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Research Article

An Optimal Identification of the Input-Output Disturbances in Linear Dynamic Systems by the Use of the Exact Observation of the State

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A new methodology for the identification of the values of unknown disturbance signals acting in the input and output measurements of the dynamic linear system is presented. For the solution of this problem, the new idea of the use of two different state observers, which are coworking simultaneously in parallel, was elaborated. Special integral type observers are operating on the same finite time window of the width T and both can reconstruct the exact value of the vector state x(T) on the basis of input-output measurements in this interval [0, T]. If in the input-output signals the disturbances are absent (measurements of I/O signals are perfect) then both observers (although they are different) reconstruct the exact and the same state x(T). However, if in the measurement signals the disturbances are present then these observers will reconstruct the different values of the final states x1(T)=x(T)+e1(T) and x2(T)=x(T)+e2(T). It is because they have different norms and hence they generate different errors in both estimated states. Because of the disturbances, the real state x(T) is unknown, but it is easy to calculate the state difference: x1(T)-x2(T)=e1(T)-e2(T). It occurs that, based on this difference, the values of all the disturbances acting during the control process can be identified. In the paper, the theory of the exact state observation and application of such observers in online mode is recalled. The new methodology for disturbances identification is presented.

1. Introduction

In the classic control theory and its applications, a common practice in the estimation of inaccessible for measurement state vector of a linear observable system is the use of Luenberger type asymptotic observers. D.G. Luenberger in [1] proposed the pole placement technique for calculation of the observer gain matrix. The structure of such an observer was derived directly from the differential form of the Kalman Filter, KF [2, 3]. Calculation of the optimal gain matrix in KF was based on the known stochastic properties of the disturbances and the least-squares estimation error approach. In both types of these estimators, their structures are given by ordinary linear differential equations. Hence under assumption that real initial state is unknown the solution of the estimation problem by the use of these observers could give only a state estimate, which tends to the real state asymptotically and reaches them theoretically in infinity. Calculation of the current estimation error is impossible, because the real state is unknown.

Nowadays in many real-time control applications and fault detection, the finishing of the observation tasks in assumed and possibly short time T is an important requirement. The asymptotic state estimators may not be sufficient for this purpose. The power of the modern microcomputers makes the design of the other online observation algorithms possible. They reconstruct the value of the current state vector **exactly** in finite time T. The calculations are based on the finite time history of measurement samples of input and output. What is more, the width $T > 0$ of the measurement window can be almost freely chosen.

The general theory of the exact reconstruction of the finite dimension state for linear systems in Hilbert spaces as well as the rules for designing of the exact state observers with minimal norm was formulated and presented by W. Byrski and S. Fuksa in 1984 [4, 5]. This theory originates from the definition of the state observability and formulates the extension of the method presented in [6] with the use of functional analysis technique. The authors of [4, 5] proposed a deterministic approach to disturbances characterization and to the exact and optimal state observation for which the relations were formulated generally in function Hilbert spaces U, Y. Such type of the observer must have the structure of two linear continuous functionals. It is because based on two fragments of two continuous functions u and y, which are given on finite time interval T, the observer should calculate the real unknown vectors x(0) or x(T), where $x \in R^n$. On the other hand from the Riesz Representation Theorem it follows that every linear continuous functional in Hilbert space can be expressed as the inner product. Hence the structure of the observer must have a form of two inner products: one product of continuous output function $y \in Y$ and special observation function $G_1(\tau) \in Y$ and the second one with the input function $u \in U$ and special observation function $G_2(\tau) \in U$. After the first observation interval [0,T] the observer reconstructs x(T) and next can also reconstruct the exact values of the state x(t) for $\forall t>T$, based on continuous moving window (in online mode). Choosing different input-output Hilbert spaces one can obtain different formulas for the finite state exact observers.

In engineering sciences, commonly used is the function space $L^2[0.T]$. This space is defined as a space of functions for which the second power is integrable in Lebesgue sense, on the interval [0, T]. The existence of such integral enables definition of the norm of the function as the square root of this integral. In many cases, it can represent the square root of the signal's energy. The space $L^2[0.T]$ belongs to class of Hilbert spaces; hence the inner product of its two elements has the form of the integral operator.

The studies on the exact state observation were undertaken by various authors, although they may be considered as particular cases of the general theory of the exact state observation [4]. In 1966 J.D. Gilchrist [7] proposed mixed version of the exact state observation with the use of discrete measurement of the output signal and continuous measurements of the control. Similar approach was presented in [8–11]. In 1992 A. Medvedev and H.T. Toivonen in [12, 13] continued the study of such mixed types of state observers and called them "finite memory deadbeat observers." Some other version of the finite memory deadbeat observer was also presented in [14]. However, all of the above observer's versions were based on standard least squares approach and are only subcases of the general form of the exact and optimal state observer.

As it was stated before, the exactness in reconstruction of the state is possible by the use of the exact state observers but only under assumption that no input-output noise or disturbances occur (i.e., in the case of perfect input-output measurements). Hence, in practical case of noisy measurements the use of the exact observers gives also a reconstruction error. In the most popular version of the exact observers it was assumed that the input function (control) is perfectly known and for state calculation there is no need for its extra measurement. The last assumption however, in practical application, is not always proper because, e.g., the actuator and the control valve produce the system input signal and it may differ from the control signal u, which is generated by the computer (the number of impulses). From the general theory of the exact and optimal observation one can obtain the observer with minimal norm. It guarantees that, based on the perfect input-output measurements, the state x will be reconstructed exactly and for the measurements with bounded disturbances on y and u (disturbances with bounded norm, from unit balls) the norm of the state reconstruction error will be minimal.

In publications [4, 5] it was assumed that spaces Y and U are chosen as $L^2[0,T]$; hence the inner product is represented by the integral operator. Therefore the name "integral observers" was also frequently used to underline its contrary type to differential structure of the Kalman Filter (LQF) or Luenberger observer [15–17]. Particularly important theorem of a closed loop stability with the use of the integral observer and linear quadratic regulator (LQR) was given in [4]. It was also proven that integral observer and LQR controller in closed loop create the dynamic system of the order n, and not 2n like in linear quadratic Gaussian system (LQG=LQF+LQR).

The extended results of the online exact observation and application were presented in [15, 16, 18]. For instance in the paper [18], the integral observers with Expanding Window as well as Moving Window of Observation (sliding window) and their differential versions were given. In [16] the application of Moving Window Observer (MWO) and LQR to stabilization of distillation column was presented. In [19, 20] the authors presented the use of the double window exact state observer for detection and isolation of abrupt faults in system parameters.

The most important properties and features of the integral observers used for the exact reconstruction of the continuous state are

(i) problem was formulated for Multi-Input Multi-Output (MIMO) continuous linear time invariant (LTI) systems given by the standard state matrix equation model,

(ii) the state observer has integral description in space $L^2[0,T]$ and can be used online (in interval [t-T, t]),

(iii) the possible existence of the disturbance in both the control u(t) and output y(t) signals is assumed,

(iv) the optimal formulas of the observer are obtained by the minimization of its norm which is the function of assumed weighting coefficients,

(v) the independence of the state observation from the initial conditions of the real state (unknown) occurs,

(vi) fixed finite observation time interval [0,T] is used.

A brief explanation regarding the first point may be useful. The theory of the exact state observation in function space $L^2[0, T]$ can use the idea of continuous functions and all the mathematical proofs as well as the final results relate to continuous functions. However, of course for the real applications the term continuous measurements of such functions means that for computer calculations it is enough to have standard discrete measurements, although with frequency, according to Nyquist-Shannon sampling theorem. This establishes a sufficient condition for a sample rate that permits a discrete sequence of samples to capture all the information from a continuous-time signal of finite bandwidth. In that case numerical calculations of integrals, e.g., by Simpson's rule, will guarantee accuracy which will correspond to continuous version.

In this paper the quite new idea of the exact state observers application will be presented. Using two different exact state observers (with different norms) working simultaneously in parallel structure on the same time interval, it is possible to calculate the unknown values of some disturbances, which affect the input-output measurements. The calculation is possible either in the batch mode or in online observation version.

If in the input-output measurements the disturbances are absent (measurements of I/O signals are perfect), then two different observers reconstruct the exact and the same state x(T). However, if in the measurement signals the disturbances are present ($y+z_1$, $u+z_2$) then the first observer reconstructs the value $x_1(T)$ and the second $x_2(T)$. This is because the observers have different norms and in the case of disturbances they generate various errors $e_1(T)$ and $e_2(T)$; i.e., they generate the estimates $x_1(T)=x(T)+e_1(T)$ and $x_2(T)=x(T)+e_2(T)$. The real undisturbed state x(T) is unknown, but it is easy to find the states difference $\varepsilon(T) = x_1(T) - x_2(T) = e_1(T) - e_2(T)$. It occurs that, based on this difference, the values of all the disturbances acting during the control process can be identified. In the next sections the first observer will be marked as the observer (a) and the second observer as the observer (b).

