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Research Article

A Distributional Identity for the Bivariate Brownian Bridge: A Nontensor Gaussian Field

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The bivariate Brownian bridge, a nontensor Gaussian Field, is defined by $B(t_1, t_2) = W(t_1, t_2)|_{W(1,1)=0} = W(t_1, t_2) - t_1 t_2 W(1, 1)$, where $t_1, t_2 \in I = [0, 1]$ and $W(t_1, t_2)$ is a Brownian sheet. We obtain a distributional identity, a consequence of the Karhunen-Loève expansion for the bivariate Brownian bridge by Fredholm integral equation and Laplace transform approach.

1. Introduction

Let $X = \{X(\mathbf{t}), \mathbf{t} \in [0, 1]^2\}$ be a mean zero Gaussian process on $C([0, 1]^2)$ with covariance function $K_X(\mathbf{t}, \mathbf{s}) = \mathbb{E}X(\mathbf{t})X(\mathbf{s})$, $\mathbf{t}, \mathbf{s} \in [0, 1]^2$. Then the well-known Karhunen-Loève (KL) expansion is

$$X(\mathbf{t}) = \sum_{k\geq 1, j\geq 1} \eta_{k,j} \sqrt{\lambda_{k,j}} f_{k,j}(\mathbf{t