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Research Article

Strong Solutions and Global Attractors for Kirchhoff Type Equation

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We study the long-time behavior of the Kirchhoff type equation with linear damping. We prove the existence of strong solution and the semigroup associated with the solution possesses a global attractor in the higher phase space.

1. Introduction

We consider the following nonlinear Kirchhoff type equation with the initial-boundary conditions:

$$\begin{aligned} u_{tt} + u_t + \triangle^2 u + \varphi(u) &= f(x), \\ u(0) &= u_0, \\ u_t(0) &= u_1, \\ u|_{\partial\Omega} &= \triangle u|_{\partial\Omega} = 0, \end{aligned} \tag{1}$$

where Ω is a bounded domain in R^n with smooth boundary, $\triangle$ denotes the Laplace operator, f is a given function lying in $L^2(\Omega)$, independent of time, and $\varphi \in C^1(R)$ with $\varphi(0) = 0$ fulfills the dissipation inequality

$$\liminf_{|u|\longrightarrow\infty} \frac{\varphi(u)}{u} > -\lambda_1^2, \tag{2}$$

and

$$\varphi' \geq -l, \tag{3}$$

where $l > 0$ is a real number and λ_1 is the first eigenvalue of $-\triangle$ on $L^2(\Omega)$ with Dirichlet boundary conditions

$$\begin{aligned} -\triangle w &= \lambda w, \\ w|_{\partial\Omega} &= 0. \end{aligned} \tag{4}$$

When $n = 1$, this problem describes, for instance, the motion of a vibrating string with fixed boundary in a viscous medium. In particular, the function represents the displacement from equilibrium, u_t is the velocity, and the term $\varphi(u) - f$ may correspond to a (nonlinear) elastic force. For more details on the model of Kirchhoff, one can refer to [1–3] and the reference therein.

When the coefficient of u_t is a positive function $g(u)$, which depends on u, then the term $g(u)u_t$ is a resistance force and the model impresses that the viscous medium embedding the string is stratified. In this case, the existence of the global and exponential attractors has been proven by S. Kolbasin in [4]. The attractor is in the phase space $H_0^2(\Omega) \times L^2(\Omega)$ and it is bounded in $[H^4(\Omega) \cap H_0^2(\Omega)] \times H_0^2(\Omega)$, where Ω is a bounded domain in R^3 with smooth boundary.

Similar models have been considered in [5–9], such as plate equation

$$u_{tt} + \sigma(u) u_t + \triangle^2 u + \lambda u + f(u) = g(x), \tag{5}$$

when $\sigma(u) = \alpha(x)$, the existence, regularity, and finite dimensionality of a global attractor in $H_2^2(R^n) \times L^2(R^n)$ with a localized damping and a critical exponent were proven in [5]. For Ω a three-dimensional unbounded domain and under suitable conditions on $\sigma(u)$ and $f(u)$, A.Kh. Khanmamedov in [6] showed that this equation possesses a global attractor in $H^2(R^3) \times L^2(R^3)$. About Kirchhoff models, long-time dynamics properties were studied by Yang and I. Chueshov in

[7–9] and their references. For example, Yang et al. discussed the long-time behavior of solutions to the Cauchy problem of some Kirchhoff type equations with a strong dissipation in [7, 8] and proved that the dynamical system possesses a global attractor under suitable conditions in the phase space $X_{1+\delta}$, where $X_{1+\delta} = V_{1+\delta} \times V_\delta, 0 < \delta \le 1$.

Thus, to the best of our knowledge, the research about global attractors of the weak solutions for problem (1) with respect to the norm of $H^2 \times L^2$ is much more; however, the results about the existence of the strong solutions and strong global attractors for (1) are relatively fewer. The purpose of this paper is to supplement some conclusions for the above problem. In particular, we do not demand the function $\varphi(u)$ to be bounded by polynomials and present here a method different from [9–14]. Furthermore, the global attractor is established in the higher energy space in $H^4(\Omega) \cap H_0^2(\Omega)$.

The paper is organized as follows. In Section 2, we recall some notations and some general facts about the dynamical systems theory. In Section 3, we prove well-posedness and the existence of a bounded absorbing set for problem (1). Sections 4 and 5 contain our main results, and we prove the existence of strong solution and a global attractor in the space of higher order.