Such identification of disturbances cannot be performed using classical asymptotic state estimators like Kalman Filter, due to the unknown real value (even theoretical) of the state in both estimators. Such an approach, by the use of classical estimators (e.g., bank of Kalman Filters) for the disturbance isolation, was tested however in [21–23] with the use of "fault signatures" table. Other works [24–26] use disturbance distribution matrices and apply observers for diagnosis [27, 28].

In Section 2 we will start with a reminder of the theory of the exact state observation and the application of such observers in online mode.

2. The General and Optimal Form of the Exact Integral Observer in $L^2[0,T]$ Space

2.1. The Existence Conditions of the Exact Observer. Let a linear state observable MIMO system be given

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x(t)\in R^n$, $u(t)\in R^r$, and $y(t)\in R^m$ for $\forall t \geq 0$, m<n. Matrices A, B, C are of compatible dimensions.

Assume that we perfectly measure the control u and the output y on the interval [0, T], where T is the fixed observation time.

Our purpose is to determine the state x(T). We assume the following.

The state space $X = R^n$ (for state vectors), the output space $Y=(L^2(0,T))^m$ (for output functions $y\in Y$), and the control space $U=(L^2(0,T))^r$ (for input functions, $u\in U$). The output of the system (1) has the well-known form

$$y(t) = Ce^{-A(T-t)}x(T) - C\int_t^T e^{A(t-s)}Bu(s)\,ds \tag{2}$$

The general exact state observer should have the form of two inner products in $L^2[0,T]$:

$$x(T) = \int_0^T G_1(T,\tau)\,y(\tau)\,d\tau + \int_0^T G_2(T,\tau)\,u(\tau)\,d\tau \tag{3}$$

where the dimensions of matrices $G_1(T,\tau)$ and $G_2(T,\tau)$ are (n×m) and (n×r), respectively. The elements of these matrices are functions of time $\tau \in [0, T]$. They are also functions of assumed observation time T. However in the sequel we will omit the first argument T in writing, $G_1(\tau)$, $G_2(\tau)$.

For these assumptions, the general conditions for the observation matrices G_1, G_2 should be determined, in such a manner that formula (3) could represent an accurate state observer. To this end, one must substitute (2) to (3). We obtain

$$\begin{aligned} x(T) = {} & \int_0^T G_1(T,\tau)\,Ce^{-A(T-\tau)}d\tau \cdot x(T) \\ & - \int_0^T G_1(T,\tau)\,C\left[\int_\tau^T e^{A(\tau-s)}Bu(s)\,ds\right]d\tau \\ & + \int_0^T G_2(T,\tau)\,u(\tau)\,d\tau \end{aligned} \tag{4}$$

or after changing the order of integration in internal integral

$$\begin{aligned} x(T) = {} & \int_0^T G_1(T,\tau)\,Ce^{-A(T-\tau)}d\tau \cdot x(T) \\ & - \int_0^T \left[\int_0^s G_1(T,\tau)\,Ce^{A(\tau-s)}Bd\tau\right]u(s)\,ds \\ & + \int_0^T G_2(T,\tau)\,u(\tau)\,d\tau \end{aligned} \tag{5}$$

The left hand side of the above equation is equal to right hand side if and only if matrices G_1 and G_2 fulfill conditions

$$\int_0^T G_1(\tau)\,Ce^{-A(T-\tau)}d\tau = I, \tag{6}$$

where I is n×n identity matrix and

$$G_2(\tau) = \int_0^\tau G_1(s)\,Ce^{-A(\tau-s)}Bds. \tag{7}$$

Equation (6) should be treated as the constraint for all possible observation matrices $G_1(\tau)$ in (3). For the chosen matrix G_1, the matrix G_2 should fulfill the second constraint (7). For the observable linear system (1) there is infinite number of G_1, G_2 matrix pairs which fulfill (6) and (7) and are matrices for exact observation in (3). Therefore, one can additionally assume some quality index and find the observer, which will fulfill the minimum of this index. A very reasonable quality index of observation is the observer's norm.

2.2. Interpretation of the Observer Norm. Any exact state observer perfectly reconstructs the state of the system (1) regardless of the initial or final conditions, if no disturbances occur in the input-output measurements. Otherwise, any exact state observer will reconstruct the state with some observation error. If the observer norm will be minimal, the norm of observation error will be also minimal (under some assumptions). Let us assume that in measurements y and u additive bounded norm disturbances occur, $z_1 \in L^2[0,T]$, $z_2 \in L^2[0,T]$, $\|z_1\| \le 1$, and $\|z_2\| \le 1$.

Then the state estimate is given by

$$\hat{x}(T) = \int_0^T G_1^o (y + z_1)\, d\tau + \int_0^T G_2^o (u + z_2)\, d\tau \tag{8}$$

Hence, the vector of the state reconstruction errors will have a value:

$$\varepsilon(T) = \int_0^T G_1^o(\tau) z_1(\tau)\, d\tau + \int_0^T G_2^o(\tau) z_2(\tau)\, d\tau. \tag{9}$$

We can estimate the norm of the error $\|\varepsilon\|$ in the space $L^2[0,T]$ assuming that disturbances are bounded and normalized to unit balls in $L^2[0,T]$, $\|z_1\| \le 1$, and $\|z_2\| \le 1$. Inner product is denoted by $\langle \bullet \mid \bullet \rangle$.

$$\begin{aligned} \max_{(z_1,z_2)} \|\varepsilon\|^2_{R^n} &= \max_{(z_1,z_2)} \left\| \langle G_1 \mid z_1 \rangle + \langle G_2 \mid z_2 \rangle \right\|^2_{R^n} \\ &\le 2\left[\|G_1\|^2_Y + \|G_2\|^2_U\right] = 2\left\|(G_1, G_2)\right\|^2_{Y\times U} \\ &= 2J \end{aligned} \tag{10}$$

Such assumed observer's norm J estimates maximal observation error in most pessimistic scenario. This min-max approach gives interpretation of the optimization task.

$$\min_{(G_1,G_2)} J = J^o \tag{11}$$

Obviously, an observer with a minimum norm still exactly reconstructs the state in the case of perfect and undisturbed input-output measurements.

2.3. The Exact State Observer with Minimal Norm. From continuity and linearity in (6) and (7), it follows that the set of all observers (pairs of matrices G_1, G_2) is closed, linear manifold in the space $Y^n \times U^n$

$$Y^n \times U^n = \left[L^2[0,T]\right]^{m\times n} \times \left[L^2[0,T]\right]^{r\times n}. \tag{12}$$

In this space one can introduce a seminorm of the observer like in (13):

$$\begin{aligned} &\left\|(G_1, G_2)\right\|^2 \\ &= \int_0^T \left[\sum_{i=1}^n \alpha_i \sum_{j=1}^m \left(g_1^{ij}(\tau)\right)^2 + \sum_{i=1}^n \beta_i \sum_{j=1}^r \left(g_2^{ij}(\tau)\right)^2\right] d\tau \\ &= J \end{aligned} \tag{13}$$

where $g_1^{ij}(\tau), g_2^{ij}(\tau)$ are i-th row and j-th column elements of matrices G_1,G_2 and α_i, β_i are weighting coefficients for the whole i-th rows of corresponding matrices.

For simplification (without loss of generality) we will assume identity weighting coefficients $\alpha_i = 1$. The weight β may be interpreted as the relative norm of the possible disturbance z_2. The task of the optimization is minimization of the norm

$$J^o = \min_{(G_1,G_2)} J, \tag{14}$$

under constraints (6) and (7). The norm (13) of the observer is the function of observation time T.

Because of (6) the Lagrange functional is of the form

$$L = J + 2\sum_{i=1}^n \left[\left[e_i^T - \left[\langle g_1^i, h_1^1 \rangle, \dots, \langle g_1^i, h_1^n \rangle\right]\right] \cdot \lambda_i\right] \tag{15}$$

(i) where e_i are basis vectors in R^n with one in i-th row,

(ii) g_1^i denotes column vector which is transposition of i-th row of matrix G_1,

(iii) h_1^i denotes i-th column vector of matrix $Ce^{-A(T-t)}$,

(iv) the symbol $\langle g_1^i, h_1^j \rangle$ stands for the integral inner products,

(v) $\lambda_i \in R^n$ are vectors of Lagrange multipliers.

From constraint (7) it follows that the squared norm J and hence Lagrange functional L are functions of matrix G_1 rows g_1^i, only.

Hence from optimality condition $\delta L / \delta g_1^i = 0$ after some calculation one can obtain the main formula

$$G_1'(t) = -\int_t^T Ce^{A(t-\tau)} B G_2'(\tau)\, d\tau \beta + Ce^{-A(T-t)} \lambda \tag{16}$$

(i) where β is diagonal matrix with β_i elements,

(ii) $\boldsymbol{\lambda}$ is Lagrange multipliers matrix with vector columns λ_i,

(iii) apostrophe ' denotes matrix transposition.

