty in the loads.

In Section 4, we consider some optimization problems with and without probabilistic constraints. The existence of optimal solutions is based on the analysis we did in Section 3. This should build a base for further works about optimal control problems with probabilistic constraints using the example of gas networks.

In Section 5 we present three numerical examples. The first is a minimal graph without inner control, which shows the idea of the spheric-radial decomposition. The second one is a minimal graph with inner control and the third one is based on real data, for which we present real values for a network.

In the last section we present a few ideas of extending this paper. One main aspect is the turnpike-theory.

2. Mathematical Modeling

We start with the introduction of the model that is also used in [8]. This model does not include compressor stations. We extend this model to include compressor stations and we discuss the network analysis. Then the main tool, the so called spheric-radial decomposition, is introduced to handle the stochastic uncertainty.

2.1. Model Description. We consider a connected, directed graph $G = (\mathcal{V}^+, \mathcal{E})$ which represents a pipeline gas transport network. We assume that there are no cycles in the graph, so the network is a tree. We set $|\mathcal{V}^+| := n + 1$ and $|\mathcal{E}| := m = n$ ($n, m \in \mathbb{N}$). In our model, every node is either a influx node (gas enters the network) or an efflux node (gas leaves the network). An edge can either be a flux edge, so the pressure decreases along the edge, or a compressor edge, so the pressure increases along the edge. We define $\mathcal{E}_F$ as the set of all flux edges and $\mathcal{E}_C$ as the set of compressor edges, it follows $\mathcal{E} = \mathcal{E}_F \cup \mathcal{E}_C$ with $\mathcal{E}_F \cap \mathcal{E}_C = \emptyset$.

Let $b^+ \in \mathbb{R}^{n+1}$ with $\mathbb{1}_{n+1}^T b^+ = 0$ denote the balanced load vector and assume $b_i \leq 0$ for nodes with gas influx and $b_i \geq 0$ for nodes with gas efflux ($i \in \{0, \dots, n\}$). The vector $\mathbb{1}_{n+1}$ denotes the vector of all ones in the dimension $n + 1$. Furthermore, a vector $q \in \mathbb{R}^m$ describes the flows through the edges resp. through the pipes. The pressures at the nodes are defined in a vector $p^+ \in \mathbb{R}^{n+1}$. Let pressure bounds $p^{+,min}, p^{+,max} \in \mathbb{R}^{n+1}$ be given. For the pressure we consider the constraints $p^+ \in [p^{+,min}, p^{+,max}]$. For further modeling we need the following definition:

Definition 1. Consider the graph $G = (\mathcal{V}^+, \mathcal{E})$:

- (i) $h(e)$ denotes the head node of an edge e and $f(e)$ denotes the feet of an edge e for all $e \in \mathcal{E}$.
- (ii) $E_0(v) := \{e \in \mathcal{E} \mid h(e) = v \vee f(e) = v\}$ denotes all edges which are connected to node v .
- (iii) The matrix $A_{ij}^+ \in \mathbb{R}^{(n+1) \times m}$, $A_{i,j}^+ := \sigma(v_i, e_j)$ with

$$\sigma(v, e) := \begin{cases} -1 & \text{if } f(e) = v \text{ and } e \in E_0(v) \\ 1 & \text{if } h(e) = v \text{ and } e \in E_0(v) \\ 0 & \text{if } e \notin E_0(v) \end{cases} \quad (1)$$

is called the incidence matrix of the graph G .

- (iv) For $v \in \mathcal{V}^+$ is $\Pi(v) := \{v\} \cup \bigcup_{\{k \in \mathcal{V}^+ \mid \exists e \in \mathcal{E}: f(e)=k \wedge h(e)=v\}} \Pi(k)$ the (unique) directed path from the root to v .
- (v) For $v, w \in \mathcal{V}^+$ is $\Pi_w(v) := \Pi(v) \setminus \Pi(w)$ the (unique) directed path from w to v .

Note that definition (iv) only makes sense since we assumed the graph to be tree-structured, so there are no cycles and the union is finite. The model is based upon the conservation equations of mass and momentum. The mass equation for the nodes is formulated for every node by

$$\sum_{e \in E_0(v)} q^e = b_v^+ \quad \forall v \in \mathcal{V}^+. \quad (2)$$

This is equivalent to *Kirchhoff's first law* (see [9]) and by using the definition of the incidence matrix, the equation for mass conservation for the whole graph is

$$A^+ q = b^+ \text{ resp. } Aq = b \text{ with} \quad (3)$$

$$A^+ = \begin{bmatrix} A_0 \\ A \end{bmatrix} \in \begin{matrix} \mathbb{R}^{1 \times m} \\ \mathbb{R}^{n \times m} \end{matrix} \text{ and } b^+ = \begin{bmatrix} b_0 \\ b \end{bmatrix} \in \begin{matrix} \mathbb{R} \\ \mathbb{R}^n \end{matrix}.$$

As balance laws we use the isothermal Euler equations (see [3, 4, 10]) for a horizontal pipe, so the flow through every flux edge is modeled by

$$\rho_t^e + q_x^e = 0$$

$$q_t^e + \left(\frac{(q^e)^2}{\rho^e} + a^2 \rho^e \right)_x = -\frac{\lambda^e}{2D^e} \frac{|q^e| q^e}{\rho^e} \quad (4)$$

$$\forall e \in \mathcal{E}_F$$

Here, ρ is the gas density, a is the speed of sound, λ is a (constant) friction coefficient and D is the diameter of the pipe.

We assume first, that the network is in a steady state, so the time derivatives are equal to zero. And second, we assume the gas flow to be slow, so the coefficient $(q^e/a)^2$ is negligible. These assumptions simplify the equations enormously. With the ideal gas equation $p = \rho R_S T$ (see [9]) we get a direct dependency between pressure and density. Solving this simplified equation leads us to the so called Weymouth-Equation (see [3]):

$$p_{f(e)}^2 - p_{h(e)}^2 = \phi^e |q^e| q^e \text{ with } \phi^e = (R_S T)^2 \frac{\lambda^e L^e}{D^e a^2} \quad (5)$$

$$\forall e \in \mathcal{E}_F.$$

Here, R_S is the specific gas constant, T is the temperature of the gas and L^e is the length of the pipe e . This equation shows the pressure drop along the pipes (see Figure 1). Some articles consider the isothermal Euler equations in gas transport without these simplifications, e.g., [5, 10].

Compressor stations counteract the pressure drop along the pipes. These stations are modeled as pipes without

FIGURE 1: Pressure drop along a pipe for $D = 0.762m$, $T = 289.15K$, $R_s = 500.27J/kgK$, $\lambda = 0.005$, $a = 429.17m/s$, $q = 795kg/m^2s$ and $p(0) = 60 \cdot 10^5 Pa$.

friction. We use the following model equation (see [1], section 7.4.7; [11], section 2.3):

$$\frac{c}{\gamma} \left(\left(\frac{p_{out}}{p_{in}} \right)^\gamma - 1 \right) = \tilde{u}. \quad (6)$$

Here, p_{out} and p_{in} are the pressures at the end resp. at the beginning of one pipe, c and γ are constants and $\tilde{u}$, which is our control, is the specific change in adiabatic enthalpy. More information about the parameters can be found in [11]. In addition, [2] contains more information about the functionality of compressor stations and a deduction of the equation. A short transformation of the equation leads to our representation of the pressure rise along the compressor edges:

$$\left(\frac{p_{h(e)}}{p_{f(e)}} \right)^2 = u^e \text{ with } u^e = \left(\frac{\gamma}{c} \tilde{u} + 1 \right)^{2/\gamma} \quad \forall e \in \mathcal{E}_C. \quad (7)$$

It holds $u^e \geq 1$ where $u^e = 1$ means that the compressor station is switched off (see Figure 2).

Now, we define the set of feasible loads. We say that a vector $b^+ \in \mathbb{R}^{n+1}$ is feasible, if (3), (5), and (7) are fulfilled. So the feasible set (here called $\widehat{M}$) is defined as follows:

$$\begin{aligned} \widehat{M} = & \{ b^+ \in \mathbb{R}^{n+1} \mid \mathbb{1}_{n+1}^T b^+ = 0 \text{ and } \exists (p^+, q) \in \mathbb{R}^{n+1} \\ & \times \mathbb{R}^m : p^+ \\ & \in [p^{+,min}, p^{+,max}] \text{ and (3), (5), (7) are fulfilled} \}. \end{aligned} \quad (8)$$

2.2. Stochastic Uncertainty in the Loads. In reality, the future demand for gas can never be known exactly a priori. It depends on various physical effects like e.g., the temperature of the environment. Also personal reasons can be responsible for a higher or lower use of gas. Thus the load vector is

FIGURE 2: Compressor property for $c = 144652.6J/kg$, $\gamma = 0.225$, and $p_{in} = 40 \cdot 10^5 Pa$.

considered to be a random vector. Many papers study how the distribution functions of the load vector can be identified, e.g., [12]. This article also describes, why a multivariate Gaussian distribution is a good choice for random gas demand. So we use some random vector $\xi(\omega)$ with $\xi \sim \mathcal{N}(\mu, \Sigma)$ for a mean vector μ and a positive definite covariance matrix Σ on a suitable probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and our aim is to calculate the following probability:

$$\mathbb{P} \{ \omega \in \Omega \mid \xi(\omega) \in M \} \quad (9)$$

for a suitable set M .

In our case, we assume that the tree-structured graph has only one influx node, which gets the number zero, all remaining nodes are efflux nodes. Then the graph is numerated with breadth-first search or depth-first search and every edge $e \in \mathcal{E}$ gets the number $\max\{h(e), f(e)\}$. Our set M is defined as follows:

$$M := \{ b \in \mathbb{R}_{\geq 0}^n \mid (-\mathbb{1}_n^T b, b) \in \widehat{M} \}. \quad (10)$$

There are many studies for special sets M , e.g., for polyhedral sets (see [13]), but our set is very general. So we need a very general method for calculating the probability (9). Our method of choice is the so called *spheric-radial decomposition*. It became popular for dealing with probabilistic constraints (e.g., see [8, 14–19]).

