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Research Article

Improved Approach to Robust Control for Type-2 T-S Fuzzy Systems

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This paper is concerned with the robust stability conditions to stabilize the type 2 Takagi-Sugeno (T-S) fuzzy systems. The conditions effectively handle parameter uncertainties using lower and upper membership functions. To improve the solvability of the stability conditions, we establish a multigain controller with comprehensive information of the lower and upper membership grades. In addition, a well-organized relaxation technique is proposed to fully exploit relationship among fuzzy weighting functions and their lower and upper membership grades, which enlarges a set of feasible solutions. Therefore, we derive a less conservative stabilization condition in terms of linear matrix inequalities (LMIs) than those in the literature. Two simulation examples illustrate the effectiveness and robustness of the derived stabilization conditions.

1. Introduction

Over the past few decades, the type 1 Takagi-Sugeno (T-S) fuzzy model has attracted much attention because it can systematically represent nonlinear systems via an interpolation method that smoothly connects some local linear systems based on fuzzy weighting functions [1–3]. The main advantage of type 1 T-S fuzzy systems is that they allow us to apply the well-established linear system theory for the analysis and synthesis of nonlinear systems. For this reason, the type 1 T-S fuzzy model has been a popular choice not only in consumer products but also in industrial processes, such as power converters [4], motors [5], and solar power generator systems [6].

For the stability analysis and synthesis of type 1 fuzzy control systems, Lyapunov stability theory is widely used [7–10]. Fundamental stability conditions in terms of linear matrix inequalities (LMIs) are derived from Lyapunov stability condition. The conditions guarantee the stability of the fuzzy control systems if there exists a solution to a set of LMIs. Many researchers introduced the stability conditions and relaxed stability conditions using parallel distributed compensation (PDC) concept [8]. Using the information of type 1 fuzzy membership functions, the stability conditions can be further relaxed [11–14].

Although the type 1 fuzzy control system can effectively handle the nonlinear systems, it cannot guarantee the stability of the nonlinear systems with parameter uncertainties. Recently, type 2 fuzzy systems have attracted a lot of research attention [15] because they are better at handling uncertainties than the conventional type 1 fuzzy systems [16, 17]. Hence, for the stability analysis and controller synthesis of nonlinear systems with parameter uncertainties, it is essential to use type 2 fuzzy systems. Several researchers have researched such type 2 fuzzy systems [18–20]. However, all the aforementioned papers have seldom studied stability analysis and controller synthesis for type 2 T-S fuzzy systems. This motivates the study of the stability analysis and controller synthesis of type 2 T-S fuzzy systems.

Recently, some researchers have studied stability analysis and controller synthesis for type 2 T-S fuzzy systems [21–25]. In [21], an interval type 2 T-S fuzzy controller was proposed using a common controller gain that collectively depends on the sum of lower upper membership grades. In [22], the controller design for the interval type 2 T-S fuzzy system

was introduced using a membership function different from a membership function of the system.

In the above studies based on the type 2 T-S fuzzy systems, the stability conditions for the design of the type 2 fuzzy controller have some tuning parameters, which can result in increasing the implementation effort. It motivates the study of the controller synthesis for type 2 T-S fuzzy system. This paper studies the robust stability conditions to stabilize type 2 T-S fuzzy systems. The conditions effectively handle parameter uncertainties using lower and upper membership functions. To improve the solvability of the stability conditions, we establish a multigain controller with comprehensive information of the lower and upper membership grades. In addition, we propose a well-organized relaxation technique that fully exploits relationship among fuzzy weighting functions and their lower and upper membership grades, which enlarges a set of feasible solutions. Therefore, we derive a less conservative stabilization condition in terms of LMIs than those in the literature. The proposed condition has a simple structure without tuning parameters. Finally, two simulation examples are given to illustrate the effectiveness and robustness of the derived stabilization condition.

Notation. The notations $X \geq Y$ and $X > Y$ mean that $X - Y$ is positive semidefinite and positive definite, respectively. In symmetric block matrices, $(*)$ is used as an ellipsis for terms that are induced by symmetry. Furthermore, $\mathrm{Sym}(X) = X + X^T$ stands for any matrix X.

2. System Description and Preliminaries

Let us consider the following type 2 T-S fuzzy model [21] that represents a continuous-time nonlinear system: for $i \in \mathscr{R} = \{1, 2, \ldots, p\}$,

$$\begin{aligned} &\text{Plant Rule } i\text{:} \quad \textbf{IF } f_1(x(t)) \text{ is } \widetilde{\mathscr{F}}_{i1} \text{ and } \ldots f_s(x(t)) \text{ is } \widetilde{\mathscr{F}}_{is}, \\ &\qquad \textbf{THEN } \dot{x}(t) = A_i x(t) + B_i u(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathscr{R}^n$, $u(t) \in \mathscr{R}^m$ denote the state and control input, respectively; $\widetilde{\mathscr{F}}_{ij}$ denotes a type 2 fuzzy set of rules i corresponding to the function $f_j(x(t))$; and p denotes the number of IF-THEN rules. The firing interval of the ith rule is as follows:

$$\begin{aligned} \widetilde{\theta}_i(x(t)) = \Big[&\underline{\mu}_{\widetilde{\mathscr{F}}_{i1}}(f_1(x(t))) \times \cdots \\ &\times \underline{\mu}_{\widetilde{\mathscr{F}}_{is}}(f_s(x(t))), \overline{\mu}_{\widetilde{\mathscr{F}}_{i1}}(f_1(x(t))) \times \cdots \\ &\times \overline{\mu}_{\widetilde{\mathscr{F}}_{is}}(f_s(x(t)))\Big] \equiv \left[\theta_i^L(x(t)), \theta_i^U(x(t))\right], \end{aligned} \tag{2}$$

