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Research Article

An Approximate Solution of the Time-Fractional Fisher Equation with Small Delay by Residual Power Series Method

Ali Demir,[1] **Mine Aylin Bayrak**,[1] **and Ebru Ozbilge**[2]

[1] *Department of Mathematics, University of Kocaeli, Kocaeli, Turkey*
[2] *American University of the Middle East, Department of Mathematics and Statistics, Egaila, Kuwait*

Correspondence should be addressed to Ali Demir; ademir@kocaeli.edu.tr

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An analytical solution of the time-fractional Fisher equation with small delay is established by means of residual the residual power series method (RPSM) where the fractional derivative is taken in the Caputo sense. Taking advantage of small delay, the time-fractional Fisher equation is expanded in powers series of delay term ϵ. By using RPSM analytical solution of time-fractional of Fisher equation is constructed. The final results and graphical consequences illustrate that the proposed method in this study is very efficient, effective, and reliable for the solution of the time-fractional Fisher equation with small delay.

1. Introduction

In last few decades, fractional calculus growing considerable interest is used in bioengineering, thermodynamics, viscoelasticity, control theory, aerodynamics, electromagnetics, signal processing, chemistry, and finance [1–7]. Many numerical methods have been applied and analyzed for differential equations with fractional order derivative of Riemann-Liouville or Caputo sense [6–12]. Delay differential equations (DDEs) can be considered as the generalization of the ordinary differential equations which are appropriate for modelling physical systems with memory. The delay is an important fact of the physical systems such as kinetics, controllers, signal processing, and damping behavior of viscoelastic materials.

The RPSM was established as a powerful method for fuzzy differential equations [13]. It has been successfully put into practice in various fields [14–20]. The solution of problems by RPSM is obtained in the form of Maclaurin series. It is an efficient method to find out the coefficients of the series solutions. Construction of multidimensional and multiple solutions for fractional differential equations in the form of power series is an important advantage of RPSM. RPSM is effective and easy to use for solving linear and nonlinear FPDEs without linearization, perturbation, or discretization.

In this study, we extend the application of the RPSM in order to establish an approximate solution to time-fractional Fisher equation with small delay

$$D_t^{\alpha} u(x,t) = D_{xx} u(x,t) + 6u(x,t-\epsilon)(1-u(x,t-\epsilon)), \quad x \in R,\ t > 0,\ 0 < \alpha \leqslant 1, \tag{1}$$

subject to initial condition

$$u(x,0) = A_1(x) \tag{2}$$

which was originally proposed by Fisher [20] which describes the spatial and temporal propagation of a virile gene in an infinite medium. The time-fractional Fisher equation can be readily solved by many methods [21–25]. Moreover, there are various studies on nonlocal of action of fractional differential equations including space fractional derivative [26–30]. In this article, by making use of the residual power series method (RPSM), we find series solution for the time-fractional Fisher equation with small delay. By taking advantage of the small delay ϵ we expand the term including small delay to Taylor series which reduced the problem to a perturbation problem. Applying RPSM we determine the solution of equations which are the obtained from the

coefficients of ϵ. Replacing these solutions in the Taylor series the solution of the original problem is obtained.

2. Preliminaries

In this section, the basic definitions and various features for fractional calculus theory are shown [6, 31–35].

Definition 1. The Riemann-Liouville fractional integral of order α $(\alpha \geq 0)$ is given as [14]

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \quad \alpha > 0,\ x > 0, \tag{3}$$

$$J^0 f(x) = f(x). \tag{4}$$

Definition 2. The Caputo fractional derivative with order α is given as [11, 12]

$$\begin{aligned} D^{\alpha} f(x) &= J^{m-\alpha} D^m f(x) \\ &= \int_0^x (x-t)^{m-\alpha-1} \frac{d^m}{dt^m} f(t)\, dt, \\ &\qquad m-1 < \alpha < m,\ x > 0, \end{aligned} \tag{5}$$

where D^m is the classic differential operator with order m.

By the Caputo derivative, we get

$$D^{\alpha} x^{\beta} = 0, \quad \beta < \alpha, \tag{6}$$

$$D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, \quad \beta \geq \alpha. \tag{7}$$

Definition 3. Caputo's time-fractional derivative of order α of $u(x,t)$ is defined as [11, 12]

$$D_t^{\alpha} u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial t^m} d\xi, & m-1 < \alpha < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N. \end{cases} \tag{8}$$

Definition 4. A power series expansion of the form

$$\begin{aligned} \sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} &= c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} \\ &\quad + \cdots, \\ &\qquad 0 \leq m-1 < \alpha \leq m,\ t \geq t_0 \end{aligned} \tag{9}$$

is called fractional power series about $t = t_0$ [25].