Transposition of (7) gives the result:

$$G_2'(\tau) = \int_0^\tau B' e^{-A'(\tau-s)} C' G_1'(s)\, ds. \tag{17}$$

Equations (16) and (17) give the set of two integral equations, which can be solved. To this end let us notice that we have some boundary relationships for rectangular matrices:

$$\begin{aligned} G_1^{'}(T) &= C\lambda, \\ G_2(0) &= 0, \\ G_2(T) &= B. \end{aligned} \quad (18)$$

Let us introduce two square matrices P_1 and P_2 of [n×n] dimension.

$$\begin{aligned} G_1^{'}(t) &= CP_1^{'}(t), \\ G_2^{'}(t) &= B^{'}P_2^{'}(t) \end{aligned} \quad (19)$$

Then one can obtain two new equations:

$$P_1^{'}(t) = e^{-A(T-t)}\lambda - \int_t^T e^{A(t-\tau)} BB^{'}P_2^{'}(\tau)\,d\tau \cdot \beta \quad (20)$$

$$P_2^{'}(t) = \int_0^t e^{-A^{'}(t-\tau)} C^{'}CP_1^{'}(\tau)\,d\tau \quad (21)$$

with boundary condition:

$$\begin{aligned} P_1^{'}(T) &= \lambda, \\ P_2(0) &= 0, \\ P_2^{'}(T) &= I. \end{aligned} \quad (22)$$

It is easy to see that (20) represent the solution of two differential equations.

$$\begin{aligned} \dot{P}_1^{'}(t) &= AP_1^{'}(t) + BB^{'}P_2^{'}(t)\beta \\ \dot{P}_2^{'}(t) &= C^{'}CP_1^{'}(t) - A^{'}P_2^{'}(t) \end{aligned} \quad (23)$$

with the above mentioned mixed boundary conditions.

Denote by $p_1^i(t), p_2^i(t)$ the columns of matrices $P_1^{'}(t), P_2^{'}(t)$, and introduce fundamental matrices $\Phi_i(t)$, for i = 1,. . .,n.

$$\Phi_i(t) = e^{W_i t} = \begin{bmatrix} \Phi_{11}^i(t) & \Phi_{12}^i(t) \\ \Phi_{21}^i(t) & \Phi_{22}^i(t) \end{bmatrix} \quad (24)$$

$$\text{where } W_i = \begin{bmatrix} A, & \beta_i BB^{'} \\ C^{'}C, & -A^{'} \end{bmatrix}. \quad (25)$$

And under the condition $p_2^i(0) = 0$, we have the solution of (23) for 2n dimensional problem:

$$\begin{aligned} \begin{bmatrix} p_1^i(t) \\ p_2^i(t) \end{bmatrix} &= \begin{bmatrix} \Phi_{11}^i(t) & \Phi_{12}^i(t) \\ \Phi_{21}^i(t) & \Phi_{22}^i(t) \end{bmatrix} \begin{bmatrix} p_1^i(0) \\ p_2^i(0) \end{bmatrix} \\ &= \begin{bmatrix} \Phi_{11}^i(t) & \Phi_{12}^i(t) \\ \Phi_{21}^i(t) & \Phi_{22}^i(t) \end{bmatrix} \begin{bmatrix} p_1^i(0) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \Phi_{11}^i(t) & p_1^i(0) \\ \Phi_{21}^i(t) & p_1^i(0) \end{bmatrix} \end{aligned} \quad (26)$$

One can find initial conditions for $p_1^i(0)$ by substitution (26) to the constraint (6).

$$\begin{aligned} &\int_0^T e^{-A^{'}(T-\tau)} C^{'}C \left[\Phi_{11}^1(\tau)\, p_1^1(0),, \ldots, \Phi_{11}^n(\tau)\, p_1^n(0)\right] d\tau \\ &\quad = I \end{aligned} \quad (27)$$

Hence denote n different square matrices by M_i, i=1,. . .n:

$$\begin{aligned} M_i^{'} &= \int_0^T e^{-A^{'}(T-\tau)} C^{'}C\Phi_{11}^i(\tau)\,d\tau \\ M_i^{'-1} &= \left[\int_0^T e^{A^{'}\tau} C^{'}C\Phi_{11}^i(\tau)\,d\tau\right]^{-1} e^{A^{'}T}. \\ \text{Then } M_i &= \left[\int_0^T \Phi_{11}^{i\,'}(\tau)\, C^{'}Ce^{A\tau}d\tau\right] e^{-AT}, \\ \text{and } M_i^{-1} &= e^{AT}\left[\int_0^T \Phi_{11}^{i\,'}(\tau)\, C^{'}Ce^{A\tau}d\tau\right]^{-1}. \end{aligned} \quad (28)$$

The initial conditions have the form

$$\begin{aligned} M_1 p_1^1(0) &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{'}, \\ M_2 p_1^2(0) &= \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix}^{'}, \\ &\vdots \\ M_n p_1^n(0) &= \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^{'}. \end{aligned} \quad (29)$$

Finally, the solution for columns of matrices $P_1^{'}(t), P_2^{'}(t)$ has the form

$$\begin{aligned} p_1^i(t) &= \Phi_{11}^i(t)\left[M_i^{'}\right]^{-1} e_i \\ p_2^i(t) &= \Phi_{21}^i(t)\left[M_i^{'}\right]^{-1} e_i \end{aligned} \quad (30)$$

where e_i are basis vectors in R^n with one on i-th row.

The vectors $p_1^i(t), p_2^i(t)$ after transposition will form the rows of optimal matrices $G_1^0(t)$ i $G_2^0(t)$ according to equations

$$\begin{aligned} G_1^o(t) &= P_1(t)\,C^{'}, \\ G_2^o(t) &= P_2(t)\,B \end{aligned} \quad (31)$$

It is the most general optimal form of the exact state observer with minimal norm (13) and weighting coefficients β_i.

2.4. Two Special Cases of Minimal Norm Observer. Two cases will be considered:

(a) The weighting coefficients β=I form unity matrix. This is reasonable case if in the measurements of the output y(t) and the input control u(t) the norms of the possible disturbances are the same.

(b) The weighting coefficients β=0. This is reasonable case if in measurement of the control u(t) the disturbances are absent.

The Solution to the Case (a). For symmetric case β=I, the solution of the optimization task has simpler and more compact form than (30) and (31).

$$\begin{aligned} G^{o}_{1a}(T,\tau) &= M^{-1}\Phi'_{11}(\tau)C' \\ G^{o}_{2a}(T,\tau) &= M^{-1}\Phi'_{21}(\tau)B. \end{aligned} \tag{32}$$

All the matrices M_i are the same $M=M_i$ and M^{-1} has the form

$$M = \left[\int_0^T \Phi'_{11}(\tau)C'Ce^{A\tau}d\tau\right]e^{-AT} \tag{33}$$

$$M^{-1} = e^{AT}\left[\int_0^T \Phi'_{11}(\tau)C'Ce^{A\tau}d\tau\right]^{-1} \tag{34}$$

Two submatrices $\Phi_{11}(\tau)$, $\Phi_{21}(\tau)$ of the fundamental matrix $\Phi(\tau)$ are calculated from (35):

$$\begin{aligned} W &= \begin{bmatrix} A & BB' \\ C'C & -A' \end{bmatrix}, \\ \Phi(t) &= e^{W\tau} = \begin{bmatrix} \Phi_{11}(\tau) & \Phi_{12}(\tau) \\ \Phi_{21}(\tau) & \Phi_{22}(\tau) \end{bmatrix} \end{aligned} \tag{35}$$

Then, matrices G (32) for the given observation time T can be calculated offline in interval [0,T] and applied online in optimal observer moving window [2].

