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Research Article

Incorporation of Inefficiency Associated with Link Flows in Efficiency Measurement in Network DEA

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Data Envelopment Analysis (DEA) is a mathematical programming approach to measure the relative efficiency of peer decision making units (DMUs) which use multiple inputs to produce multiple outputs. One of the drawbacks of traditional DEA models is the neglect of internal structures of the DMUs. Network DEA models are able to overcome the shortcoming of the traditional DEA models. In network DEA a DMU is made up of some divisions linked together by intermediate products. An intermediate product has the dual role of output from one division and input to another one. Improving the efficiency of one process may reduce the efficiency of another process. To address the conflict caused by the dual role of intermediate measures, this paper presents a new approach which categorizes the intermediate measures into either input or output type endogenously, while keeping the continuity of link flows between divisions. This categorization allows us to measure the inefficiencies associated with intermediate measures and account their indirect effects on the objective function. In this paper we propose a new Slacks-based measure which includes any nonzero slacks identified by the model and inherits the properties of monotonicity in slacks and units invariance from the conventional SBM approach.

1. Introduction

Data Envelopment Analysis (DEA), developed by Charnes et al. [1] based on the seminal work of Farrell [2], is a mathematical programming approach to measure the relative performance of peer decision making units (DMUs) which use multiple inputs to produce multiple outputs. Conventional DEA models consider the DMUs as *black boxes* and neglect the operations and interrelations of the processes within the DMU. Recently, a number of studies have looked inside the *black box* and modeled it as a network of subtechnologies.

The simplest structure of network systems is a two-stage system composed of two processes connected in series. Besides inputs and outputs, there are a set of intermediate measures that link these two stages together. The intermediate measures play the role of outputs from the first stage and inputs to the second stage at the same time. Several models have been proposed to measure the efficiency of this type of system (see the review of Cook et al. [3]). The major problem in measuring efficiency of the DMUs with two-stage structure is that the outputs of the first stage are the inputs to the second, because improving the efficiency of the first stage by increasing its output may damage the efficiency of the second stage.

Many researchers propose solutions to address the potential conflict caused by the dual role of intermediate measures. There are four types of papers that use various approaches for measuring efficiency of DMUs with two-stage processes.

In the first type, two separate DEA runs are applied to the stages to measure the relative efficiency of each stage separately. [4–7]. Such an approach does not treat intermediate measures in an organized manner. Improving the efficiency of one division by controlling intermediate measures reduces the efficiency of the other one.

Another type of researches is called "Efficiency Decomposition Methodology," as in Kao and Hwang [8] who define a two-stage efficiency score as the weighted sum of final outputs to the weighted sum of initial inputs. Their approach finds a set of multipliers that maximize either the first or the second stage efficiency score while maintaining the overall efficiency score [9, 10].

The third type of modeling called "Game theoretic approaches" originated from the work of Liang et al. [11]. They applied game theory to develop number of DEA models. They proposed a leader-follower model game and assumed the "same weights" for the intermediate products as outputs and inputs as a perfect coordination between the two subtechnologies.

In the case that there are additional independent inputs to second stage and the second stage has its own inputs not linked with the first stage, "Network DEA" approach is introduced to the literature of DEA. Färe and Grosskopf [12, 13] are pioneered in this line of research. They developed two-stage model into a general multistage model with intermediate products. Their representation of the flow of product is consistent with the industrial engineering and operations research literature on multistage systems (e.g., [14–17]).

Despotis et al. [18] presented a network DEA approach in the framework of multiobjective programming to assess the efficiency score of two-stage processes. They estimated efficiencies of the stages without a prior definition of the overall efficiency of the system. The overall efficiency is obtained by aggregating the stage efficiencies a posteriori.

Tone and Tsutsui [19] present a slacks-based NDEA model that measures the overall efficiency of the DMU and its components. The overall efficiency score is defined as the weighted average of the components that make up the DMU. The weight of each component is determined exogenously and represents the importance of that component. In their study they called the intermediate measures as links and define two possible cases for the linking constrains, the "fixed" link value case and the "free" link value case. In the latter case, the linking activities are freely determined and their target values can be smaller or greater than their observed values.

Tone and Tsutsui [20] propose slacks-based dynamic DEA model by extending their slacks-based NDEA model and taking carry-over activities into account. Network and dynamic model which is combination of the network structure by means of carry-over activities between two succeeding periods is also proposed in Tone and Tsutsui [21].

Lozano [22] proposes a slacks-based measure (SBM) model for general networks of processes that differs from the existing SBM Network Data Envelopment Analysis (NDEA) approaches. He enhances the discriminating power of his proposed model by relaxing the linking constraints proposed by Tone and Tsutsui [19]. Moreover, the model considers the exogenous inputs and outputs at the system level instead of at the process level.

F.-h. F. Liu and Y.-c. Liu [23] introduced a procedure to solve dynamic network DEA based on a Virtual Gap Measurement Model. They proposed a two-phase approach to resolve the problem of dual role of intermediate products and measure the nonzero slacks of intermediate measures.

As we discussed above the dual role of intermediate products is an issue that needs to be addressed in network DEA. In this paper we propose two new network DEA models in the slacks-based measure (SBM) framework, called Model (I) and Model (II), in which the intermediate products are categorized into either input or output type. The proposed models compute the input excesses and output shortfalls associated with intermediate measures and keep the continuity of link flows between divisions. Model (II) is able to take into account the inefficiency associated with the link variables.

The rest of this paper is structured as follows; Section 2 presents some preliminaries. In Section 3 we propose our new models and the new slack based measure. A numerical example is presented in Section 4 and to verify our proposed models we compare the results with the results of some existing approaches. Finally, Section 5 closes this paper with a few concluding remarks and some suggestions for further research.

2. Preliminaries

In this section the network SBM approaches of Tone and Tsutsui [19] and the separation approach are explained. All the preliminaries are taken from Cook et al. 2014.

2.1. Separation Approach. In this approach the divisional efficiency is evaluated individually. The weighted average of each division gives the overall efficiency of a DMU. In this case, for evaluating the efficiency of div k individually, we consider the all intermediate products consumed by div k as inputs and all intermediate products produced by div k as outputs and we evaluate the efficiency of div k with these inputs and outputs and the exogenous inputs used and outputs produced by div k. In this way, we can evaluate efficiency of each division of a company among the set of DMUs and can find benchmarks for each division. The separation model takes into account the inefficiency associated with the link variables. However, this approach does not account for the continuity of links between divisions.

2.2. NSBM Approach. Suppose that there are a set of n DMUs indexed by ($j = 1, \ldots, n$) consisting of K divisions ($k = 1, \ldots, K$)and that division k (div k) consumes m_k number of inputs denoted by x_{ij}^k ($i = 1, \ldots, m_k$) and produces r_k number of outputs denoted by y_{rj}^k ($r = 1, \ldots, r_k$). Intermediate products from div k to div h are also denoted by $z_{dj}^{(k,h)}$ ($d \in L_{(k,h)}$) where $L_{(k,h)}$ is the set of links between div k and div h and $l_{(k,h)}$ is the number of items in $L_{(k,h)}$.