2. Preliminaries

Throughout the paper we will denote

$$\begin{aligned} H &= L^2(\Omega), \\ V &= H_0^1(\Omega), \\ V_1 &= H^2(\Omega) \cap H_0^1(\Omega), \\ V_2 &= D(A) = \left\{u \in H^2(\Omega) \mid Au \in L^2(\Omega)\right\}, \\ &\qquad \text{where } A = \triangle^2. \end{aligned} \tag{6}$$

The norm and the scalar product in $L^2(\Omega)$ is denoted by $\|\cdot\|$ and $(,)$, respectively, the norm in V_i is denoted by $\|\cdot\|_{V_i}$, ($i = 0, 1, 2$), and the Hilbert spaces E_1, E_2 are

$$\begin{aligned} E_1 &= V_1 \times H, \\ E_2 &= V_2 \times V_1. \end{aligned} \tag{7}$$

For any given function $u(t)$, we will write for short $\xi(t) = (u(t), u_t(t))$ and endow space E_1, E_2 with the standard inner product and norm $\|\xi_u\|_{E_1}^2 = \|u\|_{V_1}^2 + \|u_t\|_{L^2}^2$, $\|\xi_u\|_{E_2}^2 = \|u\|_{V_2}^2 + \|u_t\|_{V_1}^2$.

For convenience, the letters C and Q present different positive constants and different positive increasing functions, respectively.

We collect some basic concepts and general theorems, which are important for getting our main results. We refer to [14–18] and the references therein for more details.

Definition 1 (see [16, 18]). Let X be a Banach space and $\{S(t)\}_{t\ge0}$ be a family operator on X. We say that $\{S(t)\}_{t\ge0}$ is a norm-to-weak continuous semigroup on X, if $\{S(t)\}_{t\ge0}$ satisfies

(i) $S(0) = \mathrm{Id}$ (the identity);

(ii) $S(t)S(s) = S(t+s), \forall t, s \ge 0$;

(iii) $S(t_n)x_n \rightharpoonup S(t)x$, if $t_n \longrightarrow t, x_n \longrightarrow x$ in X.

Definition 2 (see [15]). A C^0 semigroup $\{S(t)\}_{t\ge0}$ in a Banach space X is said to satisfy the condition (C) if, for any $\varepsilon > 0$ and for any bounded set B of X, there exists $t(B) > 0$ and a finite dimensional subspace X_1 of X, such that $\{\|PS(t)x\|_X, x \in B, t \ge t(B)\}$ is bounded and

$$\|(I-P)\,S(t)\,x\|_X < \varepsilon, \quad t \ge t(B), \ x \in B, \tag{8}$$

where $P : X \longrightarrow X_1$ is a bounded projector.

Theorem 3 (see [14]). *Let X be a Banach space and $\{S(t)\}_{t\ge0}$ be a norm-to-weak continuous semigroup on X. Then $\{S(t)\}_{t\ge0}$ has a global attractor in X provided that the following conditions hold true:*

(i) $\{S(t)\}_{t\ge0}$ has a bounded absorbing set B_0 in X.

(ii) $\{S(t)\}_{t\ge0}$ satisfies the condition (C).

Theorem 4 (see [16, 18]). *Let X and Y be two Banach spaces such that $X \subset Y$ with a continuous injection. If a function φ belongs to $L^\infty(0,T;X)$ and is weakly continuous with values in Y, then φ is weakly continuous with values in X.*

Theorem 5 (see [16, 18]). *Let X, Y be two Banach spaces and X^*, Y^* be their dual spaces, respectively, such that*

$$\begin{aligned} X &\hookrightarrow Y, \\ Y^* &\hookrightarrow X^* \end{aligned} \tag{9}$$

where the injection $i : X \longrightarrow Y$ is continuous and its adjoint, $i^* : Y^* \hookrightarrow X^*$, is a densely injective. Let $\{S(t)\}_{t\ge0}$ be a semigroup on X and Y, respectively, and be a continuous semigroup or a weak continuous semigroup on Y. Then for any bounded subset B of X, $\{S(t)\}_{t\ge0}$ is norm-to-weak continuous on $S(B)$.

Theorem 6 (see [16–18]). *Assume that $\{S(t)\}_{t\ge0}$ is a semigroup on Banach space X and satisfies that*

(i) $\{S(t)\}_{t\ge0}$ has a bounded absorbing set B_0 in X;

(ii) $\{S(t)\}_{t\ge0}$ satisfies condition (C) or $\{S(t)\}_{t\ge0}$ is ω-limit compact in X.