The Solution to the Case (b). For the special values of weight factors β=0, one can obtain from (25) the special form of the exact state observer. It means that one can obtain this form of the observer (3) by minimization of simplified form of the norm (13) which corresponds only to the output y measurements. It is reasonable case if in the measurement of the control signal u(t) the disturbances are absent.

$$\left\|(G_{1b})\right\|^2 = \int_0^T \left[\sum_{i=1}^{n}\sum_{j=1}^{m}\left(g_1^{ij}(\tau)\right)^2\right]d\tau. \tag{36}$$

Equation (23) takes the form

$$\begin{aligned} \dot{P}'_1(t) &= AP'_1(t) \\ \dot{P}'_2(t) &= C'CP'_1(t) - A'P'_2(t) \end{aligned} \tag{37}$$

$$\begin{aligned} P'_1(T) &= \lambda, \\ P_2(0) &= 0, \\ P'_2(T) &= I. \end{aligned} \tag{38}$$

The solution for matrices $P'_1(t), P'_2(t)$ has the form

$$\begin{aligned} \begin{bmatrix} P'_1(t) \\ P'_2(t) \end{bmatrix} &= \begin{bmatrix} e^{At} & 0 \\ \Phi_{21}(t) & e^{-A't} \end{bmatrix}\begin{bmatrix} P'_1(0) \\ P'_2(0) \end{bmatrix} \\ &= \begin{bmatrix} e^{At} & P'_1(0) \\ \Phi_{21}(t) & P'_1(0) \end{bmatrix} \end{aligned} \tag{39}$$

$$\begin{aligned} P'_1(t) &= e^{At}P'_1(0), \\ P_1(t) &= P_1(0)e^{A't} \end{aligned} \tag{40}$$

One can find the initial conditions in (40) by substitution of (40) to the constraint (6) remembering that

$$\begin{aligned} \int_0^T G_1(\tau)Ce^{-A(T-\tau)}d\tau &= I, \\ G_1(t) &= P_1(t)C', \\ G_2(t) &= P_2(t)B \end{aligned} \tag{41}$$

Hence,

$$\begin{aligned} &P_1(0)\int_0^T e^{A'\tau}C'Ce^{A\tau}d\tau\cdot e^{-AT} = I \\ &P_1(0) = e^{AT}\left[\int_0^T e^{A'\tau}C'Ce^{A\tau}d\tau\right]^{-1}\cdot \\ &P_1(t) = P_1(0)e^{A't} = e^{AT}\left[\int_0^T e^{A'\tau}C'Ce^{A\tau}d\tau\right]^{-1}\cdot e^{A't} \end{aligned} \tag{42}$$

Let us assume that the square matrix M with symmetric Gram matrix M_G has a form as in (33)

$$\begin{aligned} M &= \int_0^T e^{A'\tau}C'Ce^{A\tau}d\tau\cdot e^{-AT} = M_G\cdot e^{-AT} \\ M^{-1} &= e^{AT}M_G^{-1}, \\ \left[M^{-1}\right]' &= M_G^{-1}e^{A'T} \end{aligned} \tag{43}$$

Then $P_1(0) = M^{-1}$ and $P'_1(T) = \lambda = e^{AT}P'_1(0) = e^{AT}[M^{-1}]'$

$$P'_1(t) = e^{At}\cdot P'_1(0) = e^{At}\cdot\left[M^{-1}\right]' \tag{44}$$

For $P_2(0) = 0$, the solution for P_2 from (37) and (44) is

$$\begin{aligned} P'_2(t) &= \int_0^t e^{-A'(t-\tau)}C'CP'_1(\tau)d\tau \\ &= \int_0^t e^{-A'(t-\tau)}C'Ce^{A\tau}d\tau P'_1(0) \\ &= e^{-A't}\int_0^t e^{A'\tau}C'Ce^{A\tau}d\tau\left[M^{-1}\right] \end{aligned} \tag{45}$$

From the above the form of $\Phi_{21}(t)$ in (39) is visible.

Finally we have the matrices P

$$P_1(t) = M^{-1} e^{A^{'} t} = e^{AT} M_G^{-1} e^{A^{'} t}$$
$$P_2(t) = M^{-1} \int_0^t e^{A^{'} \tau} C^{'} C e^{A\tau} d\tau \cdot e^{-At} = e^{AT} M_G^{-1} \int_0^t e^{A^{'} \tau} C^{'} C e^{A\tau} d\tau \cdot e^{-At} \quad (46)$$

and we have the matrices G_1 and G_2 (marked finally as G_{1b}, G_{2b}) of the optimal observer for the case (b):

$$G_{1b}(t) = e^{AT} M_G^{-1} e^{A^{'} t} C^{'}, \quad (47)$$

$$G_{2b}(t) = e^{AT} M_G^{-1} \int_0^t e^{A^{'} \tau} C^{'} C e^{A\tau} d\tau \cdot e^{-At} B. \quad (48)$$

For the observation time T, these matrices can be calculated offline in interval [0,T] in as many samples as needed and then applied online in the observer moving window.

Interestingly, the same forms of G_1, G_2 as in (47), (48) can be obtained by using the easy and standard least squares approach, i.e., by the multiplying of both sides of the output equation (2) by the transposition of the appropriate matrix and integration of this equation on [0, T].

$$\int_0^T e^{-A^{'}(T-t)} C^{'} y(t)\, dt = \int_0^T e^{-A^{'}(T-t)} C^{'} C e^{-A(T-t)} dt \cdot x(T) - \int_0^T \left[e^{-A^{'}(T-t)} C^{'} C \int_t^T e^{A(t-s)} Bu(s)\, ds \right] dt \quad (49)$$

Then after changing the order of integration in internal integral we obtain the matrices G as in (47) and (48) and those of the form of the observer for the case (b):

$$x(T) = \int_0^T G_{1b}(\tau)\, y(\tau)\, d\tau + \int_0^T G_{2b}(\tau)\, u(\tau)\, d\tau. \quad (50)$$

It confirms the correctness of all the above derived formulas (47) and (48) and the general theory of the optimal observation (30), (31).

By the way, it means that the use of the above version of the observers in different applications of the exact state observers [12–14] may give the result, which is not in general the optimal observation solution, but only for the special observer norm (36).

In authors' research, it turned out that the norm (13) of the observer decreases to some small value with increasing of the observation time T and increases to infinity with decreasing of time T to zero, like in Figure 1.

$$\begin{aligned} &\text{for } T \longrightarrow \infty, \quad \left\| (G_1, G_2) \right\| (\infty) \longrightarrow \text{small value} \\ &\text{for } T \longrightarrow 0 \quad \text{the norm } \left\| (G_1, G_2) \right\| (0) \longrightarrow \infty, \end{aligned} \quad (51)$$

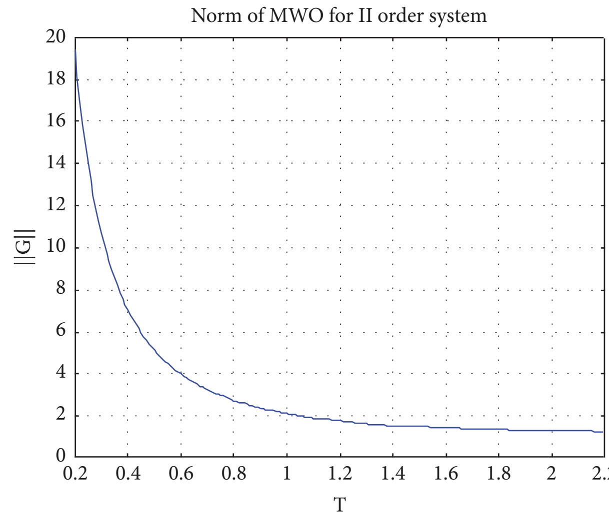

FIGURE 1: An exemplary shape of the norm, as the function of observation time T, for the second-order system.

3. Integral Observers in Online Mode as Moving Window Observers

Presented integral form of the exact observers given on finite time interval [0,T]

$$x(T) = \int_0^T G_1(\tau)\, y(\tau)\, d\tau + \int_0^T G_2(\tau)\, u(\tau)\, d\tau \quad (52)$$

can be applied in online observation and in control systems.

To this end one can design the structure of Moving Window Observer (MWO). Equation (52) is valid for any linear time invariant (LTI) systems and for any fragment of measured functions y and u and hence for shifted input/output functions, also.

One can use two possible representations of MWSO at time t:

$$x(t) = \int_0^T G_1(\tau)\, y(t-T+\tau)\, d\tau + \int_0^T G_2(\tau)\, u(t-T+\tau)\, d\tau \quad (53)$$

$$x(t) = \int_{t-T}^t G_1(T-t+\tau)\, y(\tau)\, d\tau + \int_{t-T}^t G_2(T-t+\tau)\, u(\tau)\, d\tau \quad (54)$$

The form (54) represents moving window of width T shifted along time axis against measurements and after integrals calculations giving current and exact state x(t) for t≥T. Such type of the observer has characteristic delay in starting of the observation connected with the first window for $0 \le t \le T$.

The matrices G_1, G_2 do not depend on current time t and can be calculated only once and offline in interval [0, T]. Then they may be stored in memory registers in as many samples as needed for accurate calculation of integrals, depending on discretization time of measurements of y and u.