where $\underline{\mu}_{\widetilde{\mathscr{F}}_{ij}}(f_j(x(t))) \in [0,1]$ and $\overline{\mu}_{\widetilde{\mathscr{F}}_{ij}}(f_j(x(t))) \in [0,1]$ denote the lower and upper grades of membership associated with the lower and upper membership functions, respectively. Here, $\underline{\mu}_{\widetilde{\mathscr{F}}_{ij}}(f_j(x(t))) \leq \overline{\mu}_{\widetilde{\mathscr{F}}_{ij}}(f_j(x(t)))$, which leads to $\theta_i^L(x(t)) \leq \theta_i^U(x(t))$ for all i. The overall type 2 T-S fuzzy system is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{p} \theta_i(x(t)) \left(A_i x(t) + B_i u(t)\right), \tag{3}$$

where $\theta_i(x(t)) = \eta(x(t))\theta_i^L(x(t)) + (1 - \eta(x(t)))\theta_i^U(x(t))$ denotes a fuzzy weighting function in which $\eta(x(t)) \in [0,1]$ is a nonlinear function and not necessary to be considered in this paper. Now, consider a multigain controller that is *individually* dependent on the lower and upper membership grades such as

$$u(t) = \sum_{j=1}^{p} \left(\underline{\theta}_j(x(t))\underline{K}_j + \overline{\theta}_j(x(t))\overline{K}_j\right) x(t), \tag{4}$$

where

$$\begin{aligned} \underline{\theta}_j(x(t)) &\triangleq \frac{\theta_j^L(x(t))}{\sum_{l=1}^{p}\left(\theta_l^L(x(t)) + \theta_l^U(x(t))\right)}, \\ \overline{\theta}_j(x(t)) &\triangleq \frac{\theta_j^U(x(t))}{\sum_{l=1}^{p}\left(\theta_l^L(x(t)) + \theta_l^U(x(t))\right)}, \end{aligned} \tag{5}$$

and $\underline{K}_j$ and $\overline{K}_j$ are the controller gains associated with the lower and upper membership grades. By the above relations, $\theta_i(x(t))$, $\underline{\theta}_i(x(t))$, and $\overline{\theta}_i(x(t))$ satisfy the following conditions:

$$\sum_{i=1}^{p} \theta_i(x(t)) = \sum_{j=1}^{p} \left(\underline{\theta}_j(x(t)) + \overline{\theta}_j(x(t))\right) = 1, \tag{6}$$

$$0 \leq \alpha_i \leq \theta_i(x(t)) \leq \beta_i \leq 1, \tag{7}$$

$$0 \leq \underline{\alpha}_i \leq \underline{\theta}_i(x(t)) \leq \underline{\beta}_i \leq 1, \tag{8}$$

$$0 \leq \overline{\alpha}_i \leq \overline{\theta}_i(x(t)) \leq \overline{\beta}_i \leq 1, \tag{9}$$

$$0 \leq \underline{\theta}_i(x(t)) + \overline{\theta}_i(x(t)) \leq 1, \tag{10}$$

$$0 \leq \theta_i(x(t)) \left(\underline{\theta}_j(x(t)) + \overline{\theta}_j(x(t))\right), \tag{11}$$

$$\underline{\theta}_i(x(t)) \leq \overline{\theta}_i(x(t)), \tag{12}$$

where α_i, $\underline{\alpha}_i$, $\overline{\alpha}_i$, β_i, $\underline{\beta}_i$, and $\overline{\beta}_i$ are real constant values. Henceforth, for a simple description, we use the following notations: $\theta_i(x(t)) \triangleq \theta_i$, $\underline{\theta}_i(x(t)) \triangleq \underline{\theta}_i$, and $\overline{\theta}_i(x(t)) \triangleq \overline{\theta}_i$. The resulting closed-loop system under (4) is represented as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{p} \theta_i \left\{A_i x(t) + B_i \sum_{j=1}^{p} \left(\underline{\theta}_j \underline{K}_j + \overline{\theta}_j \overline{K}_j\right) x(t)\right\} \\ &= \left\{\sum_{i=1}^{p} \theta_i A_i + \sum_{i=1}^{p}\sum_{j=1}^{p} \left(\theta_i \underline{\theta}_j B_i \underline{K}_j + \theta_i \overline{\theta}_j B_i \overline{K}_j\right)\right\} x(t). \end{aligned} \tag{13}$$

3. Stabilization Conditions

Theorem 1. *Suppose that matrices* $R_0, S_0, R_i, \underline{R}_i, \overline{R}_i, S_i, \underline{S}_i, \overline{S}_i, X_i, Y_i, Z_i, \Pi_i, \Xi_{ij}, \Lambda_i \in \mathscr{R}^{n\times n}$, $\underline{L}_i, \overline{L}_i \in \mathscr{R}^{m\times n}$, *and a positive definite matrix* $\overline{P} \in \mathscr{R}^{n\times n}$ *exist such that, for* $i, j = \{1, 2, \ldots, p\}$,

$$\begin{aligned} X_i + X_i^T &> 0, \\ Y_i + Y_i^T &> 0, \\ Z_i + Z_i^T &> 0, \\ \Pi_i + \Pi_i^T &> 0, \\ \Xi_{ij} + \Xi_{ij}^T &> 0, \\ \Lambda_i + \Lambda_i^T &> 0, \\ \Sigma &< 0, \end{aligned} \tag{14}$$