Theorem 2 (spheric-radial decomposition). *Let $\xi \sim \mathcal{N}(0, R)$ be the n -dimensional standard Gaussian distribution with zero mean and positive definite correlation matrix R . Then, for any Borel measurable subset $M \subseteq \mathbb{R}^n$ it holds that*

$$\mathbb{P}(\xi \in M) = \int_{\mathbb{S}^{n-1}} \mu_\chi \{ r \geq 0 \mid rLv \in M \} d\mu_\eta(v), \quad (11)$$

where $\mathbb{S}^{n-1}$ is the $(n-1)$ -dimensional sphere in $\mathbb{R}^n$, μ_η is the uniform distribution on $\mathbb{S}^{n-1}$, μ_χ denotes the χ -distribution

with n degrees of freedom and L is such that $R = LL^T$ (e.g., Cholesky decomposition).

The difference with our problem is, that our load vector is normal distributed with mean value μ and positive definite covariance Σ . We want to use the result of Theorem 2 to general Gaussian distributions (see [8]). Therefore we set

$$\xi^* := D^{-1}(\xi - \mu) \sim \mathcal{N}(0, R) \text{ with} \quad (12)$$

$$D := \text{diag}\left(\sqrt{\Sigma_{ii}}\right)_{i=1,\dots,n} \text{ and } R := D^{-1}\Sigma D^{-1},$$

and observe that $\mathbb{P}(\xi \in M) = \mathbb{P}(\xi^* \in D^{-1}(M - \mu))$. So our aim is to compute the following probability:

$$\mathbb{P}(\xi^* \in D^{-1}(M - \mu)). \quad (13)$$

So the calculation of the probability (9) for a very general set M is equivalent to compute the integral in Theorem 2, which is not easy to compute analytically, too. But the integral can be computed numerically in a easy and nearly exact way. Therefore we formulate the following algorithm (see [8], Algorithm 4):

Algorithm 3. Let $\xi \sim \mathcal{N}(\mu, \Sigma)$ and L such that $LL^T = \Sigma$.

- (1) Sample N points $\{v_1, \dots, v_N\}$ uniformly distributed on the sphere $\mathbb{S}^{n-1}$.
- (2) Compute the one-dimensional sets $M_i := \{r \geq 0 \mid rLv_i + \mu \in M\}$ for $i = 1, \dots, N$.
- (3) Set $\mathbb{P}(\xi \in M) \approx (1/N) \sum_{i=1}^N \mu_\chi(M_i)$.

The first step is quite easy. There are many possibilities to sample a set of uniformly distributed points on the sphere $\mathbb{S}^{n-1}$, e.g., *Monte-Carlo* sampling or the *randn*-generator in *MATLAB*[®]. Normalizing these points leads us to the sampling we are looking for. Alternatively, quasi Monte-Carlo methods can be used. In [8] the authors use a quasi Monte-Carlo method for their computations and they show its advantages in Figure 6.

The main challenge is to handle the one-dimensional sets of the second step. If the set M is convex, these sets are closed intervals but M need not to be convex. The aim is to get a representation of M_i ($i = 1, \dots, N$) as a union of disjoint intervals. With this, the third step can be done efficiently. The next section deals with this representation in detail.

The third step is the approximation of the integral. The values $\mu_\chi(M_i)$ can be calculated easily with the distribution function $\mathcal{F}_\chi$ of the distribution (see [20] for details of that distribution). With the representation $M_i = \bigcup_{j=1}^t [a_j, \bar{a}_j]$ the χ -distribution can be calculated the following:

$$\mu_\chi(M_i) = \sum_{j=1}^t \mathcal{F}_\chi(\bar{a}_j) - \mathcal{F}_\chi(a_j). \quad (14)$$

This leads to the wanted probability. The χ -distribution function is given by the following (see [20], p. 132):

$$\mathcal{F}_\chi(x) = \frac{1}{2^{N/2}\Gamma(N/2)} \int_0^x v^{N/2-1} e^{-v/2} dv \quad (15)$$

FIGURE 3: Distribution function of the Chi-square distribution for 3 (blue), 5 (red), 8 (green), and 12 (purple) degrees of freedom.

with $\Gamma(1/2) = \pi$, $\Gamma(1) = 1$ and $\Gamma(r+1) = r\Gamma(r)$ for all $r \in \mathbb{R}^+$. Note that there are very precise numerical approximations for the distribution function, e.g., the *chi2cdf*-function in *MATLAB*[®] (see Figure 3).

3. Explicit Characterization of the Gas Flow

In this section we want to characterize the feasible set to use Algorithm 3. The aim is to get a representation of the one-dimensional sets of step (2) as a union of disjoint intervals. Therefore we need another representation of the feasible set as a system of inequalities. This is an idea from [8], Theorem 1. In the proof, the authors use a representation of the pressure loss for the full network graph:

$$(A^+)^T (p^+)^2 = -\Phi |q| q, \quad (16)$$

which is equivalent to (5) if and only if $\mathcal{E}_C = \emptyset$. In this paper we also consider the case $\mathcal{E}_C \neq \emptyset$. The matrix Φ is diagonal with the values ϕ^e as defined in (5) at its diagonal. This representation is not possible yet, because we extended our model by compressor edges, so this equation does not hold for all edges. So we have to find a way to use a similar equation like (16) to state a suitable theorem. The idea that allows to include compressor stations is to remove compressor edges from the graph and replace them by suitable algebraic conditions. First we consider the equation of mass:

$$\sum_{e \in E_0(v)} q^e = b_v^+ \iff \sum_{e \in E_0(v) \cap \mathcal{E}_F} q^e = b_v^+ - \sum_{e \in E_0(v) \cap \mathcal{E}_C} q^e. \quad (17)$$

We want to treat the flows through the compressor edges as loads, so we have to adapt the load vector. The new load vector $\tilde{b}^+$ is defined as follows:

$$\tilde{b}_v^+ := \begin{cases} b_v^+ + q^e & \forall e \in E_0(v) \text{ with } e \in \mathcal{E}_C \text{ and } f(e) = v \\ b_v^+ - q^e & \forall e \in E_0(v) \text{ with } e \in \mathcal{E}_C \text{ and } h(e) = v \\ b_v^+ & \text{else.} \end{cases} \quad (18)$$

The resulting graph $\tilde{G} = (\mathcal{V}^+, \mathcal{E}_F)$ is no longer connected (scheme is shown in Figure 4).

FIGURE 4: Network graph with two compressor edges (left) and new not connected network graph with three connected subgraphs after removing the compressor edges (right).

We set $|\mathcal{E}_F| := m_1$ and $|\mathcal{E}_C| := m_2$ with $m_1 + m_2 = m$, so $\tilde{G}$ consists of $m_2 + 1$ subgraphs, which are all connected. Note that q is a vector in $\mathbb{R}^{m_1}$ now. We determine that G_i is the

i -th self-connected subgraph of G (depends on numbering of the graph). Analogously $p_{i,j}^+$ is the j -th component of the i -th subgraph. The terms $q_{i,j}$ and $A_{i,j,k}$ are defined similarly. The feasible set for $\tilde{G}$ is then

$$\tilde{M} = \left\{ \tilde{b}^+ \in \mathbb{R}^{n+1} \mid \begin{array}{l} \mathbb{1}_{n+1}^T \tilde{b}^+ = 0 \\ \exists (p^+, q) \in \mathbb{R}_{\geq 0}^{n+1} \times \mathbb{R}^m \quad \text{with } p^+ \in [p^{+,min}, p^{+,max}] : \\ A_i q_i = \tilde{b}_i \quad \forall i = 1, \dots, m_2 + 1 \\ (A_i^+)^T (p_i^+)^2 = -\Phi_i |q_i| \circ q_i \quad \forall i = 1, \dots, m_2 + 1 \\ \left(\frac{p_{h(e)}}{p_{f(e)}} \right)^2 = u_e \quad \forall e \in \mathcal{E}_C \end{array} \right\} \quad (19)$$

The last equation in $\tilde{M}$ is the same as in (8), that models the pressure rise caused by the compressor. With this we can present the equation for pressure loss in the form of (16) for every subgraph. Before we state a theorem for characterizing the feasible set, we show the equivalence of $\tilde{M}$ and $\tilde{M}$.

Lemma 4. Let $\tilde{b}^+$ be as defined in (18). The feasible sets defined in (8) and (19) are equivalent in the following sense:

- (i) For all $b^+ \in \tilde{M}$ there is $\tilde{b}^+ \in \tilde{M}$
 - (ii) For all $\tilde{b}^+ \in \tilde{M}$ there is $b^+ \in \tilde{M}$
- (20)

Proof. (i) Consider a vector $b^+ \in \tilde{M}$. It holds $\mathbb{1}_{n+1}^T b^+ = 0$ and because of (18) it follows directly $\mathbb{1}_{n+1}^T \tilde{b}^+ = 0$. After removing the compressor edges, from $A^+ q = b^+$ it follows with (17) that $A_i^+ q_i = \tilde{b}_i^+$, which is equivalent to $A_i q_i = \tilde{b}_i$. The pressure loss is fulfilled for all flux edges in $\tilde{M}$ and because the flow vector q doesn't change in the relevant coefficients while removing the compressor edges, the pressure loss is fulfilled in $\tilde{M}$ too. The compressor property is independent of b^+ and q , so it also holds. From this it follows $\tilde{b}^+ \in \tilde{M}$.