Tone and Tsutsui [19] proposed the production possibility set $(x^k, y^k, z^{(k,h)})$ as follows:

$$\sum_{j=1}^{n} \lambda_j^k x_j^k \leq x^k \quad (k = 1, \ldots, K)$$

$$\sum_{j=1}^{n} \lambda_j^k y_j^k \geq y^k \quad (k = 1, \ldots, K)$$

$$\sum_{j=1}^{n} \lambda_j^k z_j^{(k,h)} = z^{(k,h)} \quad (\forall (k,h)) \ \text{(as output from div } k\text{)}$$

$$\sum_{j=1}^{n} \lambda_j^h z_j^{(k,h)} = z^{(k,h)} \quad (\forall (k,h)) \ \text{(as input to div } h\text{)}$$

$$\sum_{j=1}^{n} \lambda_j^k = 1 \quad (\forall k), \ \lambda_j^k \geq 0 \ (\forall j,k), \tag{1}$$

where $\lambda^k \in R_+^n$ is the intensity vector corresponding to div k $(k = 1, \ldots, K)$.

It should be noted that the above model assumes the variable returns-to-scale (VRS) for production and by removing the last constraint $\sum_{j=1}^{n} \lambda_j^k = 1$ changes the assumption of VRS to the constant returns-to-scale (CRS) for production.

Regarding linking constraints, they proposed two possible cases called "*fixed link*" (2) and "*free link*" (3) formulated as follows:

$$\sum_{j=1}^{n} \lambda_j^k Z_j^{(k,h)} = Z_o^{(k,h)}$$
$$\sum_{j=1}^{n} \lambda_j^h Z_j^{(k,h)} = Z_o^{(k,h)} \tag{2}$$

$$\sum_{j=1}^{n} \lambda_j^k Z_j^{(k,h)} = \sum_{j=1}^{n} \lambda_j^h Z_j^{(k,h)}. \tag{3}$$

When linking activities are beyond the control of DMUs (nondiscretionary) they are kept unchanged by applying *fixed link* case (2) and in the case that the linking activities are freely determined (discretionary) the *free link* case (3) needs to be used. Note that in both cases the continuity of link values between divisions is assured.

In the next section we propose our new network models.

3. Proposing New Network SBM Model

As we discussed in previous section the linking constraints proposed by Tone and Tsutsui [19] do not consider the slacks of the intermediate measures unless they are exogenously categorized into either input type or output type. In this section we propose new network DEA models based on SBM framework which categorize the intermediate measures into input or output type endogenously. To incorporate the inefficiency associated with intermediate measures in efficiency measurement we propose two models referred to as Model (I) and Model (II). These models have only different objective functions. In Model (I) the slacks of intermediate products do not appear in the objective function while in Model (II) they do.

Incorporation of the slacks of intermediate measures in objective function allows us to incorporate the inefficiency associated with intermediate measures in efficiency measurement directly.

3.1. Model (I). We present Model (I) as follows:

$$\tau_p^* = \min \quad \frac{\sum_{k=1}^{K} w_k \left[1 - (1/m_k)\left(\sum_{i=1}^{m_k} \left(s_{ip}^{k-}/x_{ip}^k\right)\right)\right]}{\sum_{k=1}^{K} w_k \left[1 + (1/r_k)\left(\sum_{r=1}^{r_k} \left(s_{rp}^{k+}/y_{rp}^k\right)\right)\right]} \tag{4}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j^k x_{ij}^k + s_{ip}^{k-} = x_{ip}^k \qquad (k = 1, \ldots, K), \ (i = 1, \ldots, m_k) \tag{5}$$

$$\sum_{j=1}^{n} \lambda_j^k y_{rj}^k - s_{rp}^{k+} = y_{rp}^k \qquad (k = 1, \ldots, K), \ (r = 1, \ldots, r_k) \tag{6}$$

$$\sum_{j=1}^{n} \lambda_j^h z_{dj}^{(k,h)} + s_{dp}^{(k,h)-} = z_d^{(k,h)} \qquad (d = 1, \ldots, l_{(k,h)}), \ \forall (k,h) \tag{7}$$

$$\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)} - s_{dp}^{(k,h)+} = z_d'^{(k,h)} \qquad (d = 1, \ldots, l_{(k,h)}), \ \forall (k,h) \tag{8}$$

$$\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)} = \sum_{j=1}^{n} \lambda_j^h z_{dj}^{(k,h)} \qquad (d = 1, \ldots, l_{(k,h)}), \ \forall (k,h) \tag{9}$$

$$0 \leq s_{dp}^{(k,h)-} \leq M y_d^{(k,h)} \qquad (d = 1, \ldots, l_{(k,h)}), \ \forall (k,h) \tag{10}$$

$$0 \leq s_{dp}^{(k,h)+} \leq M \left(1 - y_d^{(k,h)}\right) \qquad (d = 1, \ldots, l_{(k,h)}), \ \forall (k,h) \tag{11}$$

$$z_{dp}^{(k,h)} - M y_{dp}^{(k,h)} \leq z_d'^{(k,h)} \leq z_{dp}^{(k,h)} + M y_{dp}^{(k,h)} \qquad (d = 1, \ldots, l_{(k,h)}), \ \forall (k,h) \tag{12}$$

$$z_{dp}^{(k,h)} - M \left(1 - y_d^{(k,h)}\right) \leq z_d^{(k,h)} \leq z_{dp}^{(k,h)} + M \left(1 - y_d^{(k,h)}\right) \qquad (d = 1, \ldots, l_{(k,h)}), \ \forall (k,h) \tag{13}$$

$$\sum_{j=1}^{n} \lambda_j^k = 1 \quad (k = 1, \ldots, K) \tag{14}$$

$$y_d^{(k,h)} = \{0,1\}; \quad z_d^{(k,h)}, z_d'^{(k,h)} : \text{free}, \quad \lambda_j^k \geq 0, \quad s_{rp}^{k+} \geq 0, \quad s_{ip}^{k-} \geq 0, \tag{15}$$

where M is a large positive number. $\sum_{k=1}^{K} w_k = 1$ and $w_k \geq 0$ is the relative weight of div k which is determined corresponding to its importance. The proposed model is a mixed integer programming and we can solve this problem by transforming into a mixed integer linear programming using Charnes and Cooper transformation (see Appendix). The model presented above assumes the condition of variable returns-to-scale (VRS) for production and the production frontiers are spanned by the convex hull of the existing DMUs. If we neglect the last constraints (14) we can deal with the constant returns-to scale (CRS) case as well.

Note that if $y_d^{(k,h)} = 1$, then the utilization intermediate product $z_d^{(k,h)}$ is under the control of div h and $z_d^{(k,h)}$ is considered as an input to div h. We denote the set of those intermediate measures by $L_{(k,h)}^{\text{in}}$. In a similar manner, if $y_d^{(k,h)} = 0$, then the production of intermediate measure $z_d^{(k,h)}$ is under the control of div k and $z_d^{(k,h)}$ is considered as an output from div k. We denote the set of those intermediate measures by $L_{(k,h)}^{\text{out}}$. It is clear that

$$\begin{gathered} L_{(k,h)}^{\text{out}} \cup L_{(k,h)}^{\text{in}} = L_{(k,h)}, \\ L_{(k,h)}^{\text{out}} \cap L_{(k,h)}^{\text{in}} = \phi. \end{gathered} \tag{16}$$

In other words the proposed model classifies the intermediate measures into input or output type. The proposed model also identifies nonzero slacks and uncovers the sources of inefficiency associated with intermediate measures. Since the optimal values of intermediate measures can be equal, above, or below the observed value the proposed model corresponds to the free link case.