And assume furthermore that $\{S(t)\}_{t\ge0}$ is norm-to-weak continuous on $S(B_0)$. Then $\{S(t)\}_{t\ge0}$ has a global attractor $\mathscr{A}$ in X; i.e., $\mathscr{A}$ is nonempty, invariant, compact in X and attracts every bounded subset of X in the norm topology of X.

3. A Priori Estimate

Next we iterate some main results in [4], which are important for getting a priori estimate.

For any initial data $u_0 \in v_1, u_1 \in H$, problem (1) possesses a unique weak solution u, which satisfies $u \in C(R_+; V_1), u_t \in C(R_+; H)$, and, for any $t \ge 0$,

$$\|\xi_u\|_{E_1}^2 \le Q\left(\|\xi_u(0)\|_{E_1}\right) + Q\left(\|f\|\right). \tag{10}$$

Furthermore problem (1) generates a dynamical system of solution within the space $H_0^2(\Omega) \times L^2(\Omega)$. This system possesses a compact global attractor $\mathscr{A}$, which is bounded in $[H^4(\Omega) \cap H_0^2(\Omega)] \times H_0^2(\Omega)$.

Choosing $0 < \varepsilon < 2/3$, taking the scalar product in H of the first equation of (1) with $Av = Au_t + \varepsilon Au$, we find

$$\begin{aligned} &\frac{d}{dt}\left[\|\xi_u\|_{E_2}^2 + 2\varepsilon(u_t, Au) - 2(f, Au) + 2(\varphi(u), Au)\right] \\ &+ 2\|u_t\|_{V_1}^2 - 2\varepsilon(u_t, Au_t) + 2\varepsilon(u_t, Au) + 2\varepsilon\|u\|_{V_2}^2 \\ &+ 2\varepsilon(\varphi(u), Au) - 2\varepsilon(f, Au) - 2(\varphi'(u)u_t, Au) \\ &= 0. \end{aligned} \tag{11}$$

Owing to [4] and *Hölder* inequality, we obtain

$$\begin{aligned} 2(\varphi'(u)u_t, Au) &\le |\varphi'(u)|\,\|u_t\|\,\|Au\| \\ &\le \varepsilon\|u\|_{V_2}^2 + Q(R), \end{aligned} \tag{12}$$

where $R = Q(\|\xi_u(0)\|_{E_1}) + Q(\|f\|)$. Furthermore we see from (11) that

$$\begin{aligned} &\frac{d}{dt}\left[\|\xi_u\|_{E_2}^2 + 2\varepsilon(u_t, Au) - 2(f, Au) + 2(\varphi(u), Au)\right] \\ &+ (2 - 2\varepsilon)\|u_t\|_{V_1}^2 + 2\varepsilon(u_t, Au) + \varepsilon\|u\|_{V_2}^2 \\ &+ 2\varepsilon(\varphi(u), Au) - 2\varepsilon(f, Au) \le Q(R). \end{aligned} \tag{13}$$

We denote the energy functions as follows:

$$E = \|\xi_u\|_{E_2}^2 + 2\varepsilon(u_t, Au) - 2(f, Au) + 2(\varphi(u), Au). \tag{14}$$

Combining with (13), (12), and $0 < \varepsilon < 2/3$, we get

$$\frac{dE}{dt} + \varepsilon E \le Q(R). \tag{15}$$

By the Gronwall inequality, it follows that

$$E \le E(\xi(t_0))e^{-\varepsilon(t-t_0)} + \frac{F(R)}{\varepsilon}\left(1 - e^{-\varepsilon(t-t_0)}\right), \quad \forall t \ge t_0. \tag{16}$$

Next, we show that

$$\frac{1}{2}\|\xi_u\|_{E_2}^2 - Q(R,\varepsilon) \le E \le \frac{3}{2}\|\xi_u\|_{E_2}^2 + Q(R,\varepsilon). \tag{17}$$