where Σ *is shown as follows:*

$$\Sigma = \left[\begin{array}{c|cccc|cccc|cccc}
\mathscr{L}_0 & \mathscr{L}_{(0,1)} & \mathscr{L}_{(0,2)} & \cdots & \mathscr{L}_{(0,p)} & \underline{\mathscr{L}}_{(0,1)} & \underline{\mathscr{L}}_{(0,2)} & \cdots & \underline{\mathscr{L}}_{(0,p)} & \overline{\mathscr{L}}_{(0,1)} & \overline{\mathscr{L}}_{(0,2)} & \cdots & \overline{\mathscr{L}}_{(0,p)} \\
\hline
(*) & \Delta_{(1,1)} & \mathscr{L}_{(1,2)} & \cdots & \mathscr{L}_{(1,p)} & \underline{\Phi}_{(1,1)} & \cdots & \cdots & \underline{\Phi}_{(1,p)} & \overline{\Phi}_{(1,1)} & \cdots & \cdots & \overline{\Phi}_{(1,p)} \\
(*) & (*) & \ddots & & \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots \\
(*) & (*) & (*) & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
(*) & (*) & (*) & (*) & \Delta_{(p,p)} & \underline{\Phi}_{(p,1)} & \cdots & \cdots & \underline{\Phi}_{(p,p)} & \overline{\Phi}_{(p,1)} & \cdots & \cdots & \overline{\Phi}_{(p,p)} \\
\hline
(*) & (*) & (*) & (*) & (*) & \underline{\Delta}_{(1,1)} & \underline{\mathscr{L}}_{(1,2)} & \cdots & \underline{\mathscr{L}}_{(1,p)} & \widetilde{\Delta}_{(1,1)} & \cdots & \cdots & \widetilde{\Phi}_{(1,p)} \\
(*) & (*) & (*) & (*) & (*) & (*) & \ddots & & \vdots & \vdots & \ddots & & \vdots \\
(*) & (*) & (*) & (*) & (*) & (*) & (*) & \ddots & \vdots & \vdots & & \ddots & \vdots \\
(*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & \underline{\Delta}_{(p,p)} & \widetilde{\Phi}_{(p,1)} & \cdots & \cdots & \widetilde{\Delta}_{(p,p)} \\
\hline
(*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & \overline{\Delta}_{(1,1)} & \overline{\mathscr{L}}_{(1,2)} & \cdots & \overline{\mathscr{L}}_{(1,p)} \\
(*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & \ddots & & \vdots \\
(*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & \ddots & \vdots \\
(*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & \overline{\Delta}_{(p,p)}
\end{array}\right],$$

$$\mathscr{L}_0 = \left(R_0 + R_0^T\right) + \left(S_0 + S_0^T\right) - \sum_{i=1}^{p} \alpha_i \beta_i \left(X_i + X_i^T\right) - \sum_{i=1}^{p} \underline{\alpha}_i \underline{\beta}_i \left(Y_i + Y_i^T\right) - \sum_{i=1}^{p} \overline{\alpha}_i \overline{\beta}_i \left(Z_i + Z_i^T\right),$$

$$\mathscr{L}_{(0,i)} = A_i \overline{P} + R_i + S_i - R_0 + \left(\alpha_i + \beta_i\right) X_i,$$

$$\underline{\mathscr{L}}_{(0,i)} = \underline{R}_i + \underline{S}_i - S_0 + \left(\underline{\alpha}_i + \underline{\beta}_i\right) Y_i + \Pi_i,$$

$$\overline{\mathscr{L}}_{(0,i)} = \overline{R}_i + \overline{S}_i - S_0 + \left(\overline{\alpha}_i + \overline{\beta}_i\right) Z_i + \Pi_i,$$

$$\Delta_{(i,i)} = -\left(R_i + R_i^T\right) - \left(X_i + X_i^T\right),$$

$$\underline{\Delta}_{(i,i)} = -\left(\underline{S}_i + \underline{S}_i^T\right) - \left(Y_i + Y_i^T\right) - \left(\Pi_i + \Pi_i^T\right) - \left(\Lambda_i + \Lambda_i^T\right),$$

$$\overline{\Delta}_{(i,i)} = -\left(\overline{S}_i + \overline{S}_i^T\right) - \left(Z_i + Z_i^T\right) - \left(\Pi_i + \Pi_i^T\right) + \Lambda_i + \Lambda_i^T,$$

$$\widetilde{\Delta}_{(i,i)} = -\left(\overline{S}_i^T + \underline{S}_i\right) - 2\Pi_i,$$

$$\underline{\Phi}_{(i,j)} = B_i \underline{L}_j - \left(S_i + \underline{R}_j\right) + \Xi_{ij},$$

$$\overline{\Phi}_{(i,j)} = B_i \overline{L}_j - \left(S_i + \overline{R}_j\right) + \Xi_{ij},$$

$$\mathscr{L}_{(i,j)} = -\left(R_i + R_j\right),$$

$$\underline{\mathscr{L}}_{(i,j)} = -\left(\underline{S}_i + \underline{S}_j\right),$$
$$\overline{\mathscr{L}}_{(i,j)} = -\left(\overline{S}_i + \overline{S}_j\right),$$
$$\widetilde{\Phi}_{(i,j)} = -\left(\underline{S}_i + \overline{S}_j\right). \tag{15}$$

Then, the closed-loop system (13) is asymptotically stable and the corresponding fuzzy controller is given by $\underline{K}_j = \underline{L}_j \overline{P}^{-1}$ *and* $\overline{K}_j = \overline{L}_j \overline{P}^{-1}$.