Set of constraints (9) allows model to keep the continuity of link flows between divisions and lets the shadow prices for the corresponding intermediate products be free. If we relax the constraints (9) by changing them to the constraints (17) we will enlarge the production possibility set and therefore increase the discriminating power of the approach. It also guarantees that no more intermediate products are consumed than are produced.

$$\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)} \geq \sum_{j=1}^{n} \lambda_j^h z_{dj}^{(k,h)}. \tag{17}$$

The objective function of Model (I) is similar to that of NSBM model of Tone and Tsutsui [19]; hence we can define the overall and divisional input or output-oriented efficiency score similar to NSBM.

The output-oriented efficiency of DMU p can be evaluated by solving mixed integer linear programming below:

$$\frac{1}{\pi_p^*} = \max \quad \sum_{k=1}^{K} w_k \left[1 + \frac{1}{r_k} \left(\sum_{r=1}^{r_k} \frac{s_{rp}^{k+}}{y^k{}_{rp}} \right) \right] \tag{18}$$

subject to (5)–(15).

And the output-oriented divisional efficiency for div k of DMUp can be calculated as follows:

$$\frac{1}{\pi_p^{k*}} = \left[1 + \frac{1}{r_k} \left(\sum_{r=1}^{r_k} \frac{s_{rp}^{*k+}}{y^k{}_{rp}} \right) \right], \tag{19}$$

where s_{rp}^{*k+} is the optimal output-slacks obtained by minimizing (18) subject to (5)–(15).

Similarly the input-oriented efficiency of DMU p can be evaluated by solving mixed integer linear programming below:

$$\rho_p^* = \min \quad \sum_{k=1}^{K} w_k \left[1 - \frac{1}{m_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{k-}}{x_{ip}^k} \right) \right], \tag{20}$$

subject to (5)–(15).

And the input-oriented divisional efficiency for div k of DMUp can be calculated as follows:

$$\rho_p^{*k} = 1 - \frac{1}{m_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{*k-}}{x_{ip}^k} \right), \tag{21}$$

where s_{ip}^{*k-} denote the optimal output-slacks obtained by minimizing (20) subject to (5)–(15).

3.1.1. Efficiency of the Projected DMU

Projection. Let an optimal solution to our proposed model be $(\lambda_j^{*k}, s_p^{*k-}, s_p^{*k+}, s_p^{*(k,h)+}, s_p^{*(k,h)-}, y_d^{*(k,h)}, z_d^{*(k,h)}, z_d'^{(k,h)})$ $\forall j$, $\forall k$, $\forall (k,h) \in l_{(k,h)}$. Then we have the projection onto the frontier as follows:

$$x_{ip}^{*k} = x_{ip}^k - s_{ip}^{*k-} \quad (k = 1, \ldots, K), \ (i = 1, \ldots, m_k) \tag{22}$$

$$y_{rp}^{*k} = y_{rp}^k + s_{rp}^{*k+} \quad (k = 1, \ldots, K), \ (r = 1, \ldots, r_k) \tag{23}$$

$$z_{dp}^{*(k,h)} = z_{dp}^{(k,h)} - s_{dp}^{*(k,h)-} \quad \forall d \in L_{(k,h)}^{\text{in}} \tag{24}$$

$$z_{dp}^{*(k,h)} = z_{dp}^{(k,h)} + s_{dp}^{*(k,h)+} \quad \forall d \in L_{(k,h)}^{\text{out}}. \tag{25}$$

Theorem 1. *The projected DMU in Model (I) is overall efficient.*

Proof. We prove the theorem in the nonoriented case.

Let τ_p^* be the efficiency of the projected DMU $(X_p^{*k}, Y_p^{*k}, Z_p^{*k})$.

And let $(\hat{\lambda}_j^k, \hat{s}_p^{k-}, \hat{s}_p^{k+}, \hat{s}_p^{(k,h)+}, \hat{s}_p^{(k,h)-}, \hat{y}_p^{(k,h)}, \hat{z}_d^{(k,h)}, \hat{z}_d'^{(k,h)})$ $\forall j$, $\forall k$, $\forall d \in L_{(k,h)}$, $\forall (k,h)$ be an optimal solution to the proposed model. Then

$$\begin{aligned} &\sum_{j=1}^{n} \hat{\lambda}_j^k x_{ij}^k + \hat{s}_{ip}^{k-} = x_{ip}^{*k}, \\ &\sum_{j=1}^{n} \hat{\lambda}_j^k y_{rj}^k - \hat{s}_{rp}^{k+} = y_{rp}^{*k}, \\ &\sum_{j=1}^{n} \hat{\lambda}_j^h z_{dj}^{(k,h)} + \hat{s}_{dp}^{(k,h)-} = z_{dp}^{*(k,h)} \quad \forall d \in L_{(k,h)}^{\text{in}} \\ &\sum_{j=1}^{n} \hat{\lambda}_j^k z_{dj}^{(k,h)} - \hat{s}_{dp}^{(k,h)+} = z_{dp}^{*(k,h)} \quad \forall d \in L_{(k,h)}^{\text{out}}. \end{aligned} \tag{26}$$

Replacing x_{ip}^{*k} and y_{rp}^{*k} by (22) and (23) we have

$$\sum_{j=1}^{n} \hat{\lambda}_j^k x_{ij}^k + \hat{s}_{ip}^{k-} + s_{ip}^{*k-} = x_{ip}^k, \quad \sum_{j=1}^{n} \hat{\lambda}_j^k y_{rj}^k - \hat{s}_{rp}^{k+} - s_{rp}^{*k+} = y_{rp}^{*k}. \tag{27}$$

Hence we have the overall efficiency as follows:

$$\hat{\tau}_p = \min \frac{\sum_{k=1}^{K} w_k \left[1 - (1/m_k)\left(\sum_{i=1}^{m_k}\left(\left(s_{ip}^{*k-} + \hat{s}_{ip}^{k-}\right)/x_{ip}\right)\right)\right]}{\sum_{k=1}^{K} w_k \left[1 + (1/r_k)\left(\sum_{r=1}^{r_k}\left(\left(s_{rp}^{*k+} + \hat{s}_{rp}^{k+}\right)/y_{rp}\right)\right)\right]}. \tag{28}$$

If only one of $\hat{s}_{rp}^{k+}$ $(\forall k, \forall r)$ or $\hat{s}_{ip}^{k-}$ $(\forall k, \forall i)$ is positive then we have

$$\hat{\tau}_p < \tau_p^*. \tag{29}$$

And it contradicts the optimality of τ_p^*. Thus, we have $\hat{s}_{rp}^{k+} = 0$ $(\forall k, \forall r)$ and $\hat{s}_{ip}^{k-} = 0$ $(\forall k, \forall i)$. Therefore, the projected DMU is overall efficient. □

3.2. Incorporation of Inefficiency Corresponding to Intermediate Measures in the Objective Function (Model (II)). Although the slacks of intermediate products in Model (I) are not included in the objective function, their indirect effect on the objective function incorporates inefficiency corresponding to intermediate measures in efficiency measurement. In order to include the inefficiency associated with intermediate measure in the objective function directly, we propose Model (II) that minimizes the objective function (30) subject to (5)–(15).

$$\eta_p^* = \min \frac{\sum_{k=1}^{K} w_k \left[1 - \left(1/\left(m_k + \sum_{d=1}^{l_{(k,h)}} y_d^{(k,h)}\right)\right)\left(\sum_{i=1}^{m_k}\left(s_{ip}^{k-}/x_{ip}\right) + \sum_{d=1}^{l_{(k,h)}}\left(s_{dp}^{(k,h)-}/z_{dp}^{(k,h)}\right)\right)\right]}{\sum_{k=1}^{K} w_k \left[1 + \left(1/\left(r_k + l_{(k,h)} - \sum_{d=1}^{l_{(k,h)}} y_d^{(k,h)}\right)\right)\left(\sum_{r=1}^{r_k}\left(s_{rp}^{k+}/y_{rp}\right) + \sum_{d=1}^{l_{(k,h)}}\left(s_{dp}^{(k,h)+}/z_{dp}^{(k,h)}\right)\right)\right]}. \tag{30}$$

The term $l_{(k,h)} - \sum_{d=1}^{l_{(k,h)}} y_d^{*(k,h)}$ represents the number of those intermediate measures that are considered as the output from div k (i.e., the cardinal number of set $L_{(k,h)}^{\text{out}}$). Similarly the term $\sum_{d=1}^{l_{(k,h)}} y_d^{(k,h)}$ represents the number of those intermediate measures that are considered as the input to div h (i.e., the cardinal number of set $L_{(k,h)}^{\text{in}}$).