By *Hölder* inequality, Young's inequality, Poincaré inequality, and [4], we conclude that

$$\begin{aligned} E &\le \|\xi_u\|_{E_2}^2 + 2\varepsilon\|u_t\|_{V_1}\|u\|_{V_1} + 2\|f\|\,\|u\|_{V_2} \\ &\quad + 2|\varphi(u)|\,\|u\|_{V_2} \le \|\xi_u\|_{E_2}^2 + \frac{1}{2}\|\xi_u\|_{E_2}^2 + Q(R,\varepsilon) \\ &= \frac{3}{2}\|\xi_u\|_{E_2}^2 + Q(R,\varepsilon), \end{aligned} \tag{18}$$

and

$$\begin{aligned} E &\ge \|\xi_u\|_{E_2}^2 - 2\varepsilon c\|u_t\|_{V_1}\|Au\| - 2\|f\|\,\|Au\| \\ &\quad - 2|\varphi(u)|\,\|Au\|_{L^2} \\ &= \|\xi_u\|_{E_2}^2 - 2\varepsilon c\left(\frac{1}{4}\|Au\|^2 + \|u_t\|_{V_1}^2\right) - 2\varepsilon c\frac{3}{8}\|Au\|^2 \\ &\quad - 2\varepsilon c\frac{3}{8}\|Au\|^2 - Q(R,\varepsilon). \\ &= (1 - 2\varepsilon c)\|\xi_u\|_{E_2}^2 - Q(R,\varepsilon) > \frac{1}{2}\|\xi_u\|_{E_2}^2 - Q(R,\varepsilon). \end{aligned} \tag{19}$$

Select ε small enough to verify that

$$1 - 2\varepsilon c > \frac{1}{2}. \tag{20}$$

From (16) and (17), we have

$$\begin{aligned} \|\xi_u\|_{E_2}^2 &\le 2E + Q(R,\varepsilon) \\ &\le 2E(\xi(t_0))e^{-\varepsilon(t-t_0)} + \frac{F(R)}{\varepsilon}\left(1 - e^{-\varepsilon(t-t_0)}\right) \\ &\quad + Q(R,\varepsilon) \\ &\le 2\left(\frac{3}{2}\|\xi_u(t_0)\|_{E_2}^2 + Q(R,\varepsilon)\right)e^{-\varepsilon(t-t_0)} + \frac{F(R)}{\varepsilon} \\ &\quad + Q(R,\varepsilon) \le 3\|\xi_u(t_0)\|_{E_2}^2 e^{-\varepsilon(t-t_0)} + Q(R,\varepsilon). \end{aligned} \tag{21}$$

Now if $B \subset B_{E_2(P_0,\rho)}$, the ball of E_2, center at P_0 of radius ρ, then $\|\xi_u(t_0)\|_{E_2} \le \rho$, provided

$$t - t_0 > \frac{1}{\varepsilon}\ln 3\rho^2. \tag{22}$$

So

$$\|\xi_u\|_{E_2}^2 \le 1 + Q(R) = \mu. \tag{23}$$

We denote $\mu = 1 + Q(R)$.

4. Existence of the Strong Solution

Theorem 7. *Suppose $f \in L^2(\Omega), \varphi$ is a $C^2(R)$ function from R into R satisfying (2)-(3) and $\varphi(0) = 0$. Then given $T > 0$, problem (1) has a unique solution $u(x,t)$ with*

$$\begin{aligned} u &\in L^\infty(0,T;D(A)), \\ u_t &\in L^\infty(0,T;V_1) \end{aligned} \tag{24}$$

for $u_0 \in D(A), u_1 \in V_1$. Moreover (u, u_t) is a weakly continuous function from $[0,T]$ to E_2.

Proof. We prove the existence of strong solutions by using the Faedo-Galerkin schemes. Assume that there exists an orthonormal basis of $D(A)$ consisting of eigenvectors ω_i of A

in $D(A)$; simultaneously they are also orthonormal basis of V. The corresponding eigenvalues are $\lambda_i, i = 1, 2, \cdots$, satisfying

$$A\omega_i = \lambda_i \omega_i, \quad \forall i \in N. \tag{25}$$

For this purpose, according to the basis theory of ordinary differential equations, we build the sequence of Galerkin approximate solutions. They are smooth functions of the form

$$u_N = \sum_{i=1}^{N} \mu_{Ni} \omega_i, \tag{26}$$

and it satisfies

$$\begin{gathered} \partial_{tt} u_N + P_N (\partial_t u_N) + A u_N + P_N (\varphi (u_N)) = P_N f, \\ u_N (0) = P_N u_0, \\ \partial_t u_N (0) = P_N u_1, \end{gathered} \tag{27}$$

where $P_m : D(A) \longrightarrow V_m$ is the orthogonal projector in V_m, and $V_m = \operatorname{span}\{\omega_1, \omega_2 \cdots \omega_m\}$. Of course for every N there is the unique solution u_N to (27).