Definition 5. A power series expansion of the form

$$\begin{aligned} \sum_{m=0}^{\infty} f_m(x) (t-t_0)^{m\alpha} &= f_0(x) + f_1(x)(t-t_0)^{\alpha} \\ &\quad + f_2(x)(t-t_0)^{2\alpha} + \cdots, \\ &\qquad 0 \leq m-1 < \alpha \leq m,\ t \geq t_0 \end{aligned} \tag{10}$$

is called multiple fractional power series about $t = t_0$ [11, 12].

Theorem 6. *Suppose that $u(x,t)$ has a multiple fractional power series representation at $t = t_0$ of the form*

$$\begin{aligned} u(x,t) &= \sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha}, \\ &\qquad x \in I,\ t_0 \leq t \leq t_0 + R. \end{aligned} \tag{11}$$

If $D_t^{m\alpha} u(x,t)$, $m = 0, 1, 2, \ldots$, are continuous on $I \times (t_0, t_0 + R)$, then $f_m(x) = D_t^{m\alpha} u(x,t_0)/\Gamma(m\alpha+1)$.

3. RPSM of the Time-Fractional Fisher Equation with Small Delay

Consider the time-fractional Fisher equation (1) with small delay. Since the delay term is very small, we replace $u(x,t-\epsilon)$ (1) by the following Taylor series expansion:

$$u(x,t-\epsilon) = u(x,t) - \epsilon D_t u(x,t) + O(\epsilon^2), \tag{12}$$

where we ignore the higher order terms, which leads to

$$\begin{aligned} D_t^{\alpha} u(x,t) &= D_{xx} u(x,t) + 6u(x,t)(1-u(x,t)) \\ &\quad + 6\epsilon u D_t u - 6\epsilon(1-u) D_t u - 6\epsilon^2 (D_t u)^2 \end{aligned} \tag{13}$$

and assuming that $n\alpha = 1$, we obtain

$$\begin{aligned} D_t^{\alpha} u(x,\mathrm{t}) &= D_{xx} u(x,t) + 6u(x,t)(1-u(x,t)) \\ &\quad + 6\epsilon u D_t^{n\alpha} u - 6\epsilon(1-u) D_t^{n\alpha} u \\ &\quad - 6\epsilon^2 (D_t^{n\alpha} u)^2. \end{aligned} \tag{14}$$

In this study we obtain the solution of time-fractional Fisher equation with small delay for $n = 2$; hence we get the following equation:

$$\begin{aligned} D_t^{\alpha} u(x,t) &- D_{xx} u(x,t) - 6u(x,t)(1-u(x,t)) \\ &- 6\epsilon(2u-1) D_t^{2\alpha} u + 6\epsilon^2 (D_t^{2\alpha} u)^2 = 0. \end{aligned} \tag{15}$$

Using (12) in (1) we reduce the problem to a perturbation problem for which we use the following series solution:

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots \tag{16}$$

to obtain the solution of (14). Substituting (15) into (14) leads to the following equation:

$$\begin{aligned} &D_t^{\alpha} (u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots) - D_{xx} (u_0 + \epsilon u_1 \\ &\quad + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots) - 6 (u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 \\ &\quad + \cdots) + 6 (u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots)^2 \\ &\quad + 12\epsilon (u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots) (D_t^{2\alpha} u_0 \\ &\quad + \epsilon D_t^{2\alpha} u_1 + \epsilon^2 D_t^{2\alpha} u_2 + \epsilon^3 D_t^{2\alpha} u_3 + \cdots) \\ &\quad + 6\epsilon (D_t^{2\alpha} u_0 + \epsilon D_t^{2\alpha} u_1 + \epsilon^2 D_t^{2\alpha} u_2 + \epsilon^3 D_t^{2\alpha} u_3 \end{aligned}$$

$$+\cdots\Big)+6\epsilon^2\left(D_t^{2\alpha}u_0+\epsilon D_t^{2\alpha}u_1+\epsilon^2 D_t^{2\alpha}u_2 + \epsilon^3 D_t^{2\alpha}u_3+\cdots\right)=0, \tag{17}$$