Neglecting the constraints (9) in solving Model (II) causes links to be treated as ordinary (discretionary) inputs or outputs and reduces the model structurally to the separation model. We can solve this case separately division by division and it assures the existence of at least one divisionally efficient DMU for every division.

The slack based measure (30) is invariant with respect to the unit of measurement of each input output and intermediate measure item (Units invariant). It is also monotone decreasing with respect to each input, output, and intermediate product slack. It represents the ratios of average input, output mix inefficiencies with the upper limit of 1.

To measure the nonoriented divisional efficiency score applying the direct effect of intermediate slacks on efficiency score we use the following formula:

$$\eta_p^{*k} = \frac{1 - \left(1/\left(m_k + \sum_{d=1}^{l_{(f,k)}} y_d^{*(f,k)}\right)\right)\left(\sum_{i=1}^{m_k}\left(s_{ip}^{*k-}/x_{ip}\right) + \sum_{d=1}^{l_{(f,k)}}\left(s_{dp}^{*(f,k)-}/z_{dp}^{(f,k)}\right)\right)}{1 + \left(1/\left(r_k + l_{(k,h)} - \sum_{d=1}^{l_{(k,h)}} y_d^{*(k,h)}\right)\right)\left(\sum_{r=1}^{r_k}\left(s_{rp}^{*k+}/y_{rp}\right) + \sum_{d=1}^{l_{(k,h)}}\left(s_{dp}^{*(k,h)+}/z_{dp}^{(k,h)}\right)\right)}, \tag{31}$$

where $y_d^{*(k,h)}$, $y_d^{*(f,k)}$, s_{ip}^{*k-}, $s_{dp}^{*(f,k)-}$, $s_{dp}^{*(k,h)+}$, and s_{rp}^{*k+} are optimal values for the variables obtained from solution of Model (II). Note that the overall nonoriented efficiency score is a weighted mean of the divisional efficiency scores in which the weights are set exogenously and denote the importance of divisions.

To evaluate the input-oriented efficiency score of DMU$_p$ we can solve the following model.

$$\varphi_p^* = \min \sum_{k=1}^{K} w_k \left[1 - \frac{1}{m_k + \sum_{d=1}^{l_{(k,h)}} y_d^{(k,h)}}\left(\sum_{i=1}^{m_k}\frac{s_{ip}^{k-}}{x_{ip}} + \sum_{d=1}^{l_{(k,h)}}\frac{s_{dp}^{(k,h)-}}{z_{dp}^{(k,h)}}\right)\right] \tag{32}$$

subject to (5)–(15).

The efficiency score in the output-oriented case for DMU$_p$ can be evaluated from following model.

$$\psi_p^* = \text{Max} \sum_{k=1}^{K} w_k \left[1 + \frac{1}{r_k + l_{(k,h)} - \sum_{d=1}^{l_{(k,h)}} y_d^{*(k,h)}}\left(\sum_{r=1}^{r_k}\frac{s_{rp}^{*k+}}{y_{rp}} + \sum_{d=1}^{l_{(k,h)}}\frac{s_{dp}^{*(k,h)+}}{z_{dp}^{(k,h)}}\right)\right] \tag{33}$$

subject to (5)–(15).

Theorem 2. *The projected DMU in Model (II) is overall efficient.*

Proof. We prove the theorem in the nonoriented case.

Let η_p^* be the efficiency of the projected DMU $(X_p^{*k}, Y_p^{*k}, Z_p^{*k})$.

And let $(\hat{\lambda}_j^k, \hat{s}_p^{k-}, \hat{s}_p^{k+}, \hat{s}_p^{(k,h)+}, \hat{s}_p^{(k,h)-}, \hat{y}_p^{(k,h)}, \hat{z}_d^{(k,h)}, \hat{z}_d'^{(k,h)})$ $\forall j$, $\forall k$, $\forall d \in L_{(k,h)}$, $\forall (k,h)$ be an optimal solution to the proposed model. Then

$$\sum_{j=1}^{n} \hat{\lambda}_j^k x_{ij}^k + \hat{s}_{ip}^{k-} = x_{ip}^{*k},$$
$$\sum_{j=1}^{n} \hat{\lambda}_j^k y_{rj}^k - \hat{s}_{rp}^{k+} = y_{rp}^{*k},$$
$$\sum_{j=1}^{n} \hat{\lambda}_j^h z_{dj}^{(k,h)} + \hat{s}_{dp}^{(k,h)-} = z_{dp}^{*(k,h)} \quad \forall d \in \hat{L}_{(k,h)}^{\text{in}}$$
$$\sum_{j=1}^{n} \hat{\lambda}_j^k z_{dj}^{(k,h)} - \hat{s}_{dp}^{(k,h)+} = z_{dp}^{*(k,h)} \quad \forall d \in \hat{L}_{(k,h)}^{\text{out}}. \tag{34}$$

Suppose $L_{(k,h)}^{\text{in}} = \hat{L}_{(k,h)}^{\text{in}}$. By replacing x_{ip}^{*k}, y_{rp}^{*k}, and $z_{dp}^{*(k,h)}$ by (22), (23), (24), or (25) we have

$$\sum_{j=1}^{n} \hat{\lambda}_j^k x_{ij}^k + \hat{s}_{ip}^{k-} + s_{ip}^{*k-} = x_{ip}^{k},$$
$$\sum_{j=1}^{n} \hat{\lambda}_j^k y_{rj}^k - \hat{s}_{rp}^{k+} - s_{rp}^{*k+} = y_{rp}^{*k},$$
$$\sum_{j=1}^{n} \hat{\lambda}_j^h z_{dj}^{(k,h)} + \hat{s}_{dp}^{(k,h)-} + s_{dp}^{*(k,h)-} = z_{dp}^{(k,h)} \quad \forall d \in \hat{L}_{(k,h)}^{\text{in}}$$
$$\sum_{j=1}^{n} \hat{\lambda}_j^k z_{dj}^{(k,h)} - \hat{s}_{dp}^{(k,h)+} - s_{dp}^{*(k,h)+} = z_{dp}^{(k,h)} \quad \forall d \in \hat{L}_{(k,h)}^{\text{out}}. \tag{35}$$