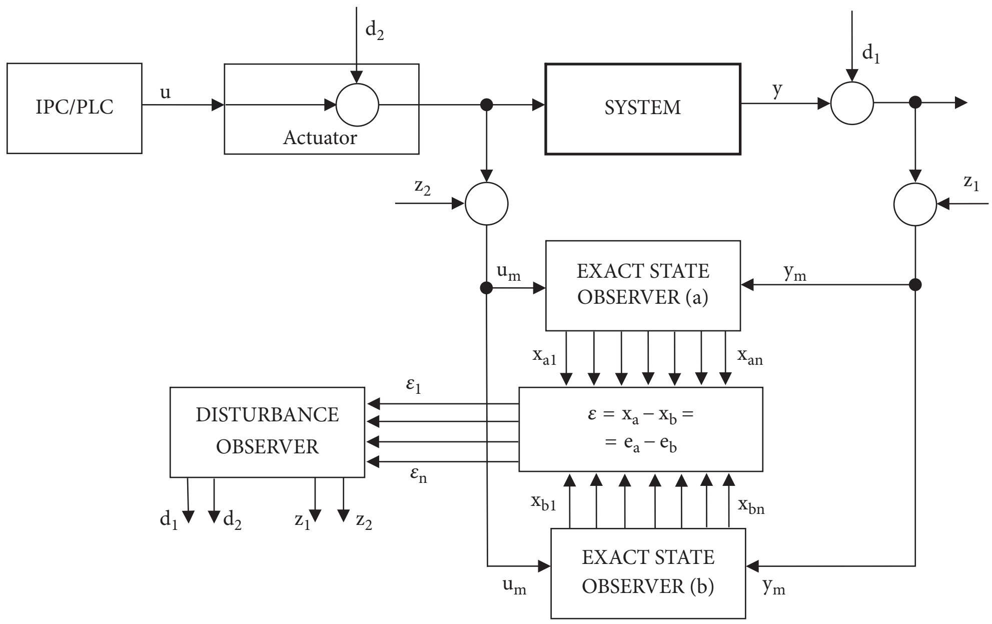

FIGURE 2: The system for reconstruction of disturbances.

Digital control equipment should have enough computation power for online calculation of thousands of multiplications and summations per second. For the nowadays industrial computers (IPC) this is no problem, let alone for those with digital signal processors (DSP).

During designing of the observer, the main problem is the right choice of the observation time T. The short interval T results in quick start of the online state reconstruction process and requires fewer calculations during the moving window mode but the observer is more highly sensitive to the disturbances (has the bigger norm). The longer time T results in bigger time delay in starting of observation and causes more calculations within the window but the observer is less sensitive to disturbances (has the smaller norm).

4. The Main Idea of the Paper

Let a linear state observable system be given

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{55}$$

We will consider the case of Single Input–Single Output (SISO) system; however one can generalize the disturbances detection method to Multi-Input Multi-Output case. The system is of n-th order, the state $x(t) \in R^n$, the control $u(t) \in R^1$, and the output $y(t) \in R^1$, for $\forall t \geq 0$. Matrices A, B, C are of compatible dimensions.

Assume that we perfectly measure the control u and the output y (without any disturbances) on the interval [0, T], where T is the fixed observation time.

For the exact state observation, we will use simultaneously working two observers (a), (32), (33), (35) and (b), (47), (48).

Both observers reconstruct the exact state $x_a(T) = x_b(T) = x(T)$.

However, if during control process the disturbances d_2 of the control and disturbances d_1 of the output occur and the measurement noises occur (measurement disturbances on input-output z_2, z_1), then we have a situation like in Figure 2.

Our purpose is to determine the state $x_a(T) = x(T)+e_a(T)$, $x_b(T) = x(T)+e_b(T)$ and the disturbance values d and z, based on the measurement signals $y_m(t)$ and $u_m(t)$.

The equation of the disturbed system has the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\left[u(t) + d_2(t)\right] \\ y(t) &= Cx(t), \end{aligned} \tag{56}$$

and the reconstructed state is given by two equations, one from the observer (a) (the pair of matrices G_{1a}, G_{2a}) and the second from the observer (b) (the pair of matrices G_{1b}, G_{2b}). Both observers use for the state calculations the disturbed measurements,

$$\begin{aligned} y_m(t) &= y(t) + d_1(t) + z_1(t), \\ u_m(t) &= u(t) + d_2(t) + z_2(t) \end{aligned} \tag{57}$$

Control $u(t) \in R^1$, the output $y(t) \in R^1$, disturbances $d_1(t) \in R^1$, $d_2(t) \in R^1$, $z_1(t) \in R^1$, $z_2(t) \in R^1$, for $\forall t \geq 0$

The control signal u(t) is known, because it is generated by the deterministic control algorithm in industrial PC or Programmable Logic Controller (PLC).

The algorithm of disturbance detection is presented in Figure 2.

The reconstructed state $x_a(T)$ by the observer (a):

$$\begin{aligned} x_a(T) &= \int_0^T G_{1a}(\tau)\, y_m(\tau)\, d\tau + \int_0^T G_{2a}(\tau)\, u_m(\tau)\, d\tau \\ &= \int_0^T G_{1a}(\tau)\left[y(\tau) + d_1(\tau) + z_1(\tau)\right] d\tau \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T G_{2a}(\tau)\left[u(\tau)+d_2(\tau)+z_2(\tau)\right]d\tau \\
&= x(T)+e_a(T)
\end{aligned} \tag{58}
$$

The reconstructed state $x_b(T)$ by the observer (b):

$$
\begin{aligned}
x_b(T) &= \int_0^T G_{1b}(\tau)\,y_m(\tau)\,d\tau+\int_0^T G_{2b}(\tau)\,u_m(\tau)\,d\tau \\
&= \int_0^T G_{1b}(\tau)\left[y(\tau)+d_1(\tau)+z_1(\tau)\right]d\tau \\
&\quad+\int_0^T G_{2b}(\tau)\left[u(\tau)+d_2(\tau)+z_2(\tau)\right]d\tau \\
&= x(T)+e_b(T)
\end{aligned} \tag{59}
$$

Taking into account the fact that the real state x(T) in (56) is given by

$$
\begin{aligned}
x(T) = &\int_0^T G_{1a}(\tau)\,y(\tau)\,d\tau \\
&+\int_0^T G_{2a}(\tau)\left[u(\tau)+d_2(\tau)\right]d\tau,
\end{aligned} \tag{60}
$$

as well as

$$
\begin{aligned}
x(T) = &\int_0^T G_{1b}(\tau)\,y(\tau)\,d\tau \\
&+\int_0^T G_{2b}(\tau)\left[u(\tau)+d_2(\tau)\right]d\tau,
\end{aligned} \tag{61}
$$

the difference of the estimated states (58), (59) gives the estimation error $\varepsilon(T)$.

$$
\begin{aligned}
\varepsilon(T) &= x_a(T)-x_b(T) \\
&= \int_0^T\left[G_{1a}(\tau)-G_{1b}(\tau)\right]\left[d_1(\tau)+z_1(\tau)\right]d\tau \\
&\quad+\int_0^T\left[G_{2a}(\tau)-G_{2b}(\tau)\right]\left[z_2(\tau)\right]d\tau
\end{aligned} \tag{62}
$$

$\varepsilon(T)\in R^n$.

The Main Assumption. Let us assume that the values of the disturbances d_1, z_1, d_2, z_2 within the interval T are constant.

Such assumption is reasonable if the interval T is small.

Let us mark two vector-matrices [nx1] for each observer, with the second indexes, which mean the numbers (items) of the state vector elements.

$$
\begin{aligned}
(G_{1a},G_{2a}) &= \left(\begin{bmatrix}G_{11,a}(\tau)\\ \vdots\\ G_{1n,a}(\tau)\end{bmatrix},\begin{bmatrix}G_{21,a}(\tau)\\ \vdots\\ G_{2n,a}(\tau)\end{bmatrix}\right), \\
(G_{1b},G_{2b}) &= \left(\begin{bmatrix}G_{11,b}(\tau)\\ \vdots\\ G_{1n,b}(\tau)\end{bmatrix},\begin{bmatrix}G_{21,b}(\tau)\\ \vdots\\ G_{2n,b}(\tau)\end{bmatrix}\right)
\end{aligned} \tag{63}
$$

Then, we have from (62) the main equation for the estimation error.

$$
\begin{aligned}
\varepsilon(T) &= \begin{bmatrix}\varepsilon_1(T)\\ \vdots\\ \varepsilon_n(T)\end{bmatrix} \\
&= \begin{bmatrix}\int_0^T\left[G_{11,a}(\tau)-G_{11,b}(\tau)\right]d\tau\\ \vdots\\ \int_0^T\left[G_{1n,a}(\tau)-G_{1n,b}(\tau)\right]d\tau\end{bmatrix}\left[d_1+z_1\right] \\
&\quad+\begin{bmatrix}\int_0^T\left[G_{21,a}(\tau)-G_{21,b}(\tau)\right]d\tau\\ \vdots\\ \int_0^T\left[G_{2n,a}(\tau)-G_{2n,b}(\tau)\right]d\tau\end{bmatrix}\left[z_2\right],
\end{aligned} \tag{64}
$$

or in compact form with the constant matrix D [nx2]:

$$
\begin{bmatrix}\varepsilon_1(T)\\ \vdots\\ \varepsilon_n(T)\end{bmatrix} = \begin{bmatrix}\int_0^T\left[G_{11,a}(\tau)-G_{11,b}(\tau)\right]d\tau, & \int_0^T\left[G_{21,a}(\tau)-G_{21,b}(\tau)\right]d\tau\\ \vdots & \vdots\\ \int_0^T\left[G_{1n,a}(\tau)-G_{1n,b}(\tau)\right]d\tau, & \int_0^T\left[G_{2n,a}(\tau)-G_{2n,b}(\tau)\right]d\tau\end{bmatrix}\begin{bmatrix}d_1+z_1\\ z_2\end{bmatrix} \tag{65}
$$

$$
\varepsilon(T) = D\begin{bmatrix}d_1+z_1\\ z_2\end{bmatrix} = D\Theta \tag{66}
$$

In this equation the vector of estimation error $\varepsilon(T) \in R^n$ is known as well as the rectangular and constant matrix D [nx2]. The unknown vector Θ of two real numbers represents the values of unknown disturbances $\Theta \in R^2$. These values Θ are valid for the entire interval T.