Hence we get the following equations:

$$D_t^{\alpha}u_0-D_{xx}u_0-6u_0+6u_0^2=0, \tag{18}$$

$$D_t^{\alpha}u_1-D_{xx}u_1-6u_1+12u_0u_1-12u_0D_t^{2\alpha}u_0 + 6D_t^{2\alpha}u_0=0, \tag{19}$$

$$D_t^{\alpha}u_2-D_{xx}u_2-6u_2+6u_1^2+12u_0u_2-12u_0D_t^{2\alpha}u_1 - 12u_1D_t^{2\alpha}u_0+6D_t^{2\alpha}u_1+6\left(D_t^{2\alpha}u_0\right)^2=0, \tag{20}$$

$$D_t^{\alpha}u_3-D_{xx}u_3-6u_3+12u_0u_3+12u_1u_2 - 12u_0D_t^{2\alpha}u_2-12u_1D_t^{2\alpha}u_1-12u_2D_t^{2\alpha}u_0 + 6D_t^{2\alpha}u_2+12D_t^{2\alpha}u_0D_t^{2\alpha}u_1=0 \tag{21}$$

and so on. We apply the RPSM to find out series solution for these equation subject to given initial conditions by replacing its fractional power series expansion with its truncated residual function. From each equation, a repetition formula for the calculation of coefficients is supplied, while coefficients in fractional power series expansion can be calculated by repeatedly fractional differentiation of the truncated residual function [13–20]. The RPSM propose the solutions for (18)-(21) as a fractional power series about the initial point $t=0$ [13]

$$u_i(x,t)=\sum_{k=0}^{\infty}f_{i,k}(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad i=0,1,2,3,\ x\in I,\ 0\le t<R. \tag{22}$$

To obtain the numerical values from this series, let $u_m(x,t)$ denote the m-th truncated series of $u(x,t)$. That is,

$$u_{i,m}(x,t)=\sum_{k=0}^{m}f_{i,k}(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad i=0,1,2,3,\ x\in I,\ 0\le t<R. \tag{23}$$

By the initial condition, the $0th$ residual power series approximate solution of $u(x,t)$ can be written as follows:

$$u_0(x,t)=f_0(x)=u(x,0)=A_1(x) \tag{24}$$

Equation (22) can be written as

$$u_m(x,t)=A_1(x)+\sum_{k=2}^{m}A_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad 0<\alpha\le 1,\ x\in I,\ 0\le t,\ k=2,3,\ldots \tag{25}$$

Define the residual function for (18) [35]

$$Res_0(x,t)=D_t^{\alpha}u_0-D_{xx}u_0-6u_0+6u_0^2 \tag{26}$$

and the mth residual function can be expressed as

$$Res_m(x,t)=D_t^{\alpha}u_m-D_{xx}u_m-6u_m+6u_m^2 \tag{27}$$

From [13–23], by making use of some results such as $Res(x,t)=0$ for each $x\in I$ and $t\ge 0$ and $D_t^{r\alpha}Res(x,0)=D_t^{r\alpha}Res_m(x,0)=0, r=0,1,2,\ldots,m$ are used to obtain the solution.

Substituting the mth truncated series of $u(x,t)$ into (18), calculating the fractional derivative $D_t^{(m-1)\alpha}$ of $Res(x,t), m=1,2,3,\ldots$, at $t=0$ and solving the following obtained algebraic system

$$D_t^{(m-1)\alpha}Res_m(x,0)=0,\quad 0<\alpha\le 1,\ m=1,2,3,\ldots, \tag{28}$$

the required coefficients $A_k(x), k=2,3,\ldots,m$ in (24) are determined.