Hence we have the overall efficiency as follows:

$$\hat{\eta}_p = \frac{\sum_{k=1}^{K} w_k \left[1 - \left(1/\left(m_k + \sum_{d=1}^{l_{(k,h)}} y_d^{*(k,h)}\right)\right)\left(\sum_{i=1}^{m_k}\left(\left(s_{ip}^{*k-} + \hat{s}_{ip}^{k-}\right)/x_{ip}\right) + \sum_{d=1}^{l_{(k,h)}}\left(\left(s_{dp}^{*(k,h)-} + \hat{s}_{dp}^{(k,h)-}\right)/z_{dp}^{(k,h)}\right)\right)\right]}{\sum_{k=1}^{K} w_k \left[1 + \left(1/\left(r_k + l_{(k,h)} - \sum_{d=1}^{l_{(k,h)}} y_d^{*(k,h)}\right)\right)\left(\sum_{r=1}^{r_k}\left(\left(s_{rp}^{*k+} + \hat{s}_{rp}^{k+}\right)/y_{rp}\right) + \sum_{d=1}^{l_{(k,h)}}\left(\left(s_{dp}^{*(k,h)+} + \hat{s}_{dp}^{*(k,h)+}\right)/z_{dp}^{(k,h)}\right)\right)\right]}. \tag{36}$$

If only one of $\hat{s}_{rp}^{k+}$ $(\forall k, \forall r)$, $\hat{s}_{ip}^{k-}$ $(\forall k, \forall i)$ $\hat{s}_{dp}^{(k,h)-}$ or $\hat{s}_{dp}^{(k,h)+}$ $\forall (k,h)$, $\forall d \in L_{(k,h)}$, is positive then we have

$$\hat{\eta}_p < \eta_p^* \tag{37}$$

and it contradicts the optimality of η_p^*. Thus, we have $\hat{s}_{rp}^{k+} = 0$ $(\forall k, \forall r)$, $\hat{s}_{ip}^{k-} = 0$ $(\forall k, \forall i)$, $\hat{s}_{dp}^{(k,h)-} = 0$ $\forall d \in \hat{L}_{(k,h)}^{\text{in}}$ and $\hat{s}_{dp}^{(k,h)+} = 0$ $\forall d \in \hat{L}_{(k,h)}^{\text{out}}$. Therefore, the projected DMU is overall efficient. □

It should be noted that, in the case $L_{(k,h)}^{\text{in}} \neq \hat{L}_{(k,h)}^{\text{in}}$, there exists $d \in \hat{L}_{(k,h)}^{\text{in}}$ that $d \notin L_{(k,h)}^{\text{in}}$. Therefore $d \in L_{(k,h)}^{\text{out}}$, and this means that the link value is free to be greater than or equal to (but not lower than) the observed one in production possibility set. On the other hand $d \in \hat{L}_{(k,h)}^{\text{in}}$ means that the link value is free to be smaller than or equal to (but not greater than) the observed one in production possibility set and it is not possible unless the link target value of both solutions is equal to the observed value. Therefore, $\hat{s}_{dp}^{(k,h)-}$ and $s^{*}{}_{dp}^{(k,h)-}$ should be equal to zero.

4. Numerical Example

In this section to illustrate our proposed models, we will use a numerical example and compare the results of our proposed models with some existing approaches in SBM framework. Table 1 exhibits the data of our numerical example.

Consider the dataset provided by Tone and Tsutsui [19]. It consists of 10 DMUs, corresponding to vertically integrated electric power companies. They illustrated the vertically integrated electric power companies as three divisions of generation, transmission, and distribution that are linked together via intermediate products as shown in Figure 1. Each division has a single exogenous input: *Labor*. The *Electric Power Sold* to large customers and to small customers are, respectively, the output of *Transmission* and *Distribution* divisions. *Electric Power Generated* by *Generation* division is consumed by *Transmission* Division therefore *Electric Power Generated* is an intermediate measure which links the *Generation* Division to *Transmission* Division. *Electric Power Sent* produced by *Transmission* Division and consumed by *Distribution* Division is used as an intermediate measure which links *Transmission* Division to *Distribution* Division.

4.1. Black Box and Proposed Model. In this section first, we solved the black box model using Inputs 1, 2, and 3 and Outputs 2 and 3 where links were neglected. The column "black box" in Table 2 exhibits the results.

Next, we solved the two proposed models explained in Sections 3.1 and 3.2. The numbers 0.4, 0.2, and 0.4 are weights to div 1, div 2, and div 3, respectively. This weight selection is just for illustrative purpose. Table 2 reports the results where "Overall score" indicates the weighted average scores of divisions.

Throughout this section, we used the input-oriented SBM (slacks-based measure) under the variable returns-to-scale (VRS) assumption for efficiency evaluation.

Figure 2 clearly illustrates that the black box model has lower discriminate power than those of our proposed

Table 1: Exhibits data for inputs, outputs, and links of the ten DMUs in their numerical example; data for inputs, outputs, and links of the ten DMUs presented by Tone and Tsutsui [19].

DMU	Generation process (div 1)	Transmission process (div 2)		Distribution process (div 3)		links	
	Input 1 (x_1)	Input 2 (x_2)	Output 2 (y_2)	Input 3 (x_3)	Output 3 (y_3)	Link 12 (z_1)	Link 23 (z_2)
DMU1	0.838	0.277	0.879	0.962	0.337	0.894	0.362
DMU2	1.233	0.132	0.538	0.443	0.180	0.678	0.188
DMU3	0.321	0.045	0.911	0.482	0.198	0.836	0.207
DMU4	1.483	0.111	0.570	0.467	0.491	0.869	0.516
DMU5	1.592	0.208	1.086	1.073	0.372	0.693	0.407
DMU6	0.790	0.139	0.722	0.545	0.253	0.966	0.269
DMU7	0.451	0.075	0.509	0.366	0.241	0.647	0.257
DMU8	0.408	0.074	0.619	0.229	0.097	0.756	0.103
DMU9	1.864	0.061	1.023	0.691	0.380	1.191	0.402
DMU10	1.222	0.149	0.769	0.337	0.178	0.792	0.187

Table 2: SBM scores for black box and proposed models.

DMU	Black Box	Model (I)				Model (II)			
	Overall efficiency score	Overall efficiency score	div 1 (0.4)	div 2 (0.2)	div 3 (0.4)	ρ^*	div 1 (0.4)	div 2 (0.2)	div 3 (0.4)
DMU1	1.00	0.385	0.383	0.383	0.389	0.441	0.383	0.659	0.389
DMU2	0.54	0.433	0.260	0.341	0.652	0.433	0.26	0.341	0.652
DMU3	1.00	0.968	1.00	1.00	0.919	0.968	1	1	0.919
DMU4	1.00	0.719	0.297	1.00	1.00	0.719	0.297	1	1
DMU5	1.00	0.456	0.263	1	0.377	0.456	0.263	1	0.377
DMU6	0.681	0.484	0.406	0.420	0.593	0.608	0.406	0.643	0.792
DMU7	1.00	0.778	0.712	0.740	0.863	0.778	0.712	0.74	0.863
DMU8	1.00	0.969	0.922	1.00	1.00	0.969	0.922	1	1
DMU9	1.00	0.832	1.00	1.00	0.581	0.832	1	1	0.581
DMU10	1.00	0.506	0.271	0.338	0.825	0.506	0.271	0.338	0.825
Average	0.9221	0.602	0.524	0.688	0.637	0.620	0.524	0.738	0.657

network models. The scores of black box are greater than the overall scores obtained by proposed models and the rank of scores of the DMUs is not corresponding. For example DMU5 is scored worse in the proposed models while best in black box model. This means that there is no significant correlation between the scores of network models and black box. There is also a sharp contrast between the trends of black box and proposed models. This is quite natural since we ignored the internal linking activities in black box model.