Proof. Choose a quadratic Lyapunov function, $V(t) = x^T(t)Px(t)$, where P is a positive definite matrix. Then the time derivative of $V(x(t))$ is given by

$$\dot{V}(t) = 2x(t)^T P\dot{x}(t) = 2x(t)^T \cdot P\left\{\sum_{i=1}^{p}\theta_i A_i + \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\theta_i\underline{\theta}_j B_i\underline{K}_j + \theta_i\overline{\theta}_j B_i\overline{K}_j\right)\right\} \cdot x(t). \tag{16}$$

For the asymptotic stability of (13), the stability criterion is given as $\dot{V}(t) < 0$. It leads to the following condition:

$$\mathrm{Sym}\left(\sum_{i=1}^{p}\theta_i PA_i + \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\theta_i\underline{\theta}_j PB_i\underline{K}_j + \theta_i\overline{\theta}_j PB_i\overline{K}_j\right)\right) < 0. \tag{17}$$

Multiplying both sides of (17) by P^{-1} yields

$$\mathrm{Sym}\left(\sum_{i=1}^{p}\theta_i A_i\overline{P} + \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\theta_i\underline{\theta}_j B_i\underline{L}_j + \theta_i\overline{\theta}_j B_i\overline{L}_j\right)\right) < 0, \tag{18}$$

where $\overline{P} = P^{-1}$, $\underline{L}_j = \underline{K}_j\overline{P}$, and $\overline{L}_j = \overline{K}_j\overline{P}$. We can then represent (18) as the following form:

$$\mathscr{M} = \sum_{i=1}^{p}\theta_i\left(M_{(0,i)} + M_{(0,i)}^T\right) + \sum_{i=1}^{p}\sum_{j=1}^{p}\theta_i\underline{\theta}_j\left(\underline{M}_{(i,j)} + \underline{M}_{(i,j)}^T\right) + \sum_{i=1}^{p}\sum_{j=1}^{p}\theta_i\overline{\theta}_j\left(\overline{M}_{(i,j)} + \overline{M}_{(i,j)}^T\right) < 0, \tag{19}$$

where $M_{(0,i)} = A_i\overline{P}$, $\underline{M}_{(i,j)} = B_i\underline{L}_j$, and $\overline{M}_{(i,j)} = B_i\overline{L}_j$.

For the relaxation technique using the relationship among fuzzy weighting functions and their lower and upper membership grades, we can convert conditions (6)–(12) for all $i \in [1, p]$, respectively, into

$$0 \equiv C_1 = \begin{bmatrix} I \\ \theta_1 I \\ \vdots \\ \theta_p I \\ \underline{\theta}_1 I \\ \vdots \\ \underline{\theta}_p I \\ \overline{\theta}_1 I \\ \vdots \\ \overline{\theta}_p I \end{bmatrix}^T \begin{bmatrix} I & I \\ -I & 0 \\ \vdots & \vdots \\ -I & 0 \\ 0 & -I \\ \vdots & \vdots \\ 0 & -I \\ 0 & -I \\ \vdots & \vdots \\ 0 & -I \end{bmatrix} \begin{bmatrix} R_0^T & S_0^T \\ R_1^T & S_1^T \\ \vdots & \vdots \\ R_p^T & S_p^T \\ \underline{R}_1 & \underline{S}_1 \\ \vdots & \vdots \\ \underline{R}_p & \underline{S}_p \\ \overline{R}_1 & \overline{S}_1 \\ \vdots & \vdots \\ \overline{R}_p & \overline{S}_p \end{bmatrix}^T \begin{bmatrix} I \\ \theta_1 I \\ \vdots \\ \theta_p I \\ \underline{\theta}_1 I \\ \vdots \\ \underline{\theta}_p I \\ \overline{\theta}_1 I \\ \vdots \\ \overline{\theta}_p I \end{bmatrix}, \tag{20}$$

$$0 \le C_{2i} = -(\theta_i - \alpha_i)(\theta_i - \beta_i)\left(X_i + X_i^T\right), \tag{21}$$

$$0 \le C_{3i} = -(\underline{\theta}_i - \underline{\alpha}_i)\left(\underline{\theta}_i - \underline{\beta}_i\right)\left(Y_i + Y_i^T\right), \tag{22}$$

$$0 \le C_{4i} = -\left(\overline{\theta}_i - \overline{\alpha}_i\right)\left(\overline{\theta}_i - \overline{\beta}_i\right)\left(Z_i + Z_i^T\right), \tag{23}$$

$$0 \le C_{5i} = -\left(\underline{\theta}_i + \overline{\theta}_i\right)\left(\underline{\theta}_i + \overline{\theta}_i - 1\right)\left(\Pi_i + \Pi_i^T\right), \tag{24}$$

$$0 \le C_{6ij} = \theta_i\left(\underline{\theta}_j + \overline{\theta}_j\right)\left(\Xi_{ij} + \Xi_{ij}^T\right), \tag{25}$$

$$0 \le C_{7i} = \left(\overline{\theta}_i^2 - \underline{\theta}_i^2\right)\left(\Lambda_i + \Lambda_i^T\right), \tag{26}$$

where condition (20) is from $\sum_{i=1}^{p}\theta_i I - I = \sum_{i=1}^{p}(\underline{\theta}_i + \overline{\theta}_i)I - I = 0$ and conditions (21)–(26) are from (7)–(12), respectively. Here, the matrices R_0, R_i, $\underline{R}_i$, $\overline{R}_i$, X_i, Y_i, Z_i, Π_i, Ξ_i, and Λ_i are slack variables to reduce conservatism of (19).