In order to determine $A_2(x)$, the 1st residual function in (26) can be written as follows:

$$Res_{0,1}(x,t)=D_t^{\alpha}u_{0,1}-D_{xx}u_{0,1}-6u_{0,1}+6u_{0,1}^2 \tag{29}$$

where $u_{0,1}(x,t)=A_1(x)+A_2(x)(t^{\alpha}/\Gamma(1+\alpha))$. Therefore,

$$Res_{0,1}(x,t)=A_2-\left(A_1''+A_2''\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) -6\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) +6\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2. \tag{30}$$

From (27), we deduce that $Res_1(x,0)=0$, which leads to

$$A_2(x)=A_1''+6A_1-6A_1^2. \tag{31}$$

Similarly, to obtain $A_3(x)$, the 2nd residual function in (26) can be written in the following form:

$$Res_{0,2}(x,t)=D_t^{\alpha}u_{0,2}-D_{xx}u_{0,2}-6u_{0,2}+6u_{0,2}^2, \tag{32}$$

where $u_{0,2}(x,t)=A_1(x)+A_2(x)(t^{\alpha}/\Gamma(1+\alpha))+A_3(x)(t^{2\alpha}/\Gamma(1+2\alpha))$. Therefore,

$$\begin{aligned}Res_{0,2}(x,t)&=\left(A_2(x)+A_3(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)\\&-\left(A_1''+A_2''\frac{t^{\alpha}}{\Gamma(1+\alpha)}+A_3''\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)\\&-6\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}+A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)\\&+6\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}+A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2.\end{aligned} \tag{33}$$

The operator D_t^α is applied on both sides of (32) as follows:

$$\begin{aligned} D_t^\alpha Res_{0,2}(x,t) &= A_3 - \left(A_2'' + A_3'' \frac{t^\alpha}{\Gamma(1+\alpha)}\right) \\ &- 6\left(A_2 + A_3 \frac{t^\alpha}{\Gamma(1+\alpha)}\right) \\ &+ 6\left(A_2 + A_3 \frac{t^\alpha}{\Gamma(1+\alpha)}\right) \\ &\cdot \left(A_1 + A_2 \frac{t^\alpha}{\Gamma(1+\alpha)} + A_3 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) \\ &+ 6\left(A_1 + A_2 \frac{t^\alpha}{\Gamma(1+\alpha)} + A_3 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) \\ &\cdot \left(A_2 + A_3 \frac{t^\alpha}{\Gamma(1+\alpha)}\right) \end{aligned} \tag{34}$$

From (27) and (33),

$$A_3(x) = A_2'' + 6A_2 - 12A_1A_2. \tag{35}$$

The same manner is repeated as above; the following recurrence results is obtained

$$A_4(x) = A_3'' + 6A_3 - 12A_1A_3 - 12A_2^2 \tag{36}$$

and so on.

Thus, we have

$$\begin{aligned} u_0 &= A_1 + \left(A_1'' + 6A_1 - 6A_1^2\right) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ &+ \left(A_2'' + 6A_2 - 12A_1A_2\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ \left(A_3'' + 6A_3 - 12A_1A_3 - 12A_2^2\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &+ \cdots \end{aligned} \tag{37}$$

Define the residual function for (19)

$$\begin{aligned} Res_1(x,t) &= D_t^\alpha u_1 - D_{xx}u_1 - 6u_1 + 12u_0u_1 \\ &- 12u_0 D_t^{2\alpha} u_0 + 6D_t^{2\alpha} u_0. \end{aligned} \tag{38}$$

Suppose that

$$\begin{aligned} u_1(x) &= B_1(x) + B_2(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + B_3(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ B_4(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \end{aligned} \tag{39}$$

By the initial condition, the $0th$ residual power series approximate solution of $u(x,t)$ can be written as follows:

$$u_1(x,0) = 0 = B_1(x) \tag{40}$$

We apply the RPSM to find out $B_k(x)$, $k = 1,2,3,\ldots,m$, in (19). Thus, we have

$$\begin{aligned} u_1 &= B_1 + \left(B_1'' + 6B_1 - 12A_1B_1 + 12A_1A_3 - 6A_3\right) \\ &\cdot \frac{t^\alpha}{\Gamma(1+\alpha)} + \left(B_2'' + 6B_2 - 12A_2B_1 - 12A_1B_2\right. \\ &\left. + 12A_2A_3 + 12A_1A_4 - 6A_4\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(B_3''\right. \\ &\left. + 6B_3 - 12A_3B_1 - 12A_1B_3 - 24A_2B_2 + 12A_3^2\right. \\ &\left. + 24A_2A_4\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \end{aligned} \tag{41}$$

Define the residual function for (20)

$$\begin{aligned} Res_2(x,t) &= D_t^\alpha u_2 - D_{xx}u_2 - 6u_2 + 6u_1^2 + 12u_0u_2 \\ &- 12u_0 D_t^{2\alpha} u_1 - 12u_1 D_t^{2\alpha} u_0 + 6D_t^{2\alpha} u_1 \\ &+ 6\left(D_t^{2\alpha} u_0\right)^2. \end{aligned} \tag{42}$$

Suppose that

$$\begin{aligned} u_2(x) &= C_1(x) + C_2(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ &+ C_3(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + C_4(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &+ \cdots \end{aligned} \tag{43}$$

By the initial condition, the $0th$ residual power series approximate solution of $u_2(x,t)$ can be written as follows:

$$u_2(x,0) = 0 = C_1(x) \tag{44}$$

We apply the RPSM to find out $C_k(x), k = 1,2,3,\ldots,m$ in (20).