4.2. Separation Approach and Proposed Models. In this section we compare our proposed model with separation model.

In order to take into account the inefficiency associated with link flows, there is another approach to evaluate divisional efficiency individually called *separation* approach (see Cook et al. 2014, p. 233). In this approach we evaluate the efficiency of div 1 of our numerical example using Input 1 as input and Link 12 as output. Similarly we evaluate the efficiency of div 2 of each DMU using *Link 12* and *Input 2* as inputs and Link 23 and Output 2 as outputs. In the same way we evaluate the efficiency of div 3 using *Link 23* and *Input 3* as inputs and *Output 3* as output.

Table 3 reports the overall and divisional efficiency scores obtained by separation approach where the overall scores are the weighted sum of the divisional scores. We utilized the numbers 0.4, 0.2, and 0.4 as the weights to div 1, div 2, and, div 3, respectively.

Table 3: SBM scores for separation model.

DMU	Separation model			
	Overall score	div 1 (0.4)	div 2 (0.2)	div 3 (0.4)
DMU1	0.659	0.633	0.662	0.684
DMU2	0.657	0.26	0.763	1.00
DMU3	0.984	1.00	1.00	0.959
DMU4	0.719	0.297	1.00	1.00
DMU5	0.547	0.202	1.00	0.665
DMU6	0.844	1.00	0.635	0.792
DMU7	0.855	0.712	1.00	0.926
DMU8	0.893	0.787	0.890	1.00
DMU9	0.915	1	1	0.786
DMU10	0.640	0.263	0.672	1

Model (II) and separation model both take into the account the inefficiency associated with link flows. In proposed models the continuity of link values between divisions is assured whereas in separation model it is not. Figure 3

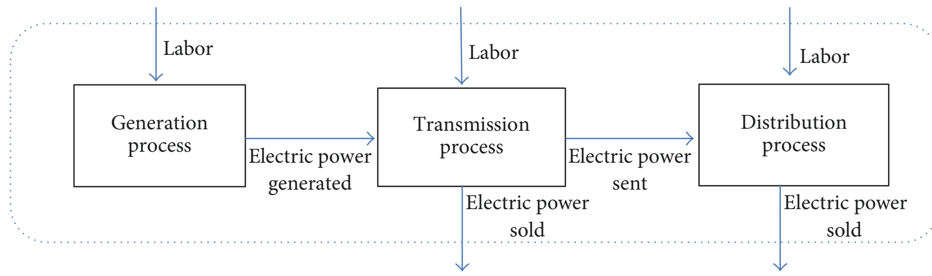

FIGURE 1: Network structure of vertically integrated electric power companies.

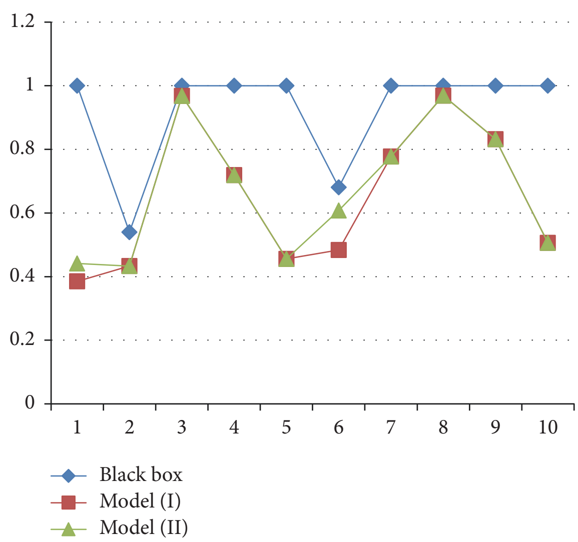

FIGURE 2: Comparisons of scores between black box and proposed models.

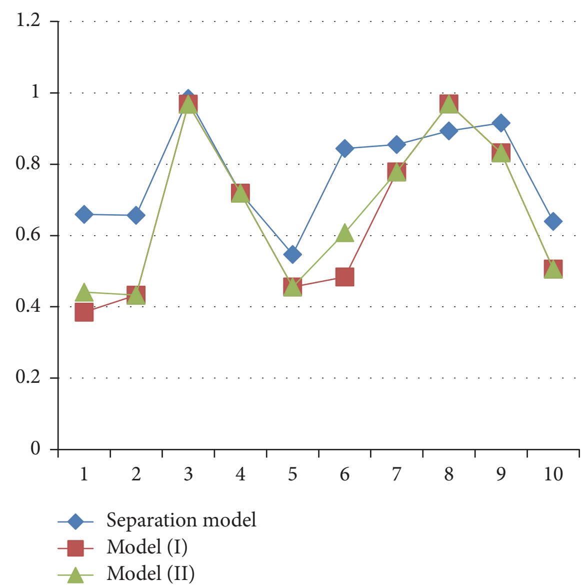

FIGURE 3: Comparisons of scores between proposed models and separation approach.

compares the overall efficiency scores of the separate model and proposed models. It can be seen from Figure 3 that the trends of the three models are close together. The gaps between proposed models and separation model must be caused by the difference of assumption on the links among divisions.

4.3. NSBM and Proposed Models. In this section we aim to compare the scores given by Slacks-Based Network DEA Models (proposed by Tone and Tsutsui [19]) and our proposed models. The overall and divisional SBM scores given by free link and fixed link case NSBM are tabulated in Table 4.

Based on the results shown in Table 4, Model (I) yields the same results as applying free link NDEA model. It was not unexpected because the projected values of the intermediate products in both models are free to be greater or lower than their observed values; both models have the same objective function and the continuity of links between divisions is assured in both models.

The advantage of applying Model (I) instead of free link case is that we can find out the intermediate products are being viewed as inputs or outputs in the system.

The linking constraints in fixed link case are tighter than free link case and the proposed models; hence the overall scores of the fixed link case exceed or are equal to those of the free case and Model (I) for every DMU. Figure 4 compares scores of the proposed models and network models (fixed and free link cases).

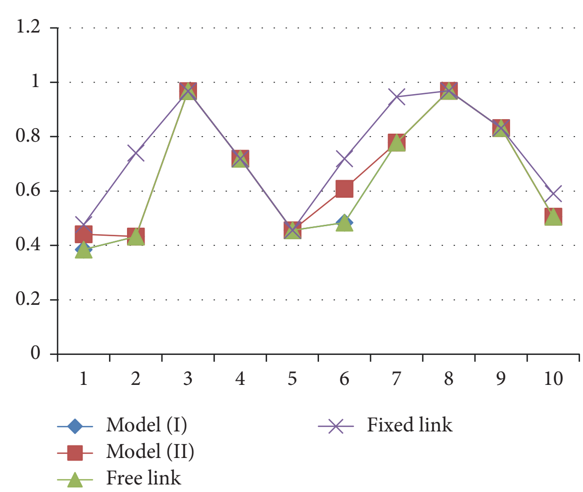

FIGURE 4: Comparisons of scores between proposed network models and NSBM models proposed by Tone and Tsutsui.

Table 4: Slacks-based network DEA.