Then, merging conditions (20)–(26) gives

$$\mathscr{N} = C_1 + C_1^T + \sum_{i=1}^{p}C_{2i} + \sum_{i=1}^{p}C_{3i} + \sum_{i=1}^{p}C_{4i} + \sum_{i=1}^{p}C_{5i} + \sum_{i=1}^{p}\sum_{j=1}^{p}C_{6ij} + \sum_{i=1}^{p}C_{7i} \ge 0. \tag{27}$$

To derive LMI condition, (27) can be represented as the following form:

$$
\begin{aligned}
0 &\le \mathcal{N} \\
&= N_0 + \sum_{i=1}^{p}\theta_i\left(N_{(0,i)} + N_{(0,i)}^T\right) + \sum_{i=1}^{p}\underline{\theta}_i\left(\underline{N}_{(0,i)} + \underline{N}_{(0,i)}^T\right) \\
&\quad + \sum_{i=1}^{p}\overline{\theta}_i\left(\overline{N}_{(0,i)} + \overline{N}_{(0,i)}^T\right) + \sum_{i=1}^{p}\theta_i^2 N_{(i,i)} + \sum_{i=1}^{p}\underline{\theta}_i^2\underline{N}_{(i,i)} \\
&\quad + \sum_{i=1}^{p}\overline{\theta}_i^2\overline{N}_{(i,i)} + \sum_{i=1}^{p}\underline{\theta}_i\overline{\theta}_i\left(\widetilde{T}_{(i,i)} + \widetilde{T}_{(i,i)}^T\right) \\
&\quad + \sum_{i=1}^{p}\sum_{j=1}^{p}\theta_i\underline{\theta}_j\left(\underline{T}_{(i,j)} + \underline{T}_{(i,j)}^T\right) \\
&\quad + \sum_{i=1}^{p}\sum_{j=1}^{p}\theta_i\overline{\theta}_j\left(\overline{T}_{(i,j)} + \overline{T}_{(i,j)}^T\right) \\
&\quad + \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\theta_i\theta_j\left(N_{(i,j)} + N_{(i,j)}^T\right) \\
&\quad + \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\theta_i\underline{\theta}_j\left(\underline{N}_{(i,j)} + \underline{N}_{(i,j)}^T\right) \\
&\quad + \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\theta_i\overline{\theta}_j\left(\overline{N}_{(i,j)} + \overline{N}_{(i,j)}^T\right) \\
&\quad + \sum_{i=1}^{p}\sum_{j=1,j\ne i}^{p}\underline{\theta}_i\overline{\theta}_j\left(\widetilde{T}_{(i,j)} + \widetilde{T}_{(i,j)}^T\right),
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
N_0 &= R_0 + S_0 + R_0^T + S_0^T - \sum_{i=1}^{p}\alpha_i\beta_i\left(X_i + X_i^T\right) \\
&\quad - \sum_{i=1}^{p}\underline{\alpha}_i\underline{\beta}_i\left(Y_i + Y_i^T\right) - \sum_{i=1}^{p}\overline{\alpha}_i\overline{\beta}_i\left(Z_i + Z_i^T\right), \\
N_{(0,i)} &= R_i + S_i - R_0 + \left(\alpha_i + \beta_i\right)X_i, \\
\underline{N}_{(0,i)} &= \underline{R}_i + \underline{S}_i - S_0 + \left(\underline{\alpha}_i + \underline{\beta}_i\right)Y_i + \Pi_i, \\
\overline{N}_{(0,i)} &= \overline{R}_i + \overline{S}_i - S_0 + \left(\overline{\alpha}_i + \overline{\beta}_i\right)Z_i + \Pi_i, \\
N_{(i,i)} &= -\left(R_i + R_i^T\right) - \left(X_i + X_i^T\right), \\
\underline{N}_{(i,i)} &= -\left(\underline{S}_i + \underline{S}_i^T\right) - \left(Y_i + Y_i^T\right) - \left(\Pi_i + \Pi_i^T\right) \\
&\quad - \left(\Lambda_i + \Lambda_i^T\right), \\
\overline{N}_{(i,i)} &= -\left(\overline{S}_i + \overline{S}_i^T\right) - \left(Z_i + Z_i^T\right) - \left(\Pi_i + \Pi_i^T\right) + \Lambda_i \\
&\quad + \Lambda_i^T, \\
\widetilde{T}_{(i,i)} &= -\left(\overline{S}_i^T + \underline{S}_i\right) - 2\Pi_i, \\
\underline{T}_{(i,j)} &= -\left(S_i + \underline{R}_j\right) + \Xi_{ij}, \\
\overline{T}_{(i,j)} &= -\left(S_i + \overline{R}_j\right) + \Xi_{ij}, \\
N_{(i,j)} &= -\left(R_i + R_j\right), \\
\underline{N}_{(i,j)} &= -\left(\underline{S}_i + \underline{S}_j\right), \\
\overline{N}_{(i,j)} &= -\left(\overline{S}_i + \overline{S}_j\right), \\
\widetilde{T}_{(i,j)} &= -\left(\underline{S}_i + \overline{S}_j\right).
\end{aligned}
\tag{29}
$$

The $\mathcal{S}$-procedure enables the condition in (19) subject to (28) to be expressed as

$$\mathcal{L} = \mathcal{M} + \mathcal{N} < 0, \tag{30}$$

which can be converted into

$$\mathcal{L} = \Theta^T\Sigma\Theta < 0, \tag{31}$$

where

$$\Theta = \left[I \mid \theta_1 \ \cdots \ \theta_p \mid \underline{\theta}_1 \ \cdots \ \underline{\theta}_p \mid \overline{\theta}_1 \ \cdots \ \overline{\theta}_p\right]^T. \tag{32}$$

□

Remark 2. The proposed LMIs are not always feasible for all type 2 T-S fuzzy system. However, the proposed method can guarantee a larger feasible solution set than the previous studies because we use the relationship among fuzzy weighting functions and their lower and upper membership grades and the proposed controller is *individually* dependent on the lower and upper membership grades.