Thus, we have

$$\begin{aligned} u_2 &= C_1 + \left(C_1'' + 6C_1 - 6B_1^2 - 12A_1C_1 + 12A_1B_3\right. \\ &\left. + 12A_3B_1 - 6B_3 - 6A_3^2\right) \frac{t^\alpha}{\Gamma(1+\alpha)} + \left(C_2'' + 6C_2\right. \\ &- 12B_1B_2 - 12A_2C_1 - 12A_1C_2 + 12A_2B_3 \\ &\left. + 12A_1B_4 + 12A_4B_1 + 12A_3B_2 - 6B_4 - 12A_3A_4\right) \\ &\cdot \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(C_3'' + 6C_3 - 12B_3B_1 - 12B_2^2\right. \\ &- 24A_2C_2 - 12A_3C_1 - 12A_1C_3 + 24A_3B_3 \\ &\left. + 24A_2B_4 + 24A_4B_2 - 12A_4^2\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \end{aligned} \tag{45}$$

Define the residual function for (11)

$$\begin{aligned} Res_3(x,t) = {} & D_t^{\alpha} u_3 - D_{xx} u_3 - 6u_3 + 12u_0 u_3 \\ & + 12u_1 u_2 - 12u_0 D_t^{2\alpha} u_2 - 12u_1 D_t^{2\alpha} u_1 \\ & - 12u_2 D_t^{2\alpha} u_0 + 6D_t^{2\alpha} u_2 \\ & + 12D_t^{2\alpha} u_0 D_t^{2\alpha} u_1. \end{aligned} \tag{46}$$

Suppose that

$$\begin{aligned} u_3(x) = {} & D_1(x) + D_2(x) \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ & + D_3(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + D_4(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ & + \cdots \end{aligned} \tag{47}$$

By the initial condition, the $0th$ residual power series approximate solution of $u_2(x,t)$ can be written as follows:

$$u_3(x,0) = 0 = D_1(x) \tag{48}$$

We apply the RPSM to find out $D_k(x)$, $k = 1, 2, 3, \ldots, m$, in (21).

Thus, we have

$$\begin{aligned} u_3 = {} & D_1 + \left(D_1'' + 6D_1 - 12A_1 D_1 - 12B_1 C_1 \right. \\ & \left. + 12A_1 C_3 + 12A_3 C_1 - 6C_3 - 12A_3 B_3\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ & + \left(D_2'' + 6D_2 - 12A_2 D_1 - 12A_1 D_2 - 12B_2 C_1 \right. \\ & - 12B_1 C_2 + 12A_2 C_3 + 12A_1 C_4 + 12B_2 B_3 \\ & + 12B_1 B_4 + 12A_4 C_1 + 12A_3 C_2 - 6C_4 - 12A_4 B_3 \\ & \left. - 12A_3 B_4\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(D_3'' + 6D_3 - 12A_3 D_1 \right. \\ & - 12A_1 D_3 - 24A_2 D_2 - 12B_3 C_1 - 12B_1 C_3 \\ & - 24B_2 C_2 + 24A_3 C_3 + 24A_2 C_4 12B_3^2 + 24B_2 B_4 \\ & \left. + 24A_4 C_2 - 24A_4 B_4\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \end{aligned} \tag{49}$$