Choosing $0 < \varepsilon < 1$, after multiplying (27) by $AV_m = Au_m' + \varepsilon_0 A u_m$, the similar process of (11) leads to

$$\begin{aligned} &\frac{d}{dt} \left[\left\| \xi_{u_N} \right\|_{E_2}^2 + 2\varepsilon (u_{Nt}, A u_N) - 2 (f, A u_N) \right. \\ &\quad \left. + 2 (\varphi (u_N), A u) \right] + 2 \left\| u_{Nt} \right\|_{V_1}^2 - 2\varepsilon (u_{Nt}, A u_{Nt}) \\ &\quad + 2\varepsilon (u_{Nt}, A u_N) + 2\varepsilon \left\| u_N \right\|_{V_2}^2 + 2\varepsilon (\varphi (u_N), A u_N) \\ &\quad - 2 (f, A u_N) - 2 \left(\varphi' (u_N) u_{Nt}, A u_N \right) = 0. \end{aligned} \tag{28}$$

Like the estimates of (12)-(21), we have

$$\left\| \xi_{u_N} \right\|_{E_2}^2 \le 1 + Q(R). \tag{29}$$

It means that the sequence $\{\xi_{u_N}(t)\}$ is bounded in the space $L^\infty(0, T; D(A))$ for an arbitrary fixed $T > 0$. From (27) we know that

$$\frac{d^2 u_N}{dt} = -P_N \frac{d u_N}{dt} - A u_N - P_N (\varphi (u_N)) + P_N f. \tag{30}$$

So $\{\partial_{tt} u_N\}$ is uniformly bounded in $L^\infty(0, T; H)$; then we can extract a subsequences, still denoted by u_N, such that

$$\begin{gathered} u_N \rightharpoonup u \quad \text{in } L^2 (0, T; D(A)), \\ u_N' \rightharpoonup u' \quad \text{in } L^2 \left(0, T; H^2\right), \\ u_N'' \rightharpoonup u'' \quad \text{in } L^2 (0, T; H), \\ \varphi (u_N) \rightharpoonup \varphi (u) \quad \text{in } L^2 (0, T; H), \\ P_N f \rightharpoonup f \quad \text{in } L^2 (0, T; H), \quad \text{as } N \longrightarrow \infty. \end{gathered} \tag{31}$$

It is easy to pass the limit in (27) and we obtain that u is a solution of (1), such that

$$\begin{gathered} u(t) \in L^\infty (0, T; D(A)), \\ u_t(t) \in L^\infty \left(0, T; V_1\right). \end{gathered} \tag{32}$$

Furthermore, from Theorem 4 and [4], we know that u is a weakly continuous function from $[0, T]$ to E_2.

Finally, uniqueness is followed from [4], since any strong solution would be a weak solution. □

Thus, the dynamical system generated by (1) can be defined in the phase space E_2, and the corresponding solution semigroup is $\{s(t)\}_{t>0}$. By Theorem 7, we have the following results.

Theorem 8. *Suppose the conditions of Theorem 7 hold; then there exists a bounded absorbing set B in E_2 for the semigroup $\{S(t)\}_{t>0}$.*

5. Global Attractors in E_2

We first prove the following compactness results and the norm-to-weak continuity of semigroup.

Lemma 9. *Suppose that (2) and (3) hold; $\varphi(0) = 0, \varphi(u) : D(A) \longrightarrow V_1$ is defined by*

$$((\varphi (u), v)) = \int_\Omega \triangle \varphi (u) \triangle v dx, \tag{33}$$

$\forall u \in D(A), v \in H_0^2(\Omega)$. Then φ is continuous compact.

Proof. Suppose that u_n is bounded sequences in $D(A)$; without lose of generality, we assume that u_n weakly converge to u_0 in $D(A)$. By the Sobolev embedding theorem, we know that u_n is bounded and converges to u_0 in $L^P(0, T)$, $W^{1,P}(0, T), W^{2,P}(0, T), \forall P \ge 1, \forall T > 0$. Denote $u_n - u_0 = \omega_n$; by the results of [4] and the Sobolev embedding theorem, we show that $\varphi'(u), \varphi''(u), \varphi'''(u)$ are uniformly bounded in L^∞; that is, there exists a constant $M > 0$, such that

$$\begin{gathered} \left| \varphi' (u) \right|_{L^\infty} \le M, \\ \left| \varphi'' (u) \right|_{L^\infty} \le M, \\ \left| \varphi''' (u) \right|_{L^\infty} \le M, \end{gathered} \tag{34}$$