(ii) The other way around is quite similar. We consider a vector $\tilde{b}^+ \in \tilde{M}$. From $\mathbb{1}_{n+1}^T \tilde{b}^+ = 0$ it follows with (18) that $\mathbb{1}_{n+1}^T b^+ = 0$. The equation for mass conservation is fulfilled in every subgraph. So with (17) it follows $A^+ q = b^+$, which implies the mass conservation in $\tilde{M}$. The pressure loss holds

in all self-connected parts of the graph too and because the relevant coefficients of q doesn't change while adding the compressor edges, the pressure loss is fulfilled in $\tilde{M}$ too. Last, as said before, the compressor property is independent of $\tilde{b}^+$ and q , it is also fulfilled in $\tilde{M}$. This completes the proof. $\square$

Because our network graph is a tree, the mass conservation is fulfilled with a full rank incidence matrix A of size $n \times n$, so q can be computed explicit by the following:

$$q = A^{-1} b, \quad (21)$$

which makes us able to use Lemma 4.

With this result, we can write the equation for momentum conservation depending on the incidence matrices for the parts of the tree. This makes us able to formulate the theorem for characterizing the feasible set $\tilde{M}$. Therefore we will use the following function:

$$g : \begin{cases} \mathbb{R}^n \rightarrow \mathbb{R} \\ x \mapsto (A^T)^{-1} \Phi |A^{-1} x| (A^{-1} x), \end{cases} \quad (22)$$

which describes the pressure loss between the nodes. We also introduce a new notation:

- (i) $u_{(i,j),k}$ is the k -th control on the path from $v_{i,0}$ to $v_{j,0}$
- (ii) $g_{(i,j),k}$ is the function g for the k -th subgraph on the path from $v_{i,0}$ to $v_{j,0}$, whereas $g_{(i,j),1}$ is the function g

of the graph G_i . According to this, $g_{(i,j),k,\ell}$ is the ℓ -th component of the function $g_{(i,j),k}$

- (iii) $\tilde{b}_{(i,j),k}$ is the load vector of the k -th subgraph in the path from $v_{i,0}$ to $v_{j,0}$, whereas $\tilde{b}_{(i,j),1}$ is the load vector of the graph G_i and $\tilde{b}_{(i,j),k,\ell}$ is the ℓ -th component of $\tilde{b}_{(i,j),k}$

Analogously we define $A_{(i,j),k}$, $P_{(i,j),k}$ and $q_{(i,j),k}$. Furthermore we define the following values:

- (i) $k_{i,j}^* := \max\{k \in \{1, \dots, m_2 + 1\} \mid v_{k,0} \in \Pi(v_{i,0}) \wedge v_{k,0} \in \Pi(v_{j,0})\}$ as the largest index of all subgraph, the paths to $v_{i,0}$ and $v_{j,0}$ pass through
- (ii) $n_{i,j}^* : |\{k \in \{1, \dots, m_2 + 1\} \mid v_{k,0} \in \Pi_{v_{i,0}}(v_{j,0})\}|$ as the number of subgraph which are on the path from $v_{i,0}$ from $v_{j,0}$ containing G_i and G_j
- (iii) $m_{i,j}^* := |\{e \in \mathcal{E}_C \mid h(e) \in \Pi_{v_{i,0}}(v_{j,0}) \wedge f(e) \in \Pi_{v_{i,0}}(v_{j,0})\}|$ as the number of controls which are between $v_{i,0}$ and $v_{j,0}$

With this notation and these values we define a sum for $i, j \in \{1, m_2 + 1\}$. If $v_{i,0} \in \Pi(v_{j,0})$ we set $\Sigma_{i,i} := 0$ and else we set

$$\Sigma_{k^*,i} := \sum_{k=1}^{n^*-2} \frac{1}{\prod_{\ell=1}^{m^*-k} u_{(k^*,i),\ell}} \cdot g_{(k^*,i),n^*-k,f(e_{u_{(k^*,i),n^*-k}})}(\tilde{b}_{(k^*,i),n^*-k}) + g_{(k^*,i),1,f(e_{u_{(k^*,i),1}})}(\tilde{b}_{(k^*,i),1}), \quad (23)$$

where we write only n^* and m^* instead of $n_{k^*,i}^*$ and $m_{k^*,i}^*$ for better reading. Last, we define a product $\Pi_{i,i} := 1$ and

$$\Pi_{k^*,i} := \prod_{k=1}^{m^*} u_{(k^*,i),k}. \quad (24)$$

From a first point of view, the notation seems to be confusing, but using this notation makes sense since we want to guarantee the feasibility of a load vector by comparing the pressure bounds at every node with each other. We explain this in a short example. Figure 5 shows a graph after removing the compressor edges.

If we want to compare the pressure bounds of node 11 and node 12, we need to know how the pressures change on the path from 2 to 11, resp., from 2 to 12. Node 11 is part of the subgraph G_5 and node 12 is part of the subgraph G_6 , so it follows $k_{5,6}^* = 2$ because node 2 is part of subgraph 2. The notation can be understood as a numbering along the path. It is $u_{(2,11),1} = u_2$, the first control on the path from node 2 to node 11 and $u_{(2,11),2} = u_4$, the second control on this path. Further it follows $n_{2,11}^* = 3$ (G_2, G_3, G_5) and $m_{2,11}^* = 2$ (u_2, u_4). The product defined before is the product of all controls along this path and the sum is the pressure loss between node 2 and node 7. Using this notation makes us able to formulate the theorem for characterizing the feasible set in a readable way.

Theorem 5. For given pressure bounds $p^{+,min}, p^{+,max} \in \mathbb{R}^{n+1}$ and controls $u_i \in \mathbb{R}$ ($i = 1, \dots, m_2$) the following equivalence holds.

A vector $\tilde{b}^+$ with $\mathbb{1}_{n+1}^T \tilde{b}^+ = 0$ is feasible, i.e., $\tilde{b}^+ \in \tilde{M}$, if and only if the following inequalities hold. For all $i = 1, \dots, m_2 + 1$ holds (feasibility inside the subgraphs):

$$(p_{i,0}^{min})^2 \leq \min_{k=1, \dots, n_i} [(p_{i,k}^{max})^2 + g_{i,k}(\tilde{b}_i)], \quad (25)$$

$$(p_{i,0}^{max})^2 \geq \max_{k=1, \dots, n_i} [(p_{i,k}^{min})^2 + g_{i,k}(\tilde{b}_i)], \quad (26)$$

$$\max_{k=1, \dots, n_i} [(p_{i,k}^{min})^2 + g_{i,k}(\tilde{b}_i)] \leq \min_{k=1, \dots, n_i} [(p_{i,k}^{max})^2 + g_{i,k}(\tilde{b}_i)]. \quad (27)$$

For all $i, j = 1, \dots, m_2 + 1$ with $i < j$ holds (feasibility between the subgraphs):

$$\frac{1}{\Pi_{k^*,i}} (p_{i,0}^{min})^2 + \Sigma_{k^*,i} \leq \frac{1}{\Pi_{k^*,j}} (p_{j,0}^{max})^2 + \Sigma_{k^*,j}, \quad (28)$$

$$\frac{1}{\Pi_{k^*,i}} (p_{i,0}^{max})^2 + \Sigma_{k^*,i} \geq \frac{1}{\Pi_{k^*,j}} (p_{j,0}^{min})^2 + \Sigma_{k^*,j}, \quad (29)$$

$$\frac{1}{\Pi_{k^*,i}} (p_{i,0}^{min})^2 + \Sigma_{k^*,i} \leq \frac{1}{\Pi_{k^*,j}} \min_{k=1, \dots, n_j} [(p_{j,k}^{max})^2 + g_{j,k}(\tilde{b}_j)] + \Sigma_{k^*,j}, \quad (30)$$

$$\frac{1}{\Pi_{k^*,i}} (p_{i,0}^{max})^2 + \Sigma_{k^*,i} \geq \frac{1}{\Pi_{k^*,j}} \max_{k=1, \dots, n_j} [(p_{j,k}^{min})^2 + g_{j,k}(\tilde{b}_j)] + \Sigma_{k^*,j}, \quad (31)$$

$$\frac{1}{\Pi_{k^*,i}} \max_{k=1, \dots, n_i} [(p_{i,k}^{min})^2 + g_{i,k}(\tilde{b}_i)] + \Sigma_{k^*,i} \leq \frac{1}{\Pi_{k^*,j}} (p_{j,0}^{max})^2 + \Sigma_{k^*,j}, \quad (32)$$

$$\frac{1}{\Pi_{k^*,i}} \min_{k=1, \dots, n_i} [(p_{i,k}^{max})^2 + g_{i,k}(\tilde{b}_i)] + \Sigma_{k^*,i} \geq \frac{1}{\Pi_{k^*,j}} (p_{j,0}^{min})^2 + \Sigma_{k^*,j}, \quad (33)$$

$$\frac{1}{\Pi_{k^*,i}} \max_{k=1, \dots, n_i} [(p_{i,k}^{min})^2 + g_{i,k}(\tilde{b}_i)] + \Sigma_{k^*,i} \leq \frac{1}{\Pi_{k^*,j}} \min_{k=1, \dots, n_j} [(p_{j,k}^{max})^2 + g_{j,k}(\tilde{b}_j)] + \Sigma_{k^*,j}, \quad (34)$$

$$\frac{1}{\Pi_{k^*,i}} \min_{k=1, \dots, n_i} [(p_{i,k}^{max})^2 + g_{i,k}(\tilde{b}_i)] + \Sigma_{k^*,i} \geq \frac{1}{\Pi_{k^*,j}} \max_{k=1, \dots, n_j} [(p_{j,k}^{min})^2 + g_{j,k}(\tilde{b}_j)] + \Sigma_{k^*,j}. \quad (35)$$

FIGURE 5: Example graph for illustration of the used notation.

FIGURE 6: Graph of Example 1.