For $n = 3$, we obtain u_0, u_1, u_2, and u_3 as follows:

$$\begin{aligned} u_0 = {} & A_1 + \left(A_1'' + 6A_1 - 6A_1^2\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ & + \left(A_2'' + 6A_2 - 12A_1 A_2\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \left(A_3'' + 6A_3 - 12A_1 A_3 - 12A_2^2\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ & + \cdots \end{aligned} \tag{50}$$

and

$$\begin{aligned} u_1 = {} & B_1 + \left(B_1'' + 6B_1 - 12A_1 B_1 + 12A_1 A_4 - 6A_4\right) \\ & \cdot \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \left(B_2'' + 6B_2 - 12A_2 B_1 - 12A_1 B_2 \right. \\ & \left. + 12A_2 A_4 + 12A_1 A_4 - 6A_4\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(B_3'' \right. \\ & \left. + 6B_3 - 12A_3 B_1 - 12A_1 B_3 - 24A_2 B_2 + 12A_3 A_4\right) \\ & \cdot \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \end{aligned} \tag{51}$$

and

$$\begin{aligned} u_2 = {} & C_1 + \left(C_1'' + 6C_1 - 6B_1^2 - 12A_1 C_1 + 12A_1 B_4 \right. \\ & \left. + 12A_4 B_1 - 6B_4 - 6A_4^2\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \left(C_2'' + 6C_2 \right. \\ & - 12B_1 B_2 - 12A_2 C_1 - 12A_1 C_2 + 12A_2 B_4 \\ & \left. + 12A_4 B_2\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(C_3'' + 6C_3 - 12B_3 B_1 \right. \\ & - 12B_2^2 - 24A_2 C_2 - 12A_3 C_1 - 12A_1 C_3 + 12A_3 B_4 \\ & \left. + 12A_4 B_3\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \end{aligned} \tag{52}$$

and

$$\begin{aligned} u_3 = {} & D_1 + \left(D_1'' + 6D_1 - 12A_1 D_1 - 12B_1 C_1 \right. \\ & \left. + 12A_1 C_3 + 12A_3 C_1 - 6C_3 - 12A_3 B_3\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ & + \left(D_2'' + 6D_2 - 12A_2 D_1 - 12A_1 D_2 - 12B_2 C_1 \right. \\ & - 12B_1 C_2 + 12A_2 C_3 + 12A_1 C_4 + 12B_2 B_3 \\ & + 12B_1 B_4 + 12A_4 C_1 + 12A_3 C_2 - 6C_4 - 12A_4 B_3 \\ & \left. - 12A_3 B_4\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(D_3'' + 6D_3 - 12A_3 D_1 \right. \\ & - 12A_1 D_3 - 24A_2 D_2 - 12B_3 C_1 - 12B_1 C_3 \\ & - 24B_2 C_2 + 24A_3 C_3 + 24A_2 C_4 12B_3^2 + 24B_2 B_4 \\ & \left. + 24A_4 C_2 - 24A_4 B_4\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \end{aligned} \tag{53}$$

For $n = 4$, we have u_0, u_1, u_2, and u_3 as follows:

$$\begin{aligned} u_0 = {} & A_1 + \left(A_1'' + 6A_1 - 6A_1^2\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ & + \left(A_2'' + 6A_2 - 12A_1 A_2\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \left(A_3'' + 6A_3 - 12A_1 A_3 - 12A_2^2\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ & + \cdots \end{aligned} \tag{54}$$

and

$$u_1 = B_1 + \left(B_1'' + 6B_1 - 12A_1B_1\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \left(B_2'' + 6B_2 - 12A_2B_1 - 12A_1B_2\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(B_3'' + 6B_3 - 12A_3B_1 - 12A_1B_3 - 24A_2B_2\right) \cdot \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \tag{55}$$

and

$$u_2 = C_1 + \left(C_1'' + 6C_1 - 6B_1^2 - 12A_1C_1\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \left(C_2'' + 6C_2 - 12B_1B_2 - 12A_2C_1 - 12A_1C_2\right) \cdot \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(C_3'' + 6C_3 - 12B_3B_1 - 12B_2^2 - 24A_2C_2 - 12A_3C_1 - 12A_1C_3\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \tag{56}$$

and

$$u_3 = D_1 + \left(D_1'' + 6D_1 - 12A_1D_1 - 12B_1C_1\right) \cdot \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \left(D_2'' + 6D_2 - 12A_2D_1 - 12A_1D_2 - 12B_2C_1 - 12B_1C_2\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(D_3'' + 6D_3 - 12A_3D_1 - 12A_1D_3 - 24A_2D_2 - 12B_3C_1 - 12B_1C_3 - 24B_2C_2\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \tag{57}$$