4. Numerical Examples

Example 1. Let us consider a simple fuzzy model [21] in the following format:

$$
\begin{aligned}
\text{Plant Rule } i\text{:} \quad & \textbf{IF } x_1(t) \text{ is } \widetilde{\mathcal{F}}_{i1}, \\
& \textbf{THEN } \dot{x}(t) = A_i x(t) + B_i u(t), \\
& \qquad i = 1, 2, 3,
\end{aligned}
\tag{33}
$$

where

$$A_1 = \begin{bmatrix} 2.78 & -5.63 \\ 0.01 & 0.33 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.2 & -3.22 \\ 0.35 & 0.12 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a & -6.63 \\ 0.45 & 0.15 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix},$$
$$B_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix}, \tag{34}$$

where $14 \le a \le 30$ and $20 \le b \le 35$. The above model is assumed operating in the domain of $x_1(t) \in [-10 \quad 10]^T$. The lower and upper membership functions are chosen as

$$\theta_1^L = \underline{\mu}_{\widetilde{\mathcal{F}}_{11}}(x_1(t)) = 0.95 - \left(\frac{0.925}{(1+e^{-((x_1+3.5)/8)})}\right),$$
$$\theta_3^L = \underline{\mu}_{\widetilde{\mathcal{F}}_{31}}(x_1(t)) = 0.025 + \left(\frac{0.925}{(1+e^{-((x_1-3.5)/8)})}\right),$$
$$\theta_2^L = \underline{\mu}_{\widetilde{\mathcal{F}}_{21}}(x_1(t)) = 1 - \theta_1^U - \theta_3^U,$$
$$\theta_1^U = \overline{\mu}_{\widetilde{\mathcal{F}}_{11}}(x_1(t)) = 0.95 - \left(\frac{0.925}{(1+e^{-((x_1+4.5)/8)})}\right),$$
$$\theta_3^U = \overline{\mu}_{\widetilde{\mathcal{F}}_{31}}(x_1(t)) = 0.025 + \left(\frac{0.925}{(1+e^{-((x_1-4.5)/8)})}\right),$$
$$\theta_2^U = \overline{\mu}_{\widetilde{\mathcal{F}}_{21}}(x_1(t)) = 1 - \theta_1^L - \theta_3^L. \tag{35}$$

For Theorem 1, $\alpha_i = 0$, $\beta_i = 1$, $\underline{\alpha}_1 = \underline{\alpha}_3 = 0.154$, $\underline{\alpha}_2 = 0.164$, $\underline{\beta}_1 = \underline{\beta}_3 = 0.65$, $\underline{\beta}_2 = 0.25$, $\overline{\alpha}_1 = \overline{\alpha}_3 = 0.169$, $\overline{\alpha}_2 = 0.204$, $\overline{\beta}_1 = \overline{\beta}_3 = 0.67$, $\overline{\beta}_2 = 0.3$. Figure 1 shows the stability regions for the proposed conditions and previous conditions in Example 1 of [21]. From Figure 1, we can clearly see that the proposed conditions outperform previous one [21] in terms of a larger stability region. It is an important observation that the proposed conditions with a multigain controller and a well-organized relaxation technique achieve much better performance than those of the previous study [21].

Example 2. Let us consider an inverted pendulum model subject to parameter uncertainties, which is adapted from [26]:

$$\dot{x}_2(t) = \frac{g\sin(x_1(t)) - am_pLx_2^2(t)\sin(2x_1(t))/2 - a\cos(x_1(t))u(t)}{4L/3 - am_pL\cos^2(x_1(t))}, \tag{36}$$

where $x_1(t)$ is the angular displacement of the pendulum; $g = 9.8\,\mathrm{m/s^2}$; m_p and M_c are uncertain in $m_{p_{\min}} = 2 \le m_p \le 5 = m_{p_{\max}}$ and $M_{c\min} = 8 \le M_c \le 18 = M_{c\max}$, respectively; $a = 1/(m_p + M_c)$; $2L = 1$ m; and $u(t)$ is the force applied to the cart. From [21], we can obtain a plant rule to describe the inverted pendulum subject to parameter uncertainties in the following format:

$$\begin{aligned} &\text{Plant Rule } i\text{:} \quad \textbf{IF } x_1(t) \text{ is } \widetilde{\mathcal{F}}_{i1}, \\ &\qquad \textbf{IF } x_1(t) \text{ is } \widetilde{\mathcal{F}}_{i2} \\ &\qquad \textbf{THEN } \dot{x}(t) = A_i x(t) + B_i u(t), \\ &\qquad\qquad i = 1,2,3,4, \end{aligned} \tag{37}$$

where

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ f_{1\min} & 0 \end{bmatrix},$$
$$A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ f_{1\max} & 0 \end{bmatrix},$$
$$B_1 = B_3 = \begin{bmatrix} 0 \\ f_{2\min} \end{bmatrix},$$
$$B_2 = B_4 = \begin{bmatrix} 0 \\ f_{2\max} \end{bmatrix},$$
$$x(t) = [x_1(t) \quad x_2(t)]^T,$$
$$x_1(t) \in [x_{1\min} \quad x_{1\max}] = \left[-\frac{5\pi}{12} \quad \frac{5\pi}{12}\right],$$
$$x_2(t) \in [x_{2\min} \quad x_{2\max}] = [-5 \quad 5],$$
$$f_{1\min} = 8.81 \le f_1(x(t)) \le f_{1\max} = 20.7,$$
$$f_{2\min} = -0.18 \le f_2(x_1(t)) \le f_{2\max} = -0.017,$$
$$f_1(x(t)) = \frac{g - am_pLx_2^2(t)\cos(x_1(t))}{4L/3 - am_pL\cos^2(x_1(t))}\left(\frac{\sin(x_1(t))}{x_1(t)}\right),$$
$$f_2(x_1(t)) = \frac{-a\cos(x_1(t))}{4L/3 - am_pL\cos^2(x_1(t))}. \tag{38}$$