Let us have a look at the inequalities. Inequalities (25)–(27) are well-known from [8] and they describe the feasibility inside a connected tree. The other inequalities (28)–(35) guarantee the feasibility between the subgraphs. Therefore we compare the pressures of the entries of two parts ((28), (29)), of the entry of one part and the exit nodes of the other parts ((30)–(33)) and we compare the pressures of all exits ((34), (35)). Because we cannot compare the pressures of G_i and G_j directly (there may be no direct connection), we have to compare them using the subgraph G_{k^*} which is the last part the paths to $v_{i,0}$ and $v_{j,0}$ pass. Therefore we defined the sums $\Sigma_{k^*,i}$ and $\Sigma_{k^*,j}$, which trace back the pressures to G_{k^*} .

Remark 6. For an implementation it is wise to define a set $U_{(i,j)}$ containing all controls on the path from $v_{i,0}$ to $v_{j,0}$ and use an index running over this set to define the sums and the products.

Now we want to proof the theorem.

Proof of Theorem 5. Because the columns of A^+ sum up to zero, it holds

$$-\mathbb{1}_{n_i}^T A_i = A_{i,0}^+ \quad (i = 1, \dots, m_2 + 1). \quad (36)$$

“ $\implies$ ” We consider a feasible vector $b^+ \in \widetilde{M}$. From the equation of mass conservation of every subgraph it follows

$$q_i = A_i^{-1} \tilde{b}_i. \quad (37)$$

We take the equation of momentum conservation for the i -th subgraph ($i \in \{1, \dots, m_2 + 1\}$) and separate A_i^+ to $A_{i,0}^+$ and A_i^- :

$$(A_{i,0}^+)^T p_{i,0}^2 + A_i^T p_i^2 = -\Phi |q_i| q_i. \quad (38)$$

Next we insert (36) into (38) and get

$$-A_{i,0}^T \mathbb{1}_{n_i} p_{i,0}^2 + A_i^T p_i^2 = -\Phi |q_i| q_i. \quad (39)$$

We multiply this equation from left with $(A_i^T)^{-1}$ and insert (37), so it follows

$$\mathbb{1}_{n_i} p_{i,0}^2 = p_i^2 + (A_i^T)^{-1} \Phi |A_i^{-1} \tilde{b}_i| A_i^{-1} \tilde{b}_i \quad (40)$$

and using the definition of the function g we have

$$\mathbb{1}_{n_i} p_{i,0}^2 = p_i^2 + g_i(\tilde{b}_i). \quad (41)$$

Note here, that p_i is the vector of pressures of the subgraph G_i and $g_i(\tilde{b}_i)$ is the vector-valued function, which describes the pressure loss in G_i . Equation (41) holds for all subgraphs G_i with $i \in \{1, \dots, m_2 + 1\}$. From this, the inequalities (25)–(27) follow directly with $p^+ \in [p^{+,min}, p^{+,max}]$. Next we use (41) component-by-component. For fix $i, j \in \{1, \dots, m_2 + 1\}$ with $i < j$ it follows

$$p_{k^*,0}^2 = p_{k^*,k}^2 + g_{k^*,k}(\tilde{b}_{k^*}) \quad (k = 1, \dots, n_{k^*}). \quad (42)$$

Remember, that $k_{i,j}^*$ is the largest index of all subgraphs, the paths to $v_{i,0}$ and $v_{j,0}$ pass through. For better reading, we only write k^* instead of $k_{i,j}^*$. We use the equation of the compressor property for the first compressor on the path from $v_{k^*,0}$ to $v_{i,0}$,

$$\frac{p_{h(e_{u_{(k^*,i),1})}}^2}{p_{f(e_{u_{(k^*,i),1})}}^2} = u_{(k^*,i),1}, \quad (43)$$

and insert it into (42) for $k = f(e_{u_{(k^*,i),1}})$:

$$\begin{aligned} p_{(k^*,i),1,0}^2 &= \frac{1}{u_{(k^*,i),1}} p_{(k^*,i),2,0}^2 \\ &+ g_{(k^*,i),1,f(e_{u_{(k^*,i),1})}}(\tilde{b}_{(k^*,i),1}). \end{aligned} \quad (44)$$

Note that $p_{k^*,0}^2 = p_{(k^*,i),1,0}^2$, $g_{k^*,k}(\tilde{b}_{k^*}) = g_{(k^*,i),1,k}(\tilde{b}_{(k^*,i),1})$ and $p_{h(e_{u_{(k^*,i),1})}}^2 = p_{(k^*,i),2,0}^2$. We use the compressor equation for the second compressor on the path from $v_{k^*,0}$ to $v_{i,0}$ and put it, together with (42) into (44):

$$\begin{aligned} p_{(k^*,i),1,0}^2 &= \frac{1}{u_{(k^*,i),1}} \left[\frac{1}{u_{(k^*,i),2}} p_{(k^*,i),3,0}^2 \right. \\ &+ g_{(k^*,i),2,f(e_{u_{(k^*,i),2})}}(\tilde{b}_{(k^*,i),2}) \\ &+ g_{(k^*,i),1,f(e_{u_{(k^*,i),1})}}(\tilde{b}_{(k^*,i),1}) \left. \right] \end{aligned} \quad (45)$$

We repeat this procedure for all controls on the path from $v_{k^*,0}$ to $v_{i,0}$. Although we write n^* and m^* instead of $n_{k^*,i}^*$ and $m_{k^*,i}^*$. Thus we get

$$\begin{aligned} p_{(k^*,i),1,0}^2 &= \frac{1}{u_{(k^*,i),1}} \left[\frac{1}{u_{(k^*,i),2}} \left[\cdots \left[\frac{1}{u_{(k^*,i),m^*}} p_{(k^*,i),n^*,0}^2 \right. \right. \right. \\ &\quad \left. \left. \left. + g_{(k^*,i),n^*-1,f(e_{u_{(k^*,i),n^*-1}})}(\tilde{b}_{(k^*,i),n^*-1}) \right] \cdots \right] \right. \\ &\quad \left. + g_{(k^*,i),2,f(e_{u_{(k^*,i),2}})}(\tilde{b}_{(k^*,i),2}) \right] \\ &\quad + g_{(k^*,i),1,f(e_{u_{(k^*,i),1}})}(\tilde{b}_{(k^*,i),1}). \end{aligned} \quad (46)$$

We take again (42) for the node $v_{k^*,0}$ and do the same procedure just like before for the path from $v_{k^*,0}$ to $v_{j,0}$. With $n^* = n_{k^*,j}^*$ and $m^* = m_{k^*,j}^*$ we get

$$\begin{aligned} p_{(k^*,j),1,0}^2 &= \frac{1}{u_{(k^*,j),1}} \left[\frac{1}{u_{(k^*,j),2}} \left[\cdots \left[\frac{1}{u_{(k^*,j),m^*}} p_{(k^*,j),n^*,0}^2 \right. \right. \right. \\ &\quad \left. \left. \left. + g_{(k^*,j),n^*-1,f(e_{u_{(k^*,j),n^*-1}})}(\tilde{b}_{(k^*,j),n^*-1}) \right] \cdots \right] \right. \\ &\quad \left. + g_{(k^*,j),2,f(e_{u_{(k^*,j),2}})}(\tilde{b}_{(k^*,j),2}) \right] \\ &\quad + g_{(k^*,j),1,f(e_{u_{(k^*,j),1}})}(\tilde{b}_{(k^*,j),1}). \end{aligned} \quad (47)$$

Expanding (46) and (47) and equalizing them (this is possible because $p_{(k^*,i),1,0} = p_{(k^*,j),1,0}$) leads to

$$\frac{1}{\Pi_{k^*,i}} p_{i,0}^2 + \Sigma_{k^*,i} = \frac{1}{\Pi_{k^*,j}} p_{j,0}^2 + \Sigma_{k^*,j} \quad (48)$$

with $\Sigma_{k^*,i}$ resp. $\Sigma_{k^*,j}$ defined before. (48) directly implies (28) and (29). Using (48) together with (42) for $v_{i,0}$ first time, for $v_{j,0}$ second time and for $v_{i,0}$ and $v_{j,0}$ a third time, the inequalities (30)–(35) follow directly. So this part of the proof is complete.

“ $\Leftarrow$ ” For this part, we consider a vector $b^+ \in \mathbb{R}^{n+1}$ with $\mathbb{1}_{n+1}^T b^+ = 0$, which fulfills the inequalities (25)–(35). We define the following sets ($i = 2, \dots, m_2 + 1$):

$$\begin{aligned} P_{1,1} &:= \left[(p_{1,0}^{\min})^2, (p_{1,0}^{\max})^2 \right], \\ P_{2,1} &:= \left[\max_{k=1,\dots,n_1} \left[(p_{1,k}^{\min})^2 + g_{1,k}(\tilde{b}_1) \right], \right. \\ &\quad \left. \min_{k=1,\dots,n_1} \left[(p_{1,k}^{\max})^2 + g_{1,k}(\tilde{b}_1) \right] \right], \\ P_{1,i} &:= \left[\frac{1}{\Pi_{1,i}} (p_{i,0}^{\min})^2 + \Sigma_{1,i}, \frac{1}{\Pi_{1,i}} (p_{i,0}^{\max})^2 + \Sigma_{1,i} \right], \end{aligned}$$

$$\begin{aligned} P_{2,i} &:= \left[\frac{1}{\Pi_{1,i}} \max_{k=1,\dots,n_1} \left[(p_{i,k}^{\min})^2 + g_{i,k}(\tilde{b}_i) \right] + \Sigma_{1,i}, \frac{1}{\Pi_{1,i}} \right. \\ &\quad \left. \cdot \min_{k=1,\dots,n_1} \left[(p_{i,k}^{\max})^2 + g_{i,k}(\tilde{b}_i) \right] + \Sigma_{1,i} \right]. \end{aligned} \quad (49)$$