DMU	Free link case				Fixed link case			
	ρ_0^*	div 1 (0.4)	div 2 (0.2)	div 3 (0.4)	ρ^*	div 1 (0.4)	div 2 (0.2)	div 3 (0.4)
DMU1	0.385	0.383	0.383	0.389	0.477	0.633	0.339	0.393
DMU2	0.433	0.260	0.341	0.652	0.740	0. 349	1.000	1.000
DMU3	0.968	1.000	1.000	0.919	0.968	1.000	1.000	0.919
DMU4	0.719	0.297	1.000	1.000	0.719	0.297	1.000	1.000
DMU5	0.456	0.263	1.000	0.377	0.456	0.263	1.000	0.377
DMU6	0.484	0.406	0.420	0.593	0.719	1.000	0.403	0.596
DMU7	0.778	0.712	0.740	0.863	0.947	1.000	1.000	0.868
DMU8	0.969	0.922	1.000	1.000	0.969	0.922	1.000	1.000
DMU9	0.832	1.000	1.000	0.581	0.832	1.000	1.000	0.581
DMU10	0.506	0.271	0.338	0.825	0.590	0.287	0.376	1.000
Avarage	0.653	0.551	0.722	0.720	0.742	0.675	0.812	0.773

Table 5: Exhibition of the role of intermediate measures in proposed models.

DMU	Model (I)		Model (II)	
	$Z1$	$Z2$	$Z1$	$Z2$
DMU1	Input to div 2	Input to div 3	Input to div 2	Output from div 2
DMU2	Output from div 1	Output from div 2	Output from div 1	Output from div 2
DMU3	Input to div 2	Input to div 3	Input to div 2	Output from div 2
DMU4	Input to div 2	Input to div 3	Input to div 2	Input to div 3
DMU5	Input to div 2	Input to div 3	Input to div 2	Output from div 2
DMU6	Input to div 2	Input to div 3	Input to div 2	Input to div 3
DMU7	Output from div 1	Input to div 3	Output from div 1	Output from div 2
DMU8	Input to div 2	Input to div 3	Input to div 2	Output from div 2
DMU9	Input to div 2	Input to div 3	Output from div 1	Output from div 2
DMU10	Output from div 1	Output from div 2	Output from div 1	Output from div 2

Model (II) takes into account the inefficiency associated with the link variables, whereas the NSBM does not.

4.4. Model (I) and Model (II). Comparing the SBM scores obtained by Model (I) and Model (II) shows that incorporation of intermediate product slacks in efficiency measurement may increase or decrease the divisional or overall efficiency (see Figure 2). To explain more, let us see how the categorization of intermediate measures into input or output type will change when we change the objective function (4) to (30).

As it can be seen from Table 4 changing the objective function (4) to (30) categorizes the intermediate measures into input or output type in a different way. For example in the optimal solution of Model (I) for DMU9 the intermediate measure z_1 is considered as an input to div 2, while in Model (II) it is considered as the outputs from div 1. It means that the target value of z_1 which links div 1 to div 2 in Model (I) is greater and in Model (II) is smaller than its observed value.

Table 5 exhibits the optimal role of intermediate measures obtained from Model (I) and Model(II). In optimal solutions of both models for DMU6 z_1, which links div 1 to div 2, is considered as input to div 2 while there are significant difference between the divisional efficiency scores of ρ_6^{*2} and φ_6^{*2}. Calculation below reveals the reason (see Appendix Tables 6 and 7):

$$\begin{aligned}\rho_6^{*2} &= 1 - \frac{1}{m_2}\left(\sum_{i=1}^{m_2}\frac{s_{i6}^{*k-}}{x_{i6}}\right) = 1 - \frac{1}{1}\left(\frac{0.081}{0.139}\right) = 0.420,\\ \varphi_6^{*2} &= 1\\ &\quad - \frac{1}{m_k + \sum_{d=1}^{l_{(f,k)}} y_d^{*(f,k)}}\left(\sum_{i=1}^{m_k}\frac{s_{i6}^{*k-}}{x_{i6}} + \sum_{d=1}^{l_{(f,k)}}\frac{s_{d6}^{*(f,k)-}}{z_{d6}^{(f,k)}}\right)\\ &= 1 - \frac{1}{1+1}\left(\frac{0.081}{0.139} + \frac{0.130}{0.966}\right) = 0.643.\end{aligned} \tag{38}$$

We can also show the efficiency calculations for div 2 of DMU1.

$$\begin{aligned}\rho_1^{*2} &= 1 - \frac{1}{m_2}\left(\sum_{i=1}^{m_2}\frac{s_{i1}^{*k-}}{x_{i1}}\right) = 1 - \frac{1}{1}\left(\frac{0.171}{0.277}\right) = 0.383,\\ \varphi_1^{*2} &= 1\\ &\quad - \frac{1}{m_k + \sum_{d=1}^{l_{(f,k)}} y_d^{*(f,k)}}\left(\sum_{i=1}^{m_k}\frac{s_{i1}^{*k-}}{x_{i1}} + \sum_{d=1}^{l_{(f,k)}}\frac{s_{d1}^{*(f,k)-}}{z_{d1}^{(f,k)}}\right)\\ &= 1 - \frac{1}{1+1}\left(\frac{0.171}{0.277} + \frac{0.058}{0.894}\right) = 0.659.\end{aligned} \tag{39}$$

Table 6: Optimum slack variables in Model (I).

	s_1^{*1-}	s_1^{*2-}	s_1^{*3-}	s_1^{*2+}	s_1^{*3+}	$s_1^{*(1,2)+}$	$s_1^{*(1,2)-}$	$s_1^{*(2,3)-}$	$s_1^{*(2,3)+}$
DMU1	0.517	0.171	0.588	0.000	0.000	0.000	0.058	0.000	0.007
DMU2	0.912	0.087	0.154	0.373	0.016	0.158	0.000	0.019	0.000
DMU3	0.00	0.000	0.039	0.000	0.000	0.000	0.000	0.000	0.000
DMU4	1.043	0.000	0.00	0.000	0.000	0.000	0.000	0.000	0.000
DMU5	1.173	0.000	0.669	0.000	0.015	0.000	0.000	0.000	0.002
DMU6	0.469	0.081	0.222	0.112	0.000	0.000	0.130	0.000	0.003
DMU7	0.130	0.019	0.050	0.341	0.000	0.189	0.000	0.000	0.000
DMU8	0.032	0.000	0.00	0.000	0.000	0.000	0.000	0.000	0.000
DMU9	0.00	0.000	0.290	0.000	0.002	0.000	0.000	0.000	0.000
DMU10	0.891	0.099	0.059	0.088	0.000	0.029	0.000	0.001	0.000

Table 7: Optimum slack variables in Model (II).

	s_1^{*1-}	s_1^{*2-}	s_1^{*3-}	s_1^{*2+}	s_1^{*3+}	$s_1^{*(1,2)+}$	$s_1^{*(1,2)-}$	$s_1^{*(2,3)-}$	$s_1^{*(2,3)+}$
DMU1	0.517	0.171	0.588	0.000	0.000	0.000	0.058	0.000	0.000
DMU2	0.912	0.087	0.154	0.373	0.016	0.158	0.000	0.019	0.000
DMU3	0.000	0.000	0.039	0.000	0.000	0.000	0.000	0.000	0.000
DMU4	1.043	0.000	0.000	0.000	0.000	0.000	0.130	0.000	0.000
DMU5	1.173	0.000	0.669	0.000	0.015	0.000	0.000	0.000	0.000
DMU6	0.469	0.081	0.222	0.112	0.000	0.000	0.000	0.000	0.002
DMU7	0.130	0.019	0.050	0.341	0.000	0.189	0.000	0.000	0.000
DMU8	0.032	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
DMU9	0.000	0.000	0.290	0.000	0.002	0.000	0.000	0.000	0.000
DMU10	0.891	0.099	0.059	0.088	0.000	0.029	0.000	0.001	0.000

Calculations above show the impact of inefficiency associated with link flows on efficiency score.