The type 2 T-S fuzzy model is represented as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{4}\theta_i(A_i x(t) + B_i u(t)) \\ &= \sum_{i=1}^{4}\left\{\eta(x(t))\theta_i^L + (1-\eta(x(t)))\theta_i^U\right\} \\ &\quad \cdot (A_i x(t) + B_i u(t)), \end{aligned} \tag{39}$$

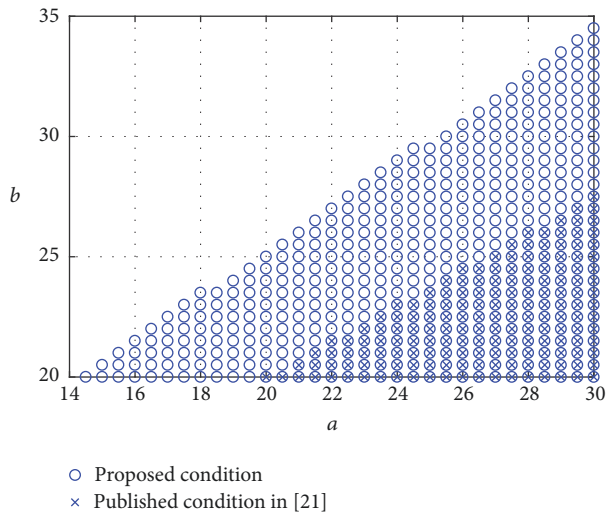

Figure 1: Stability region for the proposed (∘) and published conditions (×).

where, for all i,

$$\theta_i^L = \underline{\mu}_{\widetilde{\mathcal{F}}_{i1}}(x(t)) \times \underline{\mu}_{\widetilde{\mathcal{F}}_{i2}}(x(t)),$$
$$\theta_i^U = \overline{\mu}_{\widetilde{\mathcal{F}}_{i1}}(x(t)) \times \overline{\mu}_{\widetilde{\mathcal{F}}_{i2}}(x(t)), \tag{40}$$

and the lower and upper grades of membership for each rule are defined as follows:

(i) $x_2(t) = x_{2\max}$, $m_p = m_{p_{\max}}$, and $M_c = M_{c\min}$:

$$\underline{\mu}_{\widetilde{\mathcal{F}}_{11}}(x(t)) = \underline{\mu}_{\widetilde{\mathcal{F}}_{21}}(x(t)) = \frac{f_{1\max} - f_1(x(t))}{f_{1\max} - f_{1\min}},$$
$$\underline{\mu}_{\widetilde{\mathcal{F}}_{31}}(x(t)) = \underline{\mu}_{\widetilde{\mathcal{F}}_{41}}(x(t)) = \frac{f_1(x(t)) - f_{1\min}}{f_{1\max} - f_{1\min}}. \tag{41}$$

(ii) $x_2(t) = 0$, $m_p = m_{p_{\max}}$, and $M_c = M_{c\min}$:

$$\overline{\mu}_{\widetilde{\mathcal{F}}_{11}}(x(t)) = \overline{\mu}_{\widetilde{\mathcal{F}}_{21}}(x(t)) = \frac{f_{1\max} - f_1(x(t))}{f_{1\max} - f_{1\min}},$$
$$\overline{\mu}_{\widetilde{\mathcal{F}}_{31}}(x(t)) = \overline{\mu}_{\widetilde{\mathcal{F}}_{41}}(x(t)) = \frac{f_1(x(t)) - f_{1\min}}{f_{1\max} - f_{1\min}}. \tag{42}$$

(iii) $m_p = m_{p_{\min}}$ and $M_c = M_{c\min}$:

$$\underline{\mu}_{\widetilde{\mathcal{F}}_{12}}(x_1(t)) = \underline{\mu}_{\widetilde{\mathcal{F}}_{32}}(x_1(t)) = \frac{f_{2\max} - f_2(x_1(t))}{f_{2\max} - f_{2\min}},$$
$$\underline{\mu}_{\widetilde{\mathcal{F}}_{22}}(x_1(t)) = \underline{\mu}_{\widetilde{\mathcal{F}}_{42}}(x_1(t)) = \frac{f_2(x_1(t)) - f_{2\min}}{f_{2\max} - f_{2\min}}. \tag{43}$$

(iv) $m_p = m_{p_{\max}}$ and $M_c = M_{c\max}$:

$$\overline{\mu}_{\widetilde{\mathcal{F}}_{12}}(x_1(t)) = \overline{\mu}_{\widetilde{\mathcal{F}}_{32}}(x_1(t)) = \frac{f_{2\max} - f_2(x_1(t))}{f_{2\max} - f_{2\min}},$$
$$\overline{\mu}_{\widetilde{\mathcal{F}}_{22}}(x_1(t)) = \overline{\mu}_{\widetilde{\mathcal{F}}_{42}}(x_1(t)) = \frac{f_2(x_1(t)) - f_{2\min}}{f_{2\max} - f_{2\min}}. \tag{44}$$