We set

$$P_0 := \bigcap_{k=1}^{m_2+1} P_{1,k} \cap \bigcap_{k=1}^{m_2+1} P_{2,k} \quad (50)$$

and we will define a value $p_{1,0}^2 \in P_0$ later. So to guarantee such a value $p_{1,0}^2$ exists, we have to show first, that P_0 is not empty. Because P_0 is a finite intersection of convex intervals, we have to make sure that all intervals are not empty and every intersection between two intervals is not empty. In one dimension, this is sufficient that P_0 is not empty. For $i = 1, \dots, m_2 + 1$,

$$\begin{aligned} P_{1,i} &\neq \emptyset \quad \text{according to requirement} \\ P_{2,i} &\neq \emptyset \quad \text{because of (26)}. \end{aligned} \quad (51)$$

Furthermore its $P_{1,i} \cap P_{2,i} \neq \emptyset$ because of (25) and (26). The missing intersections are nonempty too, for $i, j \in \{1, \dots, m_2 + 1\}$ with $i < j$, as follows:

$$\begin{aligned} P_{1,i} \cap P_{1,j} &\neq \emptyset \quad \text{because of (28) and (29).} \\ P_{1,i} \cap P_{2,j} &\neq \emptyset \quad \text{because of (30) and (31).} \\ P_{2,i} \cap P_{1,j} &\neq \emptyset \quad \text{because of (32) and (33).} \\ P_{2,i} \cap P_{2,j} &\neq \emptyset \quad \text{because of (34) and (35).} \end{aligned} \quad (52)$$

So the set P_0 is not empty. We define the following values:

$$\begin{aligned} p_{1,0}^2 &\in P_0, \\ p_{i,0}^2 &:= \Pi_{1,i} (p_{1,0}^2 - \Sigma_{1,i}) \quad i = 2, \dots, m_2 + 1, \\ p_{i,k}^2 &:= p_{i,0}^2 - g_{i,k}(\tilde{b}_i) \\ &\quad i = 1, \dots, m_2 + 1, \quad k = 1, \dots, n_i, \\ q_i &:= A_i^{-1} \tilde{b}_i \quad i = 1, \dots, m_2 + 1. \end{aligned} \quad (53)$$

Now we show that our choice of pressures and flows is feasible. It is $p_{1,0} \in [p_{1,0}^{\min}, p_{1,0}^{\max}]$ because of $p_{1,0}^2 \in P_{1,1}$. Further because of $p_{i,0}^2 \in P_{1,i}$ for all $i = 2, \dots, m_2 + 1$ it holds

$$\frac{1}{\Pi_{1,i}} p_{i,0}^2 + \Sigma_{1,i} \leq \frac{1}{\Pi_{1,i}} (p_{i,0}^{\max})^2 + \Sigma_{1,i} \quad (54)$$

and

$$\frac{1}{\Pi_{1,i}} p_{i,0}^2 + \Sigma_{1,i} \geq \frac{1}{\Pi_{1,i}} (p_{i,0}^{\min})^2 + \Sigma_{1,i}. \quad (55)$$

Now we replace $p_{i,0}^2$ in its definition with $(p_{i,k}^2 + g_{i,k}(\tilde{b}_i))$ and because of $p_{i,0}^2 \in P_{2,i}$ for all $i = 1, \dots, m_2 + 1$ it follows

$$\begin{aligned} & \frac{1}{\Pi_{1,i}} (p_{i,k}^2 + g_{i,k}(\tilde{b}_i)) + \Sigma_{1,i} \\ & \leq \frac{1}{\Pi_{1,i}} \min_{k=1, \dots, n_i} [(p_{i,k}^{max})^2 + g_{i,k}(\tilde{b}_i)] + \Sigma_{1,i} \end{aligned} \quad (56)$$

and

$$\begin{aligned} & \frac{1}{\Pi_{1,i}} (p_{i,k}^2 + g_{i,k}(\tilde{b}_i)) + \Sigma_{1,i} \\ & \geq \frac{1}{\Pi_{1,i}} \max_{k=1, \dots, n_i} [(p_{i,k}^{min})^2 + g_{i,k}(\tilde{b}_i)] + \Sigma_{1,i}. \end{aligned} \quad (57)$$

So for all $i = 1, \dots, m_2 + 1$ and for all $k = 1, \dots, n_i$ it holds $p_{i,k} \in [p_{i,k}^{min}, p_{i,k}^{max}]$ which is equivalent to

$$p^+ \in [p^{+,min}, p^{+,max}]. \quad (58)$$

The equation for mass conservation follows directly from the definition of q_i ($i = 1, \dots, m_2 + 1$). For the pressure loss we use

$$p_i^2 = \mathbb{1}_{n_i} p_{i,0}^2 - g_i(\tilde{b}_i), \quad (59)$$

And multiply this equation from left with A_i^T and use (36). With this it follows the equation for momentum conservation:

$$(A_i^+)^T (p_i^+)^2 = -\Phi |q_i| q_i. \quad (60)$$

The last equation missing is the compressor property. For a control $j \in \{1, \dots, m_2\}$ we fix a vertex $v_{i,0}$ ($i \in \{1, \dots, m_2 + 1\}$) with $h(e_{u_j}) = v_{i,0}$. We use $p_{i,0}^2 = \Pi_{1,i}(p_{1,0}^2 - \Sigma_{1,i})$ and solve the formula for $p_{1,0}^2$ in the following way:

$$\begin{aligned} p_{1,0}^2 &= \frac{1}{u_{(1,i),1}} \left[\dots \left[\frac{1}{u_{(1,i),m^*}} p_{(1,i),m^*}^2 \right. \right. \\ & \quad \left. \left. + g_{(1,i),m^*-1,f(e_{u_{(1,i),m^*-1}})}(\tilde{b}_{(1,i),m^*-1}) \right] \dots \right] \\ & \quad + g_{(1,i),1,f(e_{u_{(1,i),1}})}(\tilde{b}_{(1,i),1}). \end{aligned} \quad (61)$$

Consider w.l.o.g. that the subgraphs along the path from $v_{1,0}$ to $v_{i,0}$ are numbered chronological, s.t. if $h(e_{u_j}) \in G_i$ then $f(e_{u_j}) \in G_{i-1}$. We use $p_{i-1,0}^2 = \Pi_{1,i-1}(p_{1,0}^2 - \Sigma_{1,i-1})$ to get the same representation for $p_{1,0}^2$ depending on $p_{(1,i-1),n^*,0}^2$, which is equivalent to $p_{(1,i),n^*-1,0}^2$. We equalize these representations depending on $p_{i,0}^2$ and $p_{i-1,0}^2$ and we get

$$\begin{aligned} & \frac{1}{u_{(1,i),m^*}} p_{(1,i),m^*,0}^2 + g_{(1,i),m^*-1,f(e_{u_{(1,i),m^*-1}})}(\tilde{b}_{(1,i),m^*-1}) \\ & = p_{(1,i),n^*-1,0}^2. \end{aligned} \quad (62)$$

We chose $i \in \{1, \dots, m_2 + 1\}$ s.t. u_j is the last control on the path from $v_{1,0}$ to $v_{i,0}$, so it is $u_{(1,i),m^*} = u_j$. Further we have

$$\begin{aligned} p_{(1,i),n^*,0}^2 &= p_{i,0}^2, \\ p_{(1,i),n^*-1,0}^2 &= p_{i-1,0}^2, \\ g_{(1,i),n^*-1,f(e_{u_{(1,i),n^*-1}})}(b_{(1,i),n^*-1}) &= g_{i-1,f(e_{u_j})}(b_{i-1}), \end{aligned} \quad (63)$$

and we know that $p_{i,0}^2$ resp. $p_{i-1,0}^2 - g_{i-1,f(e_{u_j})}(b_{i-1})$ are the pressures at the head resp. the foot of the compressor edge e_{u_j} . So it follows

$$\frac{1}{u_j} p_{h(e_{u_j})}^2 = p_{f(e_{u_j})}^2, \quad (64)$$

which is equivalent to the compressor property. This completes the proof. $\square$

With this theorem we have another characterization of the feasible set $\bar{M}$. The last step in this intersection is the adaption of the results to the spheric-radial decomposition. As mentioned before, the main step is to get a representation of the sets M_i in step (2) of Algorithm 3.

Because we assumed that the graph has only one entry, we define the following set:

$$M := \{b \in \mathbb{R}_{\geq 0}^n \mid (-\mathbb{1}_n^T b, b) \in \bar{M}\}. \quad (65)$$

Consider a sampled point $v_i \in \mathbb{S}^{n-1}$ of step (1) of Algorithm 3 ($i = 1, \dots, N$). We identify the load vector b with the affine linear function:

$$b_i(r) := rLv_i + \mu = r\omega_i + \mu \quad \text{with } \omega_i = Lv_i, \quad (66)$$

where μ is the mean value of the Gaussian distribution and L is such that $LL^T = \Sigma$ for positive definite covariance matrix Σ . Note that the index i here is for the sampled vector v_i , not for the i -th subgraph. Because b_i is defined only for the exit nodes, it must hold $b \geq 0$ which leads to the definition of the regular range:

$$R_{reg,i} := \{r \geq 0 \mid b_i(r) \geq 0\} \supseteq M_i. \quad (67)$$

So the feasible set is

$$M_i = \{r \in R_{reg,i} \mid b_i(r) \text{ fulfills (3), (5) and (7)}\}, \quad (68)$$

and with Theorem 5 it holds

$$M_i = \{r \in R_{reg,i} \mid \tilde{b}_i(r) \text{ fulfills (25) - (34)}\}, \quad (69)$$

where $\tilde{b}_i(r)$ is from (18). Because $\tilde{b}_i$ depends on r now, the function g depends on r , too. More exactly, g is quadratic in r , so $g_k(b_i(r))$ can be represented as follows:

$$r^2 \underbrace{\left[\sum_{j=1}^n \Psi_{k,j} \left(\sum_{\ell=1}^n A_{j,\ell}^{-1} \omega_{i,\ell} \right)^2 \right]}_{\tilde{a}_i}$$

$$\begin{aligned}
& + r \left[\underbrace{2 \sum_{j=1}^n \Psi_{k,j} \left(\sum_{\ell=1}^n A_{j,\ell}^{-1} \omega_{i,\ell} \right) \left(\sum_{\ell=1}^n A_{j,\ell}^{-1} \mu_{\ell} \right)}_{\tilde{b}_i} \right] \\
& + \left[\underbrace{\sum_{j=1}^n \Psi_{k,j} \left(\sum_{\ell=1}^n A_{j,\ell}^{-1} \mu_{\ell} \right)^2}_{\tilde{c}_i} \right].
\end{aligned} \quad (70)$$