since there exists constant $0 < \theta < 1$, such that

$$\begin{aligned} &\left(\int_0^L \left| \frac{\partial^2 (\varphi (u_n) - \varphi (u))}{\partial x^2} \right|^2 dx \right)^{1/2} \\ &\quad \le \left(\int_0^L \left| \varphi'' (u_0 + \theta \omega_n) \frac{\partial^2 (u_0 + \theta \omega_n)}{\partial x^2} \omega_n \right|^2 dx \right)^{1/2} \\ &\quad + \left(\int_0^L \left| \varphi''' (u_0 + \theta \omega_n) \left(\frac{\partial (u_0 + \theta \omega_n)}{\partial x} \right)^2 \right. \right. \\ &\quad \left. \left. \cdot \omega_n \right|^2 dx \right)^{1/2} \end{aligned}$$

$$+\left(\int_0^L\left|\varphi'\left(u_0+\theta\omega_n\right)\frac{\partial^2\omega_n}{\partial x^2}\right|^2dx\right)^{1/2}$$
$$+2\left(\int_0^L\left|\varphi''\left(u_0+\theta\omega_n\right)\right.\right.$$
$$\left.\left.\cdot\left(\frac{\partial\left(u_0+\theta\omega_n\right)}{\partial x}\right)\frac{\partial\omega_n}{\partial x}\right|^2dx\right)^{1/2}$$
$$\leq M\left(\int_0^L\left|\frac{\partial^2\left(u_0+\theta\omega_n\right)}{\partial x^2}\omega_n\right|^2dx\right)^{1/2}$$
$$+M\left(\int_0^L\left|\left(\frac{\partial\left(u_0+\theta\omega_n\right)}{\partial x}\right)^2\omega_n\right|^2dx\right)^{1/2}$$
$$+M\left(\int_0^L\left|\frac{\partial^2\omega_n}{\partial x^2}\right|^2dx\right)^{1/2}$$
$$+2M\left(\int_0^L\left|\left(\frac{\partial\left(u_0+\theta\omega_n\right)}{\partial x}\right)\frac{\partial\omega_n}{\partial x}\right|^2dx\right)^{1/2} \tag{35}$$

Because u_n converges to u_0 in $L^P(0,T), W^{1,P}(0,T), W^{2,P}(0,T), \forall P\geq 1, \forall T>0$, we have

$$\lim_{n\longrightarrow\infty}\left(\int_0^L\left|\frac{\partial^2}{\partial x^2}\left(\varphi\left(u_n\right)-\varphi\left(u_0\right)\right)\right|^2dx\right)^{1/2}=0 \tag{36}$$

The proof is completed. □

Similarly, we can prove the following Lemma.

Lemma 10. *Let $g(u,u_t) = \varphi'(u)u_t$ and (2) and (3) hold, $\varphi(0)=0$. Then $g: D(A)\times V_1 \longrightarrow H)$ and $\varphi'(u): D(A)\longrightarrow H$ are continuous compact.*

Since $D(A)\times V_1 \hookrightarrow V_1\times L^2$, from Theorem 5 and [4], we can immediately obtain the following result.

Lemma 11. *The semigroup $\{S(t)\}_{t\geq 0}$ associated with (1) is norm-to weak continuous on $S(B)$, where B is abounded absorbing set of $\{s(t)\}_{t\geq 0}$ in E_2 and $S(B)$ is the stationary set of B defined by*

$$S(B)=\{x\in B\mid S(t)\ \ x\in B,\ \forall t\geq 0\}. \tag{37}$$

Now we give our main results of the paper.

Theorem 12. *Suppose that $f\in L^2(\Omega), \varphi\in C^3(R,R), \varphi(0)=0$, and (2) and (3) hold. Then the solution semigroup $\{s(t)\}_{t\geq 0}$ of problem (1) has global attractor $\mathscr{A}$ in E_2; it attracts all bounded subset of E_2 in the norm of E_2.*

Proof. Applying Theorems 7 and 8, we only need to verify that $\{S(t)\}_{t\geq 0}$ satisfies condition (C) in E_2.

Let $\lambda_1,\lambda_2,\cdots$ be the eigenvalues of A in $D(A)$ and $\omega_1,\omega_2,\cdots$ be the corresponding eigenvectors such that

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq\cdots, \tag{38}$$

where $\lambda_n \longrightarrow \infty$, as $n\longrightarrow\infty$ and $\{\omega_1,\omega_2,\cdots\}$ forms an orthogonal basis in $D(\mathrm{A})$ and $H_0^2(\Omega)$.