4. Numerical Results

Consider the following time-fractional Fisher equation with small delay

$$D_t^{\alpha}u(x,t) = D_{xx}u(x,t) + 6u(x,t-\epsilon)(1-u(x,t-\epsilon)) \tag{58}$$

subject to initial condition

$$u(x,0) = \frac{1}{(1+e^{x})^2}. \tag{59}$$

Then, the exact solution of (1) without delay when $\alpha = 1$ is given by

$$u(x,t) = \frac{1}{(1+e^{x-5t})^2}. \tag{60}$$

It is clear from Figures 1 and 2 and Tables 1-2 that the graph of the solution $u(x,t)$ of time-fractional Fisher equation with small delay with respect to space x in some neighborhood of $x = 0$ solution acts as a critically damped oscillator for $\alpha < 1/2$ whereas it acts as an over damped oscillator for $\alpha > 1/2$. In Figures 3 and 4, as it is seen from the graph of the solution $u(x,t)$ of time-fractional Fisher equation with small delay with respect to time t, the effect of the small time delay is limited for the fractional derivative of order $\alpha < 1/2$; i.e., the solution $u(x,t)$ of time-fractional Fisher equation with small delay keeps close to the solution $u(x,t)$ of Fisher equation without delay. As a result, it is concluded that the solution $u(x,t)$ is stable under small time delay for $\alpha < 1/2$. On the other hand, the effect of the small time delay increases as time increases for $\alpha > 1/2$; i.e., the solution $u(x,t)$ of time-fractional Fisher equation with small delay gets away from the solution $u(x,t)$ of Fisher equation without delay. As a result, it is concluded that the solution $u(x,t)$ is unstable under small time delay for $\alpha > 1/2$.

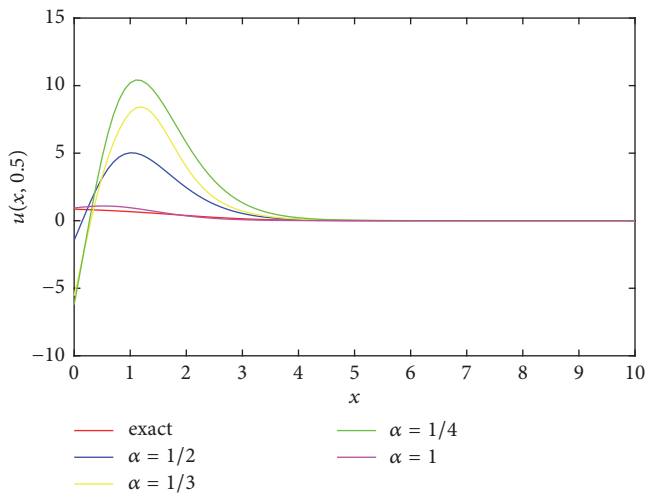

Figure 1: u solution for $t = 0.5$, $\epsilon = 0.001$, and $0 \le x \le 10$.

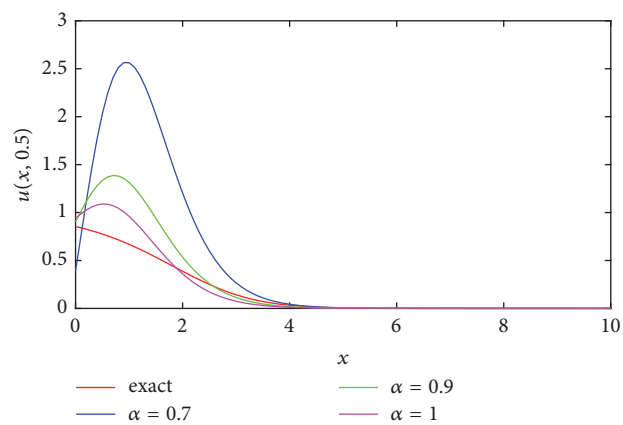

Figure 2: u solution for $t = 0.5$, $\epsilon = 0.001$, and $0 \le x \le 10$.