Here, Ψ is the product of A and Φ . So the inequations, which characterize M_i , are also quadratic in r because the minimal and maximal pressures in the inequalities only count to $\tilde{c}_i$. Now we can write the feasible sets as follows:

$$M_i = \{r \in R_{reg,i} \mid 0 \leq a_{i,t} r^2 + b_{i,t} r + c_{i,t}\}, \quad (71)$$

where t is an index for the total number of inequalities in M_i . Now the regular range can be cut with the positive intervals of the inequalities and the result is a union of disjoint intervals:

$$M_i = \bigcup_{k=1}^{s_i} [\underline{a}_{i,k}, \bar{a}_{i,k}] \quad s_i \in \mathbb{N}. \quad (72)$$

Here s_i is the number of disjoint intervals in which all inequalities are fulfilled and $\bar{a}_{i,k}$ and $\underline{a}_{i,k}$ are the interval bounds, they depend on the pressure bounds and the controls. So the third step of Algorithm 3 can be executed. With

$$\mathbb{P}(b \in M) \approx \frac{1}{N} \sum_{i=1}^N \sum_{k=1}^{s_i} \mathcal{F}_{\chi}(\bar{a}_{i,k}) - \mathcal{F}_{\chi}(\underline{a}_{i,k}), \quad (73)$$

the probability for a random load vector to be feasible can be computed.

4. Existence of Optimal Solutions

In this section we want to have a look at some optimization problems. We distinguish between problems with constant loads (without uncertainty on the demand) and problems with random loads. The latter leads us to optimization problems with so called probabilistic constraints or chance constraints (see [21]). There are two aims in optimization: Minimize the maximal pressure bounds and minimize the controls. For minimizing the maximal pressure bounds, we define the following set:

$$\bar{P} := \bigcup_{i=0}^n [p_i^{+,min}, \infty), \quad (74)$$

in which we look for them. Because the maximal pressures are in proportion to the cost of pipes, an objective function for a optimization problem could be $c^T p^{+,max}$ for a cost vector $c \in \mathbb{R}_{>0}^{n+1}$. By minimizing the controls, we use the Euclidean norm of the controls as an objective function.

Optimization with Constant Loads. Consider the two optimization problems

$$\min_{p^{+,max} \in \bar{P}} c^T p^{+,max} \quad (75)$$

$$\text{subject to } b^+ (p^{+,max}) \in \widehat{M}$$

and

$$\min_{u \in \mathbb{R}_{\geq 1}^{m_2}} \|u\|_2 \quad (76)$$

$$\text{subject to } b^+(u) \in \widehat{M}.$$

Lemma 7. *There exists a unique solution for (75).*

Proof. We have a look at the side constraint $b^+(p^{+,max}) \in \widehat{M}$. Using Theorem 5, we can write (75) as

$$\min_{p^{+,max} \in \bar{P}} c^T p^{+,max} \quad (77)$$

$$\text{subject to } (25) - (35) \text{ are fulfilled.}$$

For $i \in \{1, \dots, m_2 + 1\}$ we write inequality (25) (and all remaining analogous) in the following way:

$$0 \leq g_{i,k}(\tilde{b}_i) - (p_{i,0}^{min})^2 + (p_{i,k}^{max})^2 \quad \forall k = 1, \dots, n_i \quad (78)$$

So every inequality depends only on one upper pressure bound (but more inequalities may depend on the same bound). Because the inequalities are not strict, the values

$$\bar{p}_k := \min \{p_k^{max} \in [0, \infty) \mid (25) - (35) \text{ are fulfilled}\} \quad (79)$$

exist for all $k = 1, \dots, n + 1$ and we can define a value

$$p_{opt,k}^{+,max} := \max \{\bar{p}_k, p_k^{min}\} \quad (80)$$

which is obviously a solution of (75). Moreover, this solution is unique, because reducing one component of the upper bounds would either hurt an inequality or it would not fulfill $p^{+,max} \geq p^{+,min}$ anymore. $\square$

For the optimization problem (76) we can formulate a necessary and a sufficient condition for optimal solutions.

Lemma 8. *Consider pressure bounds $p^{+,min}, p^{+,max} \in \mathbb{R}^{n+1}$ s.t. $p^{+,min} < p^{+,max}$ and a suitable vector $b^+ \in \mathbb{R}^{n+1}$ (i.e., $\tilde{b}^+$ fulfills (25)–(27)). For all $k = 1, \dots, m_2$ we set $i(k)$ and $j(k)$ such that $f(e_k) \in \mathcal{V}_{i(k)}^+$ and $h(e_k) \in \mathcal{V}_{j(k)}^+$.*

(a) *If $u \in \mathbb{R}^{m_2}$ is a solution of (76), it holds for all $k = 1, \dots, m_2$:*

$$(p_{i(k),0}^{min})^2 - g_{i(k),f(e_k)}(\tilde{b}_{i(k)}) \leq (p_{j(k),0}^{max})^2. \quad (81)$$

(b) *If the inequalities*

$$(p_{i(k),0}^{max})^2 - g_{i(k),f(e_k)}(\tilde{b}_{i(k)}) \geq (p_{j(k),0}^{min})^2 \quad (82)$$

$$\text{and } (p_{i(k),0}^{min})^2 - g_{i(k),f(e_k)}(\tilde{b}_{i(k)}) \leq (p_{j(k),0}^{max})^2$$

are fulfilled for all $k = 1, \dots, m_2$, an optimal solution $u \in \mathbb{R}_{\geq 1}^{m_2}$ of (76) exists.

Note that a vector $b^+ \in \mathbb{R}^{n+1}$ is suitable here if the vector $\tilde{b}^+$ is feasible for the subgraphs, i.e., the inequalities (25)-(27) are fulfilled for $\tilde{b}^+$.

Proof. (a) For this part we use a proof by contradiction. Assume that

$$(p_{i(k),0}^{\min})^2 - g_{i(k),f(e_k)}(\tilde{b}_{i(k)}) > (p_{j(k),0}^{\max})^2. \quad (83)$$

Because $u_k \geq 1$ for all $k = 1, \dots, m_2$, it follows

$$(p_{i(k),0}^{\min})^2 - g_{i(k),f(e_k)}(\tilde{b}_{i(k)}) > \frac{1}{u_k} (p_{j(k),0}^{\max})^2. \quad (84)$$

It is $\Sigma_{k^*,j(k)} = 0$ and $\Pi_{k^*,j(k)} = 1$ because the subgraph with index k^* is equal to the subgraph with index $i(k)$. And because the subgraphs $i(k)$ and $j(k)$ are neighbours (i.e., directly connected), it follows $\Pi_{k^*,j(k)} = u_k$ and

$$\begin{aligned} \Sigma_{k^*,j(k)} &= g_{(k^*,j(k)),1,f(e_{u_{k^*,j(k)},1})}(b_{(k^*,j(k)),1}) \\ &= g_{i(k),f(e_k)}(\tilde{b}_{i(k)}). \end{aligned} \quad (85)$$

Thus we have a contradiction to (28) and u_k cannot be optimal.

(b) As mentioned before, the vector b^+ is such that $\tilde{b}$ is feasible for the subgraphs. So because of the conditions for $k \in \{1, \dots, m_2\}$ there exists a $p_{i(k),0} \in [p_{i(k),0}^{\min}, p_{i(k),0}^{\max}]$ and a $p_{j(k)} \in [p_{j(k)}^{\min}, p_{j(k)}^{\max}]$ so that

$$p_{i(k),0}^2 - g_{i(k),f(e_k)}(\tilde{b}_{i(k)}) \geq p_{j(k),0}^2. \quad (86)$$

With $p_{i(k),0}^2 - g_{i(k),f(e_k)}(\tilde{b}_{i(k)}) = p_{f(e_k)}^2$ and $p_{j(k),0}^2 = p_{h(e_k)}^2$ it follows

$$\left(\frac{p_{h(e_k)}}{p_{f(e_k)}} \right) \geq 1. \quad (87)$$

Therefore we can find a $u \in \mathbb{R}_{\geq 1}^{m_2}$ that the inequalities (28)-(35) are fulfilled. Now we can use the *Weierstraß theorem* and get the existence of an optimal solution. $\square$

Note that the sufficient condition in Lemma 8 is comparatively strong, there may exist optimal solutions with less strong conditions, e.g., the following.

Corollary 9. *If the graph has only one compressor edge, i.e., $|\mathcal{E}_C| = 1$, then it holds: If $u \in \mathbb{R}_{\geq 1}$ is an optimal solution of (76), u is unique.*

This conclusion follows directly from the strict monotonicity of the objective function.

Optimization with Random Loads. In this section we consider the load vector to be random as we mentioned in Section 2.2. In the optimization problems, we want to make sure that the feasibility of a random load vector is guaranteed for a probability $\alpha \in (0, 1)$. This leads us to optimization problems with so called chance or probabilistic constraints.