Let $V_m = \operatorname{span}\{\omega_1,\omega_2,\cdots\omega_m\}$ and P_m be the canonical projector on V_m and I be the identity. Then, for any $(u,u_t)\in E_2$, it has a unique decomposition $(u,u_t) = (u_1,u_{1t}) + (u_2,u_{2t})$, where

$$\begin{aligned}(u,u_t)&=\left(P_mu,P_mu_t\right),\\ \left(u_2,u_{2t}\right)&=\left(I-P_m\right)\left(u,u_t\right).\end{aligned} \tag{39}$$

Since $f\in L^2(\Omega)$ and $\varphi': D(A)\longrightarrow H(\Omega)$ are compact continuously verified by Lemma 10, then, for any $\varepsilon>0$, there exists $N>0$, such that, for any $m>N$, we have

$$\begin{aligned}&\left|\left(I-P_m\right)f\right|<\frac{\varepsilon}{8},\\ &\left\|\left(I-P_m\right)f\right\|<\frac{\varepsilon}{8},\\ &\left\|\left(I-P_m\right)\varphi'(u)\right\|<\frac{\varepsilon}{8},\quad \forall u\in B_{V_2}(0,\mu),\end{aligned} \tag{40}$$

where μ is given by (23).

Multiplying (1) by $Av = Au_{2t}+\alpha Au_2$, we can get

$$\begin{aligned}&\frac{d}{dt}\left[\left\|\xi_{u_2}\right\|_{E_2}^2+2\alpha\left(u_{2t},Au_2\right)+2\left(\varphi(u),Au_2\right)\right]\\ &\quad+2\left\|u_{2t}\right\|_{V_1}^2-2\alpha\left(u_{2t},Au_{2t}\right)+2\alpha\left(u_{2t},Au_2\right)\\ &\quad+2\alpha\left\|u_2\right\|_{V_2}^2+2\alpha\left(\varphi(u),Au_2\right)\\ &=2\left(f,Au_{2t}\right)+2\left(\varphi'(u)u_{2t},Au_2\right)+2\alpha\left(f,Au_2\right).\end{aligned} \tag{41}$$

Applying the Young inequality, *Hölder* inequality, Sobolev embedding theorem, and (40) and (41), the three terms in the right-hand side of (41) can be estimated as follows:

$$\begin{aligned}2\left(f,Au_{2t}\right)&\leq 2\left|\left(f,Au_{2t}\right)\right|\leq 2\left\|(f)_2\right\|_{V_1}\left\|u_{2t}\right\|_{V_1}\\ &\leq\frac{\varepsilon}{4}\left\|u_{2t}\right\|_{H2}\leq\left\|u_{2t}\right\|_{H2}^2+C_2\varepsilon^2;\\ 2\left(\varphi'(u),Au_2\right)&\leq 2\left|\left(\varphi'(u)u_t,Au_2\right)\right|\leq 2\cdot\frac{\varepsilon}{8}\left\|u_2\right\|_{V_2}\\ &\leq\frac{\alpha}{2}\left\|u_2\right\|_{V_2}^2+C_1\varepsilon^2;\\ 2\alpha\left(f,Au_2\right)&\leq 2\alpha\left|\left(f,Au_2\right)\right|\leq 2\left\|(f)_2\right\|_{V_1}\left\|u_2\right\|_{V_1}\\ &\leq 2C\alpha\cdot\frac{\varepsilon}{8}\left\|u_2\right\|_{V_2}\leq\frac{\alpha}{2}\left\|u_2\right\|_{V_2}^2+C_3\varepsilon^2.\end{aligned} \tag{42}$$

Using the above estimates, we transform (41) as follows:

$$\frac{d}{dt}\left[\left\|\xi_{u_2}\right\|_{E_2}^2+2\alpha\left(u_{2t},Au_2\right)+2\left(\varphi\left(u\right),Au_2\right)\right] + 2\alpha\left(u_{2t},Au_2\right)+\alpha\left\|u_2\right\|_{V_2}^2+(1-2\alpha)\left\|u_{2t}\right\|_{V_1}^2 + 2\alpha\left(\left(\varphi\left(u\right)\right)_2,Au_2\right)\leq C\varepsilon^2, \quad C=C_1+C_2+C_3. \tag{43}$$