5. Conclusion

In this paper in order to address the conflict caused by the dual role of intermediate measures we proposed two alternative slacks-based measure models, called Model (I) and Model (II). To resolve this conflict the intermediate measures are categorized into input or output type endogenously. These categorizations allow models to identify nonzero slacks and uncover the sources of inefficiency associated with intermediate measures. In Model (I) the excesses or shortfalls corresponding to intermediate measures are contributed to the optimum objective indirectly. In order to incorporate the direct effect of inefficiency associated with intermediate measures, we proposed Model (II) in which the average reduction or expansion rate of intermediate products has been taken into the account in the objective function. Keeping continuity of link flows between divisions and incorporation of link flows in efficiency measurements at the same time is the clear advantage of our proposed model over other approaches.

To verify our proposed models we provided a numerical example and we compared the results with black box model, separation model, and Slacks-based network DEA (free link and fixed link case). In comparing the scores obtained by proposed network models and black box model, no significant correlation between the efficiency scores was found and the trends of network models were in sharp contrast to that of black box model. It was not unexpected because the internal linking activities in black box approach were neglected.

Overall and divisional efficiency scores obtained by Model (I) in the numerical example are equal to those of free link case. It is quite natural because in both models the continuity of link values between divisions is assured and the target values of the intermediate products are free to be above or below their observed values. A clear advantage of using Model (I) is revealing the role of intermediate product in the system. Since the linking constraints of the fixed link case are tighter than that of Model (I), the scores of the fixed link case tend to be higher than those of Model (I) for every DMU.

The scores of separate model follow a roughly similar trend to those of the proposed network models. However there are some differences between network models and separate model that must be caused by different assumption on the linking activities. The proposed network models keep the continuity of link flows among divisions whereas the separate model does not.

Separate model and Model (II) both take into account the inefficiency associated with the link variables.

Comparing the results of Model (I) and Model (II), we can see how the inclusion of intermediate product slacks in the objective function may change the categorization of the intermediate products and exert influence over the efficiency of each division.

For further research we can suggest the following issues.

The proposed approach can be easily extended to the dynamic network models.

LP formulation of Model (II) could be analyzed and interpreted. The proposed models can be extended to the situation in which some input/output data are fuzzy numbers. Another possible line of research is to include undesirable (or bad) outputs.

Appendix

See Tables 6 and 7.

ILP Formulation of Model (I)

In Model (I) the objective function is clearly nonlinear. The objective function can be easily transformed to linear form by Charnes-Cooper transformations.

By introducing a positive scalar variable t we change the objective function 1 as follows:

$$\rho = \min \quad \sum_{k=1}^{K} w_k \left[t - \frac{1}{m_k} \left(\sum_{i=1}^{m_k} \frac{t s_{pi}^{k-}}{x_{ip}} \right) \right]$$
$$\sum_{k=1}^{K} w_k \left[1 + \frac{1}{r_k} \left(\sum_{r=1}^{r_k} \frac{s_{rp}^{k+}}{y_{rp}} \right) \right] = t. \tag{A.1}$$

Now let us define

$t\lambda_j^k = \Lambda_j^k$, $S_{ip}^{k-} = ts_{ip}^{k-}$, $S_{rp}^{k+} = ts_{ip}^{k+}$, $S_{dp}^{(k,h)+} = ts_{dp}^{(k,h)+}$, $S_{dp}^{(k,h)-} = ts_{dp}^{(k,h)-}$, $Z_d^{(k,h)} = tz_d^{(k,h)}$, $Z_d'^{(k,h)} = tz_d'^{(k,h)}$, and $M' = Mt$.

Then the model becomes as follows:

$$\rho = \min \quad \sum_{k=1}^{K} w_k \left[t - \frac{1}{m_k} \left(\sum_{i=1}^{m_k} \frac{S_{ip}^{k-}}{x_{ip}^{k}} \right) \right]$$

$$\text{s.t.} \quad \sum_{k=1}^{K} w_k \left[t + \frac{1}{r_k} \left(\sum_{r=1}^{r_k} \frac{S_{rp}^{k+}}{y_{rp}} \right) \right] = 1$$

$$(k = 1, \ldots, K), \ (r = 1, \ldots, r_k)$$

$$\sum_{j=1}^{n} \Lambda^k{}_j x_{ij}^k + S_{ip}^{k-} = t x_{ip}^k$$

$$(k = 1, \ldots, K), \ (i = 1, \ldots, m_k)$$

$$\sum_{j=1}^{n} \Lambda^k{}_j y_{rj}^k - S_{rp}^{k+} = t y_{rp}^k$$

$$(k = 1, \ldots, K), \ (r = 1, \ldots, r_k)$$

$$\sum_{j=1}^{n} \Lambda^h{}_j z_{dj}^{(k,h)} + S_{dp}^{(k,h)-} = Z_d'^{(k,h)}$$

$$(d = 1, \ldots, l_{(k,h)}), \ \forall (k,h)$$

$$\sum_{j=1}^{n} \Lambda^k{}_j z_{dj}^{(k,h)} - S_{dp}^{(k,h)-} = Z_d^{(k,h)}$$

$$(d = 1, \ldots, l_{(k,h)}), \ \forall (k,h)$$

$$\sum_{j=1}^{n} \Lambda^k{}_j z_{dj}^{(k,h)} = \sum_{j=1}^{n} \Lambda^h{}_j z_{dj}^{(k,h)}$$

$$(d = 1, \ldots, l_{(k,h)}), \ \forall (k,h)$$

$$0 \le S_{dp}^{(k,h)-} \le M' y_d^{(k,h)}$$

$$(d = 1, \ldots, l_{(k,h)}), \ \forall (k,h)$$

$$0 \le S_{dp}^{(k,h)+} \le M' \left(1 - y_d^{(k,h)}\right)$$

$$(d = 1, \ldots, l_{(k,h)}), \ \forall (k,h)$$

$$tz_{dp}^{(k,h)} - M' y_{dp}^{(k,h)} \le Z'^{(k,h)}_d$$

$$\le tz_{dp}^{(k,h)} + M' y_{dp}^{(k,h)}$$

$$(d = 1, \ldots, l_{(k,h)}), \ \forall (k,h)$$

$$tz_{dp}^{(k,h)} - Mt' \left(1 - y_d^{(k,h)}\right) \le Z_d^{(k,h)}$$

$$\le tz_{dp}^{(k,h)} + M' \left(1 - y_d^{(k,h)}\right)$$

$$(d = 1, \ldots, l_{(k,h)}), \ \forall (k,h)$$

$$\sum_{j=1}^{n} \Lambda^k{}_j = t \quad (k = 1, \ldots, K)$$

$$y_d^{(k,h)} = \{0, 1\};$$
$$Z_d^{(k,h)}, \ Z_d'^{(k,h)} : \text{free}, \tag{A.2}$$

$$\Lambda^k{}_j \ge 0,$$
$$S_{rp}^{k+} \ge 0, \tag{A.3}$$
$$S_{ip}^{k-} \ge 0.$$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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