In general problems with probabilistic constraints have the form

$$\begin{aligned} \min \quad & f(x, \xi) \\ \text{subject to} \quad & \mathbb{P}(h(x, \xi) \leq 0) \geq \alpha, \end{aligned} \quad (88)$$

with an objective function f and a vector of uncertainty ξ . There exist many works about probabilistic constraints resp. stochastic programming, e.g., [7] gives an excellent overview about chance constraints in theory and application. In our case we consider the following problem:

$$\min_{p^{+,max} \in \overline{P}} c^T p^{+,max} \quad (89)$$

$$\text{subject to} \quad \mathbb{P}(b(p^{+,max}) \in M) \geq \alpha.$$

To handle this problem resp. the probabilistic constraint, the spheric-radial decomposition gives us an explicit representation of the constraint for a Gaussian vector $b \sim \mathcal{N}(\mu, \Sigma)$ with mean value μ and positive definite covariance Σ :

$$\begin{aligned} \mathbb{P}(b(p^{+,max}) \in M) \\ = \int_{\mathbb{S}^{n-1}} \mu_\chi \{r \geq 0 \mid rLv + \mu \in M\} d\mu_\eta(v). \end{aligned} \quad (90)$$

Further, Algorithm 3 gives a way to approximate this integral in an efficient way. In Section 3 we showed a way to characterize the feasible set M and how to adapt this characterization to the spheric-radial decomposition. This changes our problem with probabilistic constraints (89) to

$$\begin{aligned} \min_{p^{+,max} \in \overline{P}} c^T p^{+,max} \\ \text{subject to} \quad \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^{s_i} \mathcal{F}_\chi(\bar{a}_{i,t}) - \mathcal{F}_\chi(\underline{a}_{i,t}) \geq \alpha. \end{aligned} \quad (91)$$

Of course the interval bounds depend on the pressure bounds and the control. For this problem we formulate the next lemma.

Lemma 10. *Let a sampling $\{v_1, \dots, v_N\}$ (from step (1) of Algorithm 3) and $\alpha \in (0, 1)$ be given. For a suitable $\mu \in \mathbb{R}^n$ problem (91) has at least one solution $p^{+,max} \in \overline{P}$.*

Proof. For the proof, we fix a $v \in \{v_1, \dots, v_N\}$ from the given sampling. From (25) and (26) it follows for all $k = 1, \dots, n_i$ with $i \in \{1, \dots, m_2 + 1\}$,

$$\begin{aligned} 0 &\leq g_{i,k}(\tilde{b}_i(r)) + (p_{i,k}^{\max})^2 - (p_{i,0}^{\min})^2, \\ 0 &\leq -g_{i,k}(\tilde{b}_i(r)) + (p_{i,0}^{\max})^2 - (p_{i,k}^{\min})^2. \end{aligned} \quad (92)$$

With a random vector $b \sim \mathcal{N}(\mu, \Sigma)$ $g_{i,k}(\tilde{b}_i(r))$ are quadratic functions in r and the upper pressure bounds only influence the constant part with a positive sign:

$$\begin{aligned} 0 &\leq a_{i,k}r^2 + b_{i,k}r + c_{i,k} + (p_{i,k}^{\max})^2 - (p_{i,0}^{\min})^2, \\ 0 &\leq -a_{i,k}r^2 - b_{i,k}r - c_{i,k} + (p_{i,0}^{\max})^2 - (p_{i,k}^{\min})^2. \end{aligned} \quad (93)$$

It follows that a rise in $p_{i,k}^{max}$ for all $k = 0, \dots, n_i$ enlarges the intervals, in which the inequalities hold. The same holds for all inequalities in Theorem 5. The cumulative distribution function of the Chi-Square-Distribution is strictly monotonic increasing and convergent to 1. Thus upper pressure bounds $p_{i,k}^{max}$ (for $i = 1, \dots, n$) can be found so that

$$\frac{1}{N} \sum_{j=1}^N \sum_{t=1}^{s_j} \mathcal{F}_\chi(\bar{a}_{j,t}) - \mathcal{F}_\chi(\underline{a}_{j,t}) \quad (94)$$

is higher than $\alpha \in (0, 1)$. The index N is for all sampled points on the sphere and the index s_j is for the union of intervals in which all inequalities hold. The interval boundaries $\bar{a}_{j,t}$ and $\underline{a}_{j,t}$ depend continuously on the pressure bounds and because the inequalities are not strict, p^{max} can be found s.t. the side constraint is fulfilled with equality. So there exists at least one solution of (91). $\square$

5. Numerical Results

In this section we show a few results of implementation. At first we show the idea of the spheric-radial decomposition by using an easy example. Next we show a easy example with one compressor edge and at last we use real data of the Greek gas network. The focus of the implementation is on the theorems proofed in Section 4.

Example 1. Our first example is an easy graph without inner control (see Figure 6).

Assume that node 1 and 2 are gas consumers with mean demand $\mu = [0.5, 0.5]^T$. The pressure bounds are given by $p^{+,min} = [2, 1, 1]^T$ and $p^{+,max} = [3, 2, 2]^T$ and $\Phi = \mathbb{E}_2$. Then the incidence matrix is

$$A^+ = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{resp. } A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \quad (95)$$

Consider the inequalities of Theorem 5 to characterize the feasible set. Because the network has no compressor edges the system of inequalities only contains the information (25)-(27), the other inequalities do not occur. The function g is given by

$$g(b) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \left| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right| \quad (96)$$

$$\circ \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right) = \begin{pmatrix} b_1^2 + 2b_1b_2 + b_2^2 \\ b_1^2 + 2b_1b_2 + 2b_2^2 \end{pmatrix}.$$

By inserting the pressure bounds, one can see that inequality (25) is always fulfilled and inequality (27) implies

$$b_2 \leq \sqrt{3}. \quad (97)$$

With this, inequality (26) follows for $0 \leq b_2 \leq \sqrt{3}$:

$$0 \leq b_1 \leq -b_2 + \sqrt{8 - b_2^2}. \quad (98)$$

So the feasible set M is the following (see Figure 7):

$$M = \left\{ b \in \mathbb{R}_{\geq 0}^2 \mid b_2 \leq \sqrt{3} \wedge b_1 \leq -b_2 + \sqrt{8 - b_2^2} \right\}. \quad (99)$$

Now consider the optimization problem:

$$\begin{aligned} \min_{p^{+,max} \in \bar{P}} \quad & \mathbb{1}_3^T p^{+,max} \\ \text{subject to} \quad & b \in M(p^{+,max}), \end{aligned} \quad (100)$$

with $b = \mu = [0.5, 0.5]^T$. Lemma 7 guarantees the existence of a unique solution of this problem. We use the *fmincon*-function in *MATLAB*®, which minimizes a constrained non-linear multivariable function. The *fmincon* default setting uses a interior-point method. For more information we refer to the official *MathWorks*® homepage (<https://www.mathworks.com/help/optim/ug/fmincon.html>). The solver returns the following result:

$$p_{opt}^{+,max} = [2, \sqrt{3}, \sqrt{2.75}]^T. \quad (101)$$

One can easily see that this is the correct solution. The gas demand at node 2 is 0.5, which is equal to the flow q_2 , so using the equation of momentum implies a difference in the quadratic pressures between node 1 and node 2 of 0.25. The flow through edge 1 is given by the two exit nodes and it is $q_1 = 1$. This implies a difference in the quadratic pressures between node 0 and node 1 of 1. So the optimal solution $p_{opt}^{+,max}$ directly follows.

Next we use the spheric-radial decomposition, especially Algorithm 3, to calculate the probability $\mathbb{P}(b \in M)$ for a random vector $b \sim \mathcal{N}(\mathbb{E}_2, [0.5, 0.5]^T)$. Therefore we view the results of 8 tests with 1000 sampled points in each one (Table 1).

The probability in all eight tests is nearly the same. The mean value is 0.3479 and the variance is $2.7723 \cdot 10^{-6}$, which is very small. This shows that the implementation is working nearly exact. The efficiency of the implementation of course is not perfect, but that is not part of this work.

Last we want to solve the following optimization problem:

$$\begin{aligned} \min_{p^{+,max} \in \bar{P}} \quad & \mathbb{1}_3^T p^{+,max} \\ \text{subject to} \quad & \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^{s_i} \mathcal{F}_\chi(\bar{a}_{i,t}) - \mathcal{F}_\chi(\underline{a}_{i,t}) \geq \alpha. \end{aligned} \quad (102)$$

We use again the *fmincon*-function in *MATLAB*® to solve the problem eight times, with 1000 sampled points in every case. Because Lemma 8 needs a suitable $\mu \in \mathbb{R}_{\geq 0}^2$, we assume $b \sim \mathcal{N}(0.1 \cdot E_2, [1.5, 1.5])$. We set $\alpha := 0.9$ and the results are shown in Table 2.

Example 2. Now we add a compressor edge to the graph of Example 1 (see Figure 8).

Assume that node 1 and 3 are consumers, node 2 is a inner node. The mean demand $\mu = [0.5, 0, 0.5]^T$. The pressure

FIGURE 7: Feasible set of Example 1.

TABLE 1: Test results of easy example.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
$\mathbb{P}(b \in M)$	0.3486	0.3481	0.3450	0.3501	0.3484	0.3499	0.3459	0.3475

TABLE 2: Test results of Example 1.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
$p_{opt,0}^{max}$	5.4514	5.4714	5.4758	5.4028	5.4767	5.3921	5.4888	5.5167
$p_{opt,1}^{max}$	2.9802	3.0126	3.0225	2.9725	3.0300	2.9485	3.0336	3.0029
$p_{opt,2}^{max}$	1.0084	1.0004	1.0776	1.0001	1.1221	1.1390	1.0285	1.0558

FIGURE 8: Graph of Example 2.