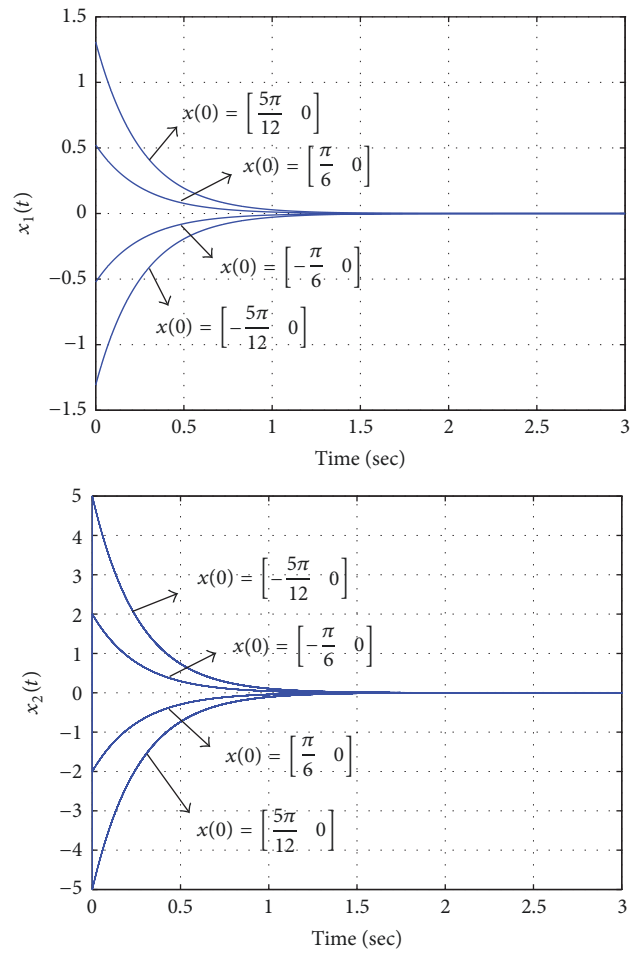

Figure 2: The trajectories of state ($m_p = m_{p_{\min}}$, $M_c = M_{c\min}$).

For Theorem 1, $\alpha_i = 0$, $\beta_i = 1$, $\underline{\alpha}_i = 0$, $\underline{\beta}_i = 0.5$, $\overline{\alpha}_1 = 0$, $\overline{\beta}_i = 0.5$. Let us demonstrate the validity of the proposed conditions for the type 2 T-S fuzzy model. Figure 2 shows the trajectories of the states with $m_p = m_{p_{\min}}$ and $M_c = M_{c\min}$ under various initial conditions. Figure 3 shows the trajectories of the states with $m_p = m_{p_{\max}}$ and $M_c = M_{c\max}$. From Figures 2 and 3, we can clearly see that the proposed controller can stabilize the inverted pendulum with different parameter values and is robust to parameter variations in the plant model. In addition, the proposed conditions lead to less conservative results because we use the larger mass ranges than those of [21].

5. Conclusion

In this paper, we proposed robust stability conditions to stabilize type 2 T-S fuzzy systems. The conditions effectively handled parameter uncertainties using lower and upper membership functions. Furthermore, by applying a multigain controller and a well-organized relaxation technique, we derived a less conservative stabilization condition in terms of LMIs than those in the literature. Our simulation results showed the effectiveness and robustness of the derived stabilization conditions.

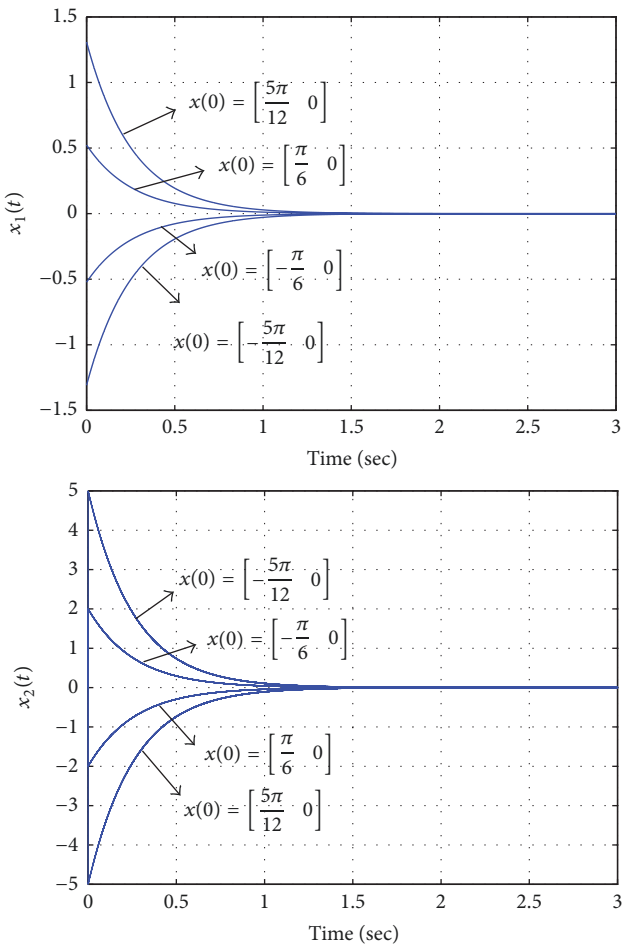

Figure 3: The trajectories of state ($m_p = m_{p_{\max}}, M_c = M_{c\max}$).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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