Hence, for the system's order n=2, D is the square matrix, D[2x2], and for the equation $\varepsilon(T) = D\theta$, we have the single solution for the vector of disturbances Θ:

$$\Theta = D^{-1}\varepsilon(T) \tag{67}$$

For n>2, the matrix D is rectangular D[nx2] and by the least squares approach it is easy to find that the best solution of (66) is given by

$$\Theta = \left[D'D\right]^{-1} D'\varepsilon(T) \tag{68}$$

$$\Theta = \begin{bmatrix} d_1 + z_1 \\ z_2 \end{bmatrix} \tag{69}$$

Then, it is easy to find that the disturbance d_2 is given by

$$d_2 = u_m(t) - u(t) - z_2. \tag{70}$$

It turned out that the separation of the disturbance sum d_1+z_1 at the time T and the exact calculation of the constant values d_1, z_1 is not possible by this method (see Conclusions).

5. The Moving Window Disturbances Observer

All the above considerations were carried out, for the one observation window [0, T] (batch mode).

If we use the idea of the Moving Window State Observer (53) working in online mode,

$$\begin{aligned} x(t) = &\int_0^T G_1(\tau) y_m(t - T + \tau) d\tau \\ &+ \int_0^T G_2(\tau) u_m(t - T + \tau) d\tau \end{aligned} \tag{71}$$

then using two MWSO, for reconstruction of $x_1(t)$ and $x_2(t)$, one can also design the Moving Window Disturbances Observer, for the online identification of the disturbances for t>T.

We can have continuous equation for the online reconstruction of the disturbances:

$$\Theta(t) = \left[D'D\right]^{-1} D'\varepsilon(t) \tag{72}$$

where

$$\varepsilon(t) = x_a(t) - x_b(t) \tag{73}$$

and the values of $x_a(t)$ and $x_b(t)$ are reconstructed online by the observers of type (71). The constant rectangular matrix D [nx2] is the same as in (66):

$$D = \begin{bmatrix} \int_0^T [G_{11,a}(\tau) - G_{11,b}(\tau)] d\tau, & \int_0^T [G_{21,a}(\tau) - G_{21,b}(\tau)] d\tau \\ \vdots & \vdots \\ \int_0^T [G_{1n,a}(\tau) - G_{1n,b}(\tau)] d\tau, & \int_0^T [G_{2n,a}(\tau) - G_{2n,b}(\tau)] d\tau \end{bmatrix} \tag{74}$$

The real matrix D as well as $[D'D]^{-1}D'$ should be calculated offline for a given interval T only once (based on (32), (47), and (48)) and used during online calculation. Of course in practice each element of the observer function matrices $G_1(t)$ and $G_2(t)$ calculated offline must be stored in computer memory within [0, T], e.g., as 100 samples, with Δ=0.1 sec each (T=10 sec). The same samples time must apply to I/O signal measurement. These measurements may be delivered in online mode and final calculation of the observers integrals is performed in real time numerically in the last window [t-T, t] for each t, i.e., in numerical version [iΔ-T, iΔ] for each i (with the use of the best integration procedure, e.g., with Simpson's rule).

There is no real-time computation problem with online calculation of such Moving Window Observer based on (71), (72), and (73). The not very new processor Pentium 4 can perform 3 GFLOPS (floating point operations per 1 sec), i.e., 3 mln FLOP/ 1 msec, while Pentium Core i7 5960X – can perform even 300 GFLOPS. Assuming that PLC cycle time is about 100 msec (sampling time of measurements), it means that during this time Pentium 4 can perform 300 mln FLOP. Two integrals in one observer with two integral windows T, each sampled, e.g., 1000 times, need 2000 multiplications (1000 multiplications of each element, e.g., $G_{11a}(t)y(t)$, and 1000 multiplications of each element, e.g., $G_{21a}(t)u(t)$) as well as 2000 summation (simplest calculation of integrals based on trapezoid rules). Hence, two observer's integrals need 8000 floating point operations for single function element. If the SISO system is of order n=10 then the disturbance reconstruction (detection) needs computation power 80 000 FLOP in each sample. This is 3% of Pentium 4 power ratio. The above theoretical and simplified estimation of the computational capabilities of standard PCs has been presented only to justify the applicability of the MWO in the real industrial processes and in online mode. All numerical experiments in this article were carried out in the Matlab/Simulink environment in which numerical integration procedures are performed with the Simpson algorithm.

6. Numerical Example

All the simulation data used to support the findings of this study are included within the article.

Assume that second-order system is given.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 2 & 0 \end{bmatrix} x(t) \end{aligned} \tag{75}$$

For this simple system, all the exact observer calculations can be done analytically.

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix};$$

$$e^{A't} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix};$$

$$M_G = \int_0^T e^{A' t} C' C e^{At} dt = \int_0^T \begin{bmatrix} 4 & 4t \\ 4t & 4t^2 \end{bmatrix} dt \tag{76}$$

We will derive the exact state x(T) optimal observer formula, for two cases of weighting coefficients β=1 and β=0.

The Case of the Observer (a), the Weighting Coefficients $\beta = 1$. After some longer calculations, one can obtain that submatrices $\Phi_{11}(t)$ and $\Phi_{21}(t)$ from (35) have the form

$$\Phi_{11}(t) = \begin{bmatrix} \cosh t \cos t, & \frac{1}{2}(\sinh t \cos t + \cosh t \sin t) \\ \sinh t \cos t - \cosh t \sin t, & \cosh t \cos t \end{bmatrix}$$

$$\Phi_{21}(t) = \begin{bmatrix} 2(\sinh t \cos t + \cosh t \sin t), & 2 \sinh t \sin t \\ -2 \sinh t \sin t, & \sinh t \cos t - \cosh t \sin t \end{bmatrix} \tag{77}$$

Matrices M and M^{-1} from (33) have the form

$$\begin{aligned} M &= \begin{bmatrix} 2 \sinh T \cos T + 2 \cosh T \sin T, & -2 \sinh T \sin T \\ 2 \sinh T \sin T, & \sinh T \cos T - \cosh T \sin T \end{bmatrix} \\ M^{-1} &= \frac{1}{2\sinh^2 T - 2\sin^2 T} \begin{bmatrix} \sinh T \cos T - \cosh T \sin T, & 2 \sinh T \sin T \\ -2 \sinh T \sin T, & 2(\sinh T \cos T + \cosh T \sin T) \end{bmatrix} \end{aligned} \tag{78}$$

The optimal observer matrices (in this case vectors) $G_{1a}(t)$ and $G_{2a}(t)$ within the interval [0, T] have the final form as in (32) and are visible in Figure 3.

$$G_{1a}(t) = M^{-1} \begin{bmatrix} 2 \cos(t) \cosh(t) \\ \cos(t) \sinh(t) + \sin(t) \cosh(t) \end{bmatrix}, \tag{79}$$

$$G_{2a}(t) = M^{-1} \begin{bmatrix} -2 \sin(t) \sinh t \\ \cos(t) \sinh(t) - \sin(t) \cosh(t) \end{bmatrix}, \tag{80}$$

and the observer (a) is given by

$$\begin{aligned} \begin{bmatrix} x_{1,a}(T) \\ x_{2,a}(T) \end{bmatrix} &= \int_0^T \begin{bmatrix} G_{11,a}(\tau) \\ G_{12,a}(\tau) \end{bmatrix} y(\tau) d\tau \\ &+ \int_0^T \begin{bmatrix} G_{21,a}(\tau) \\ G_{22.a}(\tau) \end{bmatrix} u(\tau) d\tau. \end{aligned} \tag{81}$$

The norm (13) of this observer is a function of interval T

$$\|(G_{1a}, G_{2a})\|(T) = \sqrt{\frac{3 \sinh(2T) + \sin(2T)}{4(\sinh^2(T) - \sin^2(T))}}, \tag{82}$$

and as a function of T it is presented in Figure 1.