FIGURE 9: Separated graph of Example 2.

bounds are given by $p^{+,min} = [2, 1, 1, 1]^T$ and $p^{+,max} = [3, 2, 2, 2]^T$ and $\phi_{e_1} = 1$ and $\phi_{e_3} = 1$. The incidence matrix is given by

$$A^+ = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \text{ resp. } A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \quad (103)$$

Before we have a look at the inequalities of Theorem 5, we have to remove the compressor edge from the graph. The flows can be computed by using the equation of mass conservation and the new load vector is then

$$\tilde{b} = \begin{pmatrix} b_1 + q_2 \\ b_2 - q_2 \\ b_3 \end{pmatrix} \quad \text{with} \quad (104)$$

$$q = \begin{pmatrix} b_1 + b_2 + b_3 \\ b_2 + b_3 \\ b_3 \end{pmatrix}.$$

We assume the control to be switched off, so it holds $u = 1$. With this, the control edge e_2 is treated as a flow edge without friction. The resulting graph is shown in Figure 9: from the inequalities (25)–(35) it follows

$$\tilde{b}_3 \leq \sqrt{3} \text{ and } \tilde{b}_1 \leq \sqrt{8 - \tilde{b}_3^2}, \quad (105)$$

FIGURE 10: Feasible set of the Example 2.

TABLE 3: Test results of control example.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
$\mathbb{P}(b \in M)$	0.1531	0.1500	0.1540	0.1536	0.1530	0.1550	0.1567	0.1528

so the feasible set is (see Figure 10):

$$\begin{aligned} \bar{M} = \left\{ \tilde{b}^+ \in \mathbb{R}^{n+1} \mid \mathbb{1}_4^T \tilde{b}^+ = 0 \wedge 0 \leq \tilde{b}_3 \leq \sqrt{3} \wedge \tilde{b}_2 = 0 \right. \\ \left. \wedge 0 \leq \tilde{b}_1 \leq \sqrt{8 - \tilde{b}_3^2} \right\}. \end{aligned} \quad (106)$$

This implies

$$\begin{aligned} M = \left\{ b \in \mathbb{R}_{\geq 0}^3 \mid 0 \leq b_3 \leq \sqrt{3} \wedge b_2 = 0 \wedge 0 \leq b_1 \leq -b_3 \right. \\ \left. + \sqrt{8 - b_3^2} \right\}. \end{aligned} \quad (107)$$

Solving the problem

$$\min_{p^{+,max} \in \bar{P}} \mathbb{1}_4^T p^{+,max} \quad (108)$$

subject to $\tilde{b} \in \bar{M}$

for $b = \mu = [0.5, 0, 0.5]^T$ leads us to the optimal solution

$$p_{opt}^{+,max} = [2, \sqrt{3}, \sqrt{3}, \sqrt{2.75}]^T. \quad (109)$$

The argumentation for the optimality of this vector is exactly like before in the easy example. Because we have a compressor edge in this graph, we can minimize the control. Therefore we fix the pressure bounds as we did at the beginning of this example and we set $b = [1, 0, 1.5]^T$. So inequality (31) is not fulfilled anymore for $u = 1$. We use again the *fmincon*-solver in *MATLAB*® to solve the following problem:

$$\begin{aligned} \min_{u \geq 1} \|u\|_2 \\ \text{subject to } b \in M(u). \end{aligned} \quad (110)$$

The solver returns the optimal solution $u_{opt} = 1.1818$.

Here, one can see that the sufficient condition in Lemma 8 is really strong. It's not fulfilled with the given values but there exists a solution anyway. Of course, the necessary condition is fulfilled.

Now, we use again Algorithm 3 to compute the probability $\mathbb{P}(b \in M)$ for a random load vector $b \sim \mathcal{N}(\mathbb{E}_3, [0.5, 0.5, 0.5]^T)$. Table 3 shows us the results.

Again we can see, that the probability is nearly the same in all eight cases. The mean probability is 0.1535 and the variance is $3.2369 \cdot 10^{-6}$. For the optimization, we reduce the covariance matrix and assume $b \sim \mathcal{N}(0.01 \cdot \mathbb{E}_3, [0.5, 0.5, 0.5]^T)$. For a probability $\alpha = 0.9$ Table 4 states the results.

Example 3. In this example, we use a part of a real network. The focus here is on realistic values for our model. We get these values from the GasLib (see [22]), a collection of technical gas network data.

“The goal of GasLib is to promote research on gas networks by providing a set of large and realistic benchmark instances.” (<http://gaslib.zib.de/>)

We use a part of the Greek gas network (GasLib-134) (see Figure 11).

For using real values, we first need to know the constant ϕ^e from (5):

$$\phi^e = (R_s T)^2 \frac{\lambda}{D} \frac{L}{a^2}. \quad (111)$$

The value R_s is the specific gas constant, which can be calculated by $R_s = R/M$ with the ideal gas constant $R = 8.31451 \text{ kg} \cdot \text{m}^2/\text{s}^2 \cdot \text{mol} \cdot \text{K}$ (see [23]) and the molar mass $M = 16.62 \text{ kg/kmol}$ (from real data) of the used natural gas. Also the temperature $T = 289.15 \text{ K}$ is given from real data. For the (constant) friction factor λ we use the law of Chen (see [24]):

TABLE 4: Test results of control example.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
$p_{opt,0}^{max}$	3.9013	3.9223	3.9039	3.8901	3.8879	3.9039	3.8901	3.9098
$p_{opt,1}^{max}$	1.6568	1.6565	1.6365	1.6497	1.6496	1.6365	1.6497	1.6492
$p_{opt,2}^{max}$	1.6568	1.6565	1.6365	1.6497	1.6496	1.6365	1.6497	1.6498
$p_{opt,3}^{max}$	1.0000	1.0000	1.0000	1.0000	1.0002	1.0000	1.0000	1.0000

TABLE 5: Values based on real data.

variable	R_s	T	λ	D	L	a	ϕ
value	500.27	289.15	0.0093	914.4	17.8693	438.62	$1.98 \cdot 10^7$
unit	$m^2/s^2 \cdot K$	K	–	mm	km	m/s	m^2/s^2

FIGURE 11: Greek gas network from GasLib [22], red edges are used in our example.

$$\frac{1}{\sqrt{\lambda}} = -2 \log_{10} \left(\frac{k/D}{3.7065} - \frac{5.0425}{Re} \log_{10} \left[\frac{(k/D)^{1.1098}}{2.8257} + \frac{5.8506}{Re^{0.8981}} \right] \right). \quad (112)$$

Diameter $D = 914.4mm$ and roughness value $k = 8 \cdot 10^{-6}m$ for the first pipe (see Figure 11) are known from real data. The Reynolds-Number Re can be calculated by $Re = (\nu D \rho)/\eta$ with the dynamic viscosity $\eta = 1 \cdot 10^{-5}Pa \cdot s$ (see [1]), the mean velocity $\nu = 95.48m/s$ (calculated with the mean volume flow rate which is given by the real data) of the gas and the density $\rho = 0.7433kg/m^3$ (from real data). So we get $Re = 6489590.74 \approx 6.49 \cdot 10^6$ and with this the pipe friction coefficient after Chen is $\lambda = 0.0093$.

The length of the first pipe is given by $L = 17.8693km$ from the real data. Last value missing is the sound speed in natural gas, which we assume to be ideal. The sound speed

a depends on the modulus of compressibility (see [9]). For ideal gases, the modulus of compressibility only depends on the adiabatic exponent and the pressure. Using the ideal gas equation simplifies the equation for the sound speed to

$$a = \sqrt{\kappa \frac{RT}{M}}. \quad (113)$$

The adiabatic exponent for methane, which is the main part of natural gas (> 90%) is about 1.33 (see [25]). Together, all values are shown in Table 5 (example for the pipe next to the source in Figure 11).

Now we want to optimize the inner control in the network for a fix load vector. Therefore we choose two nominations: 2011-11-01 and 2016-02-17. In both cases, the *fmincon*-function in MATLAB® returns

$$u_{opt} = 1 \quad (114)$$

as a local minimum, which satisfies the constraint $b \in M$. This means, for these nominations, a compressor station is not needed so it is switched off. With this, as a last calculation in this paper, we calculate the probability for the given nominations to be feasible. We use the values from the nominations as mean value μ and set $b \sim \mathcal{N}(\mu \cdot 10^{-3} \mathcal{E}, \mu)$. The results are shown in Tables 6 and 7.

Again one can see that the results are quite similar. The mean probability in Table 6 is 0.9860 and the mean probability in Table 7 is 0.9760. Both values are nearly 1 so there must happen something really unforeseeable that a Greek does not get his demanded gas.

6. Conclusion

In this paper, we have used an existing stationary model for gas transport based on the isothermal Euler equations. We have extended this model by compressor stations and presented a characterization of the set of feasible loads. Especially Theorem 5 contains the central idea to get an access to the optimization in Section 4 and the spheric-radial decomposition is our main tool to handle the uncertainty in the loads. We were able to show some results for optimization with probabilistic constraints in gas transport. In a next step these results should be generalized.

TABLE 6: Test results for nomination 2011-11-01.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
$\mathbb{P}(b \in M)$	0.9849	0.9849	0.9878	0.9885	0.9863	0.9845	0.9854	0.9853

TABLE 7: Test results for nomination 2016-02-17.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
$\mathbb{P}(b \in M)$	0.9744	0.9756	0.9788	0.9788	0.9754	0.9775	0.9734	0.9739

With the characterization of the feasible set, we have shown that the feasibility of a load depends significantly on the pressure bounds. These pressure bounds play a main role in our optimization problems. The other main parts are the inner controls modeled by compressor edges, which should lead in optimal control problems with probabilistic constraints. Here we have built a base for further works with this topic.

The idea of separating the graph and removing the compressor edges can be used to extend the model with other components, like, e.g., control valves and resistors. This would change the inequalities in Theorem 5, but the approach works also for these modifications. Further, [8] showed a way to deal with a network which is a cycle. Combining both methods, we get an access to model gas networks in high detail.

Finally the expansion to a dynamic model might be also possible. The idea is to use a space-mapping approach, because one can use a lot of the analysis we did here. Therefore upcoming papers should deal with these problems and present results of the turnpike-theory, which describes a relation between stationary and instationary solutions. In fact, turnpike-theory implies that the optimal dynamic state is close to the optimal static state in the interior of the time intervals for sufficiently long time horizons. A.J. Zaslavski explains the turnpike-phenomenon in [26]. This book also gives an overview about the theory.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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