5. Conclusion

Based on the analysis of the solution $u(x,t)$ of time-fractional Fisher equation with small delay, it is deduced that, at the

Table 1: The RPS solution for several values x and α for $t = 0.5$.

x	$\alpha = 1/4$	$\alpha = 1/3$	$\alpha = 1/2$
0.1	-3.95998	-3.72539	-0.046675
0.2	-1.65603	-1.82531	0.52762
0.4	2.76181	1.75026	2.37864
0.6	6.42178	4.62171	3.82190
0.8	8.93800	6.71422	4.70664
1	10.21002	8.02085	5.02524
1.5	9.17997	7.40754	4.10699
2	5.77391	3.99228	2.45076
2.5	2.95270	1.70703	1.21563
3	1.32987	0.70989	0.53796
3.5	0.55378	0.29586	0.22174
4	0.21974	0.12070	0.08746
4.5	0.08465	0.04780	0.03358
5	0.03203	0.01846	0.01268

Table 2: The RPS solution for several values t and α for $x = 0.5$.

t	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.1	-0.41551	0.75439	1.01269	0.98597
0.2	0.52501	1.11315	1.11369	1.02597
0.4	2.29109	1.76224	1.27825	1.07915
0.6	3.70390	2.24542	1.37108	1.08652
0.8	4.61711	2.51281	1.38106	1.04491
1	5.00312	2.56283	1.31405	0.96068
1.5	4.24695	2.02110	0.94079	0.65201
2	2.61056	1.20375	0.53506	0.36056
2.5	1.32027	0.59918	0.25991	0.17244
3	0.59119	0.26598	0.11381	0.07484
3.5	0.24539	0.10985	0.04664	0.03051
4	0.09719	0.04338	0.01833	0.01195
4.5	0.03740	0.01667	0.00702	0.00457
5	0.01414	0.00629	0.00265	0.00172

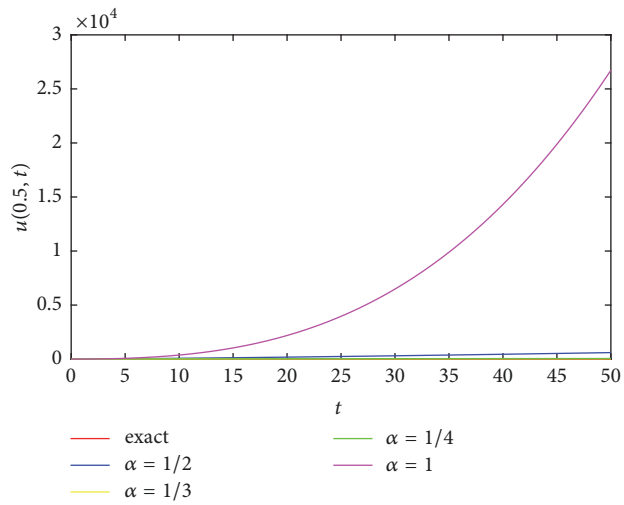

Figure 3: u solution for $x = 0.5$, $\epsilon = 0.001$, and $0 \le t \le 50$.

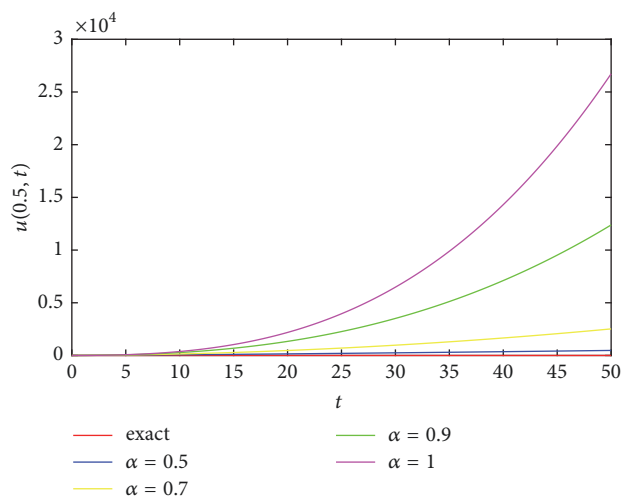

Figure 4: u solution for $x = 0.5$, $\epsilon = 0.001$, and $0 \le t \le 50$.

fractional derivative of order $\alpha = 1/2$, there is a transition of the solution $u(x, t)$ with respect to both variables x and t.

Nomenclature

$\Gamma(x)$: Gamma function
$J^{\alpha} f(x)$: Riemann-Liouville fractional integral
$D^{\alpha} f(x)$: Caputo fractional derivative
$D_t^{\alpha} f(x)$: Caputo time-fractional derivative.

Data Availability

The data in our manuscript is obtained by our calculations.

Disclosure

The previous version of the manuscript is presented in ICOMAA-2018 under the name of "On the Time-Fractional Fisher Equation with Small Delay".

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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