Choose α small enough such that $1-2\alpha>\alpha$ and $2\alpha>2\alpha^2$. Hence (43) can be rewritten as

$$\frac{d}{dt}\left[\left\|\xi_{u_2}\right\|_{E_2}^2+2\alpha\left(u_{2t},Au_2\right)+2\left(\varphi\left(u\right),Au_2\right)\right] + \alpha[\left\|\xi_{u_2}\right\|^2+2\alpha\left(u_{2t},Au_2\right)+2\left(\left(\varphi\left(u\right)\right)_2,Au_2\right) \leq C\varepsilon^2. \tag{44}$$

Denote $Y(t)=\|\xi_{u_2}\|_{E_2}^2+2\alpha(u_{2t},Au_2)+2(\varphi(u),Au_2)$ we arrive at

$$\frac{dY\left(t\right)}{dt}+\alpha Y\left(t\right)\leq C\varepsilon^2. \tag{45}$$

By Gronwall inequality

$$\begin{aligned} Y\left(t\right)&\leq Y\left(t_1\right)e^{-\alpha(t-t_1)}+\frac{C\varepsilon^2}{\alpha}\left(1-e^{-\alpha(t-t_1)}\right)\\ &\leq Y\left(t_1\right)e^{-\alpha(t-t_1)}+\frac{C\varepsilon^2}{\alpha}. \end{aligned} \tag{46}$$

Next, we show that

$$\frac{1}{2}\left\|\xi_{u_2}\right\|_{E_2}^2-C_4\varepsilon^2\leq Y\left(t\right)\leq\frac{3}{2}\left\|\xi_{u_2}\right\|_{E_2}^2+C_4\varepsilon^2. \tag{47}$$

Indeed, the right inequality is obtained using *Hölder* inequality, Young inequality, and Lemma 9:

$$\begin{aligned} Y\left(t\right)&\leq\left\|\xi_{u_2}\right\|_{E_2}^2+2\alpha\left\|u_{2t}\right\|_{V_1}\left\|u_2\right\|_{V_1} + 2\left\|\left(\varphi\left(u\right)\right)_2\right\|\left\|Au_2\right\|\\ &\leq\left\|\xi_{u_2}\right\|_{E_2}^2+2\alpha\varepsilon\left\|u_{2t}\right\|_{V_1}^2+2\varepsilon\left\|Au_2\right\|\\ &\leq\left\|\xi_{u_2}\right\|_{E_2}^2+\frac{1}{2}\left\|u_{2t}\right\|_{V_1}^2+\frac{1}{2}\left\|Au_2\right\|+2\alpha^2\varepsilon^2+2\varepsilon^2\\ &\leq\frac{3}{2}\left\|\xi_{u_2}\right\|_{E_2}^2+C_4\varepsilon^2. \end{aligned} \tag{48}$$

and the left one is the same, where $C_4=2(1+\alpha^2)$.

Thus, combining (46) and (47) and Theorem 8, we deduce

$$\begin{aligned} \left\|\xi_{u_2}\right\|_{E_2}^2&\leq 2Y\left(t\right)+2C_4\varepsilon^2\\ &\leq 2\left(\frac{3}{2}R_3^2+C_4\varepsilon^2\right)e^{-\alpha(t-t_1)}+\frac{2C\varepsilon^2}{\alpha}+2C_4\varepsilon^2\\ &\leq 3R_3^2e^{-\alpha(t-t_1)}+C_5\varepsilon^2. \end{aligned} \tag{49}$$

Taking $t-t_1>t(R_3)$, it follows that

$$\left\|\xi_{u_2}\right\|_{E_2}^2\leq\left(1+C_5\varepsilon^2\right), \tag{50}$$

where, by (29), we denote $R^3=1+Q(R)$, $C_5=C/\alpha+C_4$, $t(R_3)=t_1+(1/\alpha)\ln 3R_3^2$.

Together with Theorems 5, 7, and 8, the proof is finished. □

Remark 13. If we transform the first equation of (1) to the following form,

$$u_{tt}+\sigma\left(u\right)u_t+\triangle^2u+\varphi\left(u\right)=f, \tag{51}$$

then the term $\sigma(u)u_t$ accounts for dynamical friction, and we only need to assume the damping coefficient $\sigma(u)$ is a positive function and by adding some appropriate conditions of continuity, conclusions of Theorem 12 remain valid and the proof's process has no essential difference.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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