The Case of the Observer (b), the Weighting Coefficients $\beta = 0$. For this simpler case, the observer matrices are as (47), (48):

$$M^{-1} = \frac{1}{2T^3} \begin{bmatrix} -T^2 & 3T \\ -3T & 6 \end{bmatrix} \tag{83}$$

$$\begin{aligned} G_{1b}(t) &= \frac{1}{T^3} \begin{bmatrix} 3Tt - T^2 \\ 6t - 3T \end{bmatrix}; \\ G_{2b}(t) &= \frac{1}{T^3} \begin{bmatrix} T^2 t^2 - Tt^3 \\ 3Tt^2 - 2t^3 \end{bmatrix}; \end{aligned} \tag{84}$$

The optimal observer matrices (in this case vectors) $G_{1b}(t)$ and $G_{2b}(t)$ within the interval [0, T] have the final form as in (47), (48) and are visible in Figure 4.

The norm (13) of this observer is also function of interval T

$$\|(G_{1b}, G_{2b})\|(T) = \sqrt{\frac{T^6 + 39T^4 + 105T^2 + 315}{105T^3}} \tag{85}$$

$$\begin{aligned} \begin{bmatrix} x_{1b}(T) \\ x_{2b}(T) \end{bmatrix} &= \int_0^T \begin{bmatrix} G_{11,b}(\tau) \\ G_{12,b}(\tau) \end{bmatrix} y(\tau) d\tau \\ &+ \int_0^T \begin{bmatrix} G_{21,b}(\tau) \\ G_{22,b}(\tau) \end{bmatrix} u(\tau) d\tau. \end{aligned} \tag{86}$$

The matrix (74) has the form

$$D = \begin{bmatrix} \int_0^T [G_{11,a}(\tau) - G_{11,b}(\tau)] d\tau, & \int_0^T [G_{21,a}(\tau) - G_{21,b}(\tau)] d\tau \\ \int_0^T [G_{12,a}(\tau) - G_{12,b}(\tau)] d\tau, & \int_0^T [G_{22,a}(\tau) - G_{22,b}(\tau)] d\tau \end{bmatrix} \tag{87}$$

It is easy to calculate numerically all elements of matrix differences, e.g., $G_{11,a}(\tau) - G_{11,b}(\tau)$, on [0, T] and then integrate them obtaining the all real matrix D [2x2], and so the matrix $[D' D]^{-1} D'$ [2x2], which in this case is directly equal to the real matrix D^{-1} [2x2].

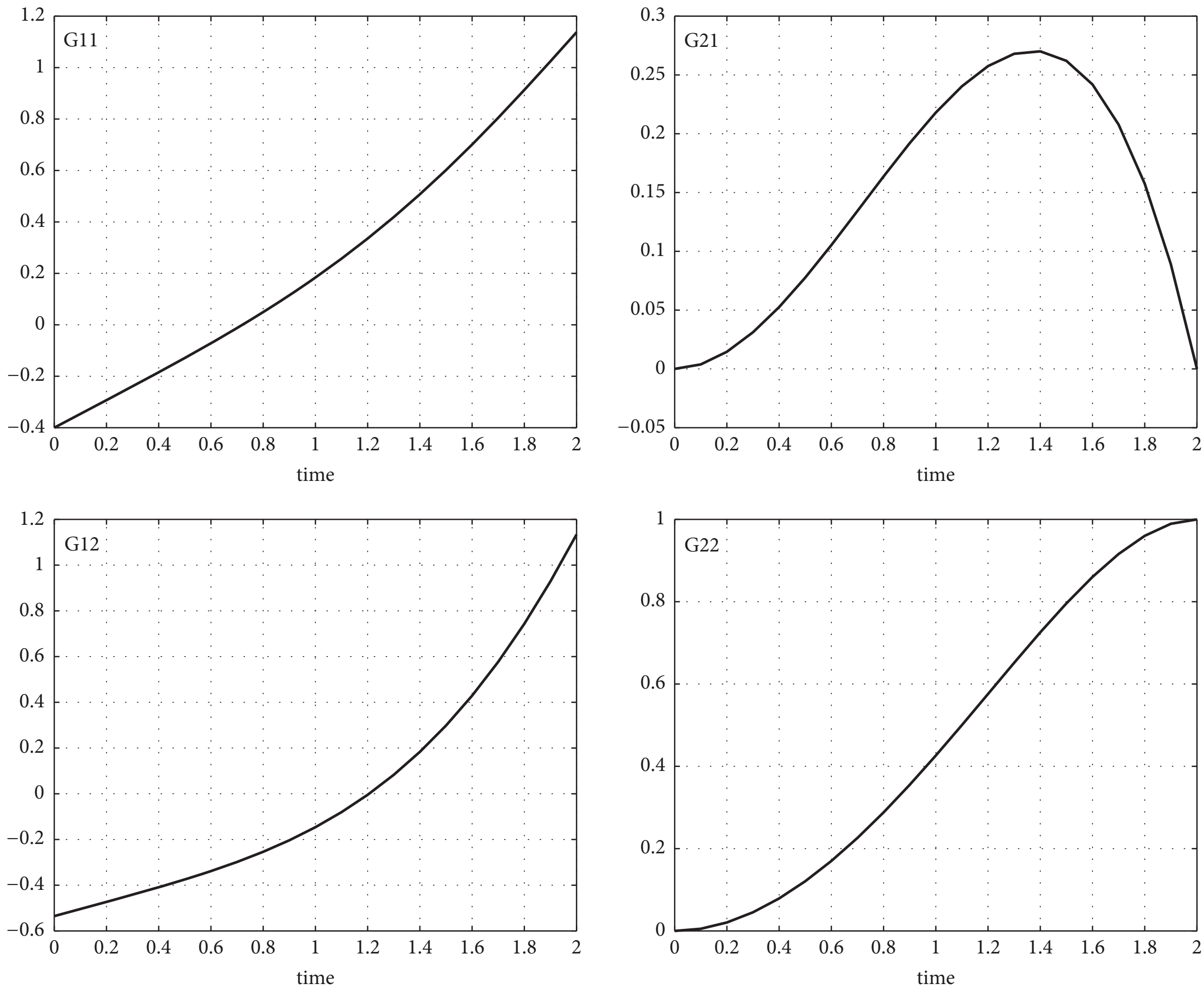

FIGURE 3: The shape of the matrix functions $G_{11,a}$ and $G_{12,a}$ as well as $G_{21,a}$ and $G_{22,a}$ of the observer (a), for T = 2.

The shapes of the integral functions (differences) in the matrix D are visible in Figure 5.

Assuming T=2 and using (80) and (84) equations, one can calculate the real matrix D and inv(D) from (87),

$$D = \begin{bmatrix} 0.0005 & -0.0337 \\ 0.00153 & -0.0788 \end{bmatrix},$$
$$D^{-1} = \begin{bmatrix} -6325,56 & 2709,57 \\ -123.11 & 40,06 \end{bmatrix}, \quad (88)$$

Simulation of $\varepsilon(T)$ using (58), (59) for some chosen u(t) as well as for some chosen constant disturbances has given value of the error

$$\varepsilon(2) = x_a(2) - x_b(2) = \begin{bmatrix} -0.0079 \\ -0.0182 \end{bmatrix}. \quad (89)$$

Hence, from (67)

$$\Theta = D^{-1}\varepsilon(2) = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \begin{bmatrix} d_1 + z_1 \\ z_2 \end{bmatrix}. \quad (90)$$

Assuming that $z_1=z_2$, we have $d_1=0.75$, $z_1=z_2=0.25$. These values are the same as those assumed in simulation.

7. Conclusions

In the paper, the quite new methodology of identification of the unknown constant values of the disturbances acting in dynamical control system was presented. To this end, the theory and application of the exact state integral observers were used. The structure of the Moving Window Disturbance Observer, which consists of two MWO, is defined.

It is possible to identify exactly three values of disturbances d_1+z_1, d_2, z_2. However, there is also the possibility of identification of the disturbance d_1 if we assume that constant values of the noises are the same; $z_1=z_2$ (e.g., if the identified

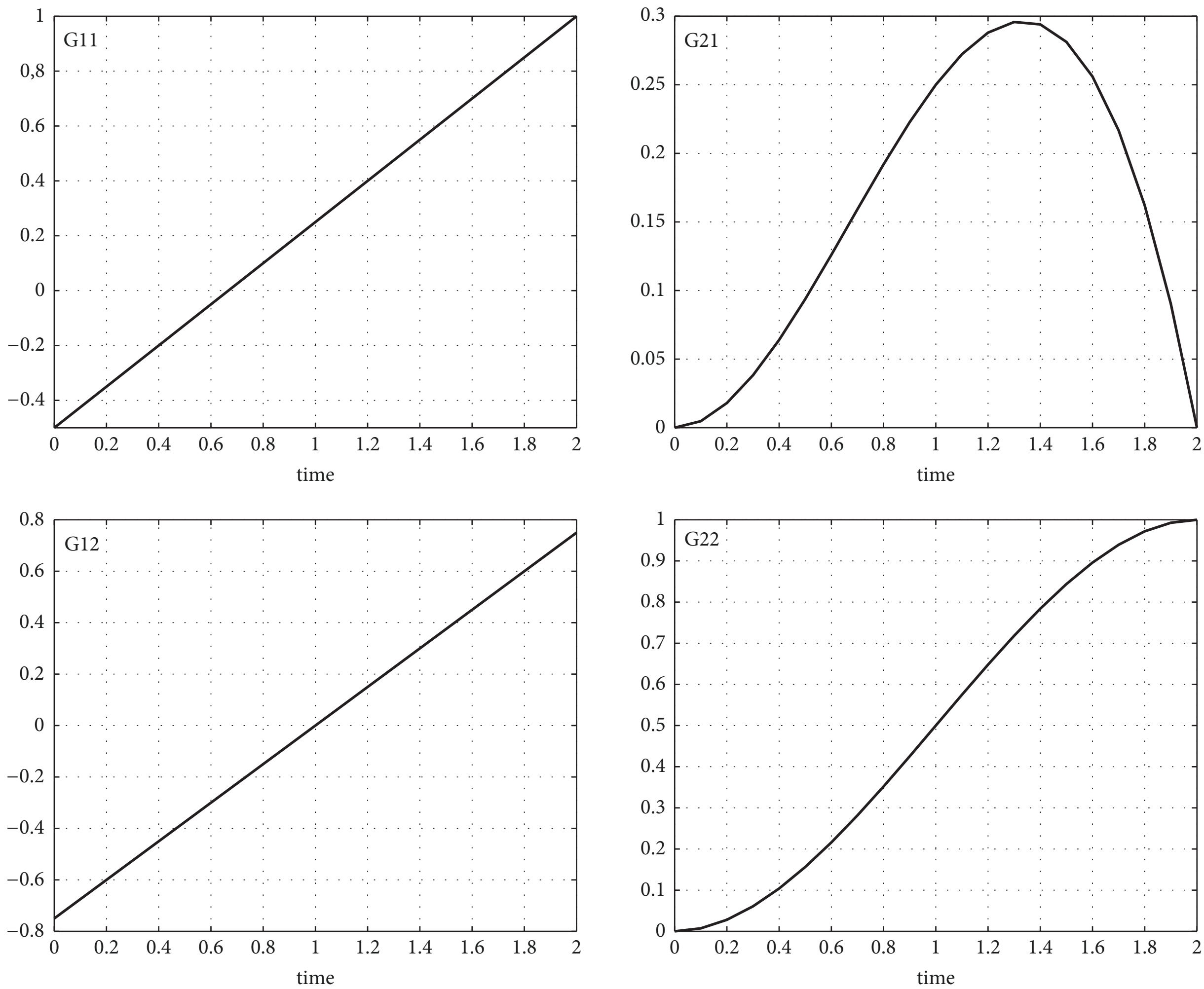

Figure 4: The shape of the matrix functions $G_{11,b}$ and $G_{12,b}$ as well as $G_{21,b}$ and $G_{22,b}$ of the observer (b), for T = 2.

value of z_2 represents the mean value of the noise $z_2(t)$ and the noise $z_1(t)$ in the interval T). Then

$$d_1 = \left(d_1 + z_1\right) - z_2. \tag{91}$$

For different signs of z_1 and z_2, there is also possible estimation of the norm of $\|d_1\|$, if we assume that at least the norm of $\|z_1\|$=$\|z_2\|$ in $L^2[0,T]$ (z_2 was calculated).

Because we know the value of $d_1 + z_1$, then we can calculate and estimate the norm $\|d_1 + z_1\|$,

$$\left\|d_1 + z_1\right\| \leq \left\|d_1\right\| + \left\|z_1\right\|, \tag{92}$$

and estimate the norm of $\|d_1\|$

$$\left\|d_1\right\| \geq \left\|d_1 + z_1\right\| - \left\|z_1\right\|. \tag{93}$$

If $\|z_1\| = \|z_2\|$, then for sure it is true that

$$\left\|d_1\right\| \geq \left\|d_1 + z_1\right\| - \left\|z_2\right\|. \tag{94}$$

Hence, because of the constant values of d_1, z_1, z_2 we have

$$\left|d_1\right| = \left|d_1 + z_1\right| - \left|z_2\right| \tag{95}$$

or

$$\left|d_1\right| = \left|d_1 + z_1\right| + \left|z_2\right|. \tag{96}$$

In this paper the general conditions and formulas for the exact observers with minimal norm in $L^2[0,T]$ spaces were recalled. In this problem, there is no need for the discussion about the convergence of the method. There is no differential equation (unlike in the Kalman Filter theory). All algorithms are based on integration operations on finite windows, and even for an unstable model, because, of the finite interval of the window, integrals cannot tend to infinity. With nondisturbed y and u measurements, the state of the unstable object will be still reconstructed exactly. In the case of disturbed measurements, there is an error in the reconstruction of the state, not due to the instability of the object, but due to measurement errors. Of course, one must assume that numerical Simpson procedures of integration are correct.

The numerical example (in Matlab/Simulink) confirms the correctness of this new method for the disturbance identification.

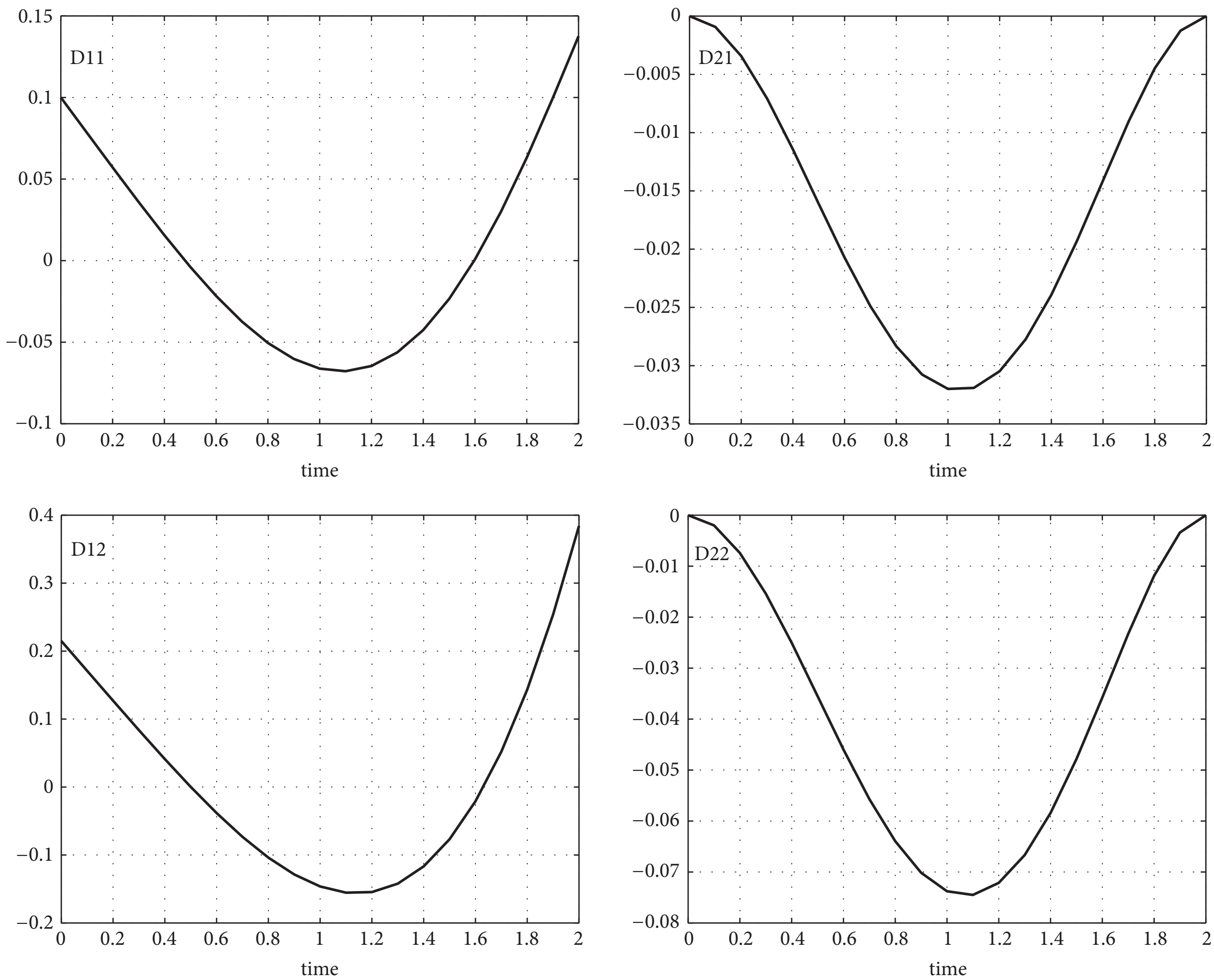

Figure 5: The shape of the integral functions D_{11}, D_{12}, D_{21}, D_{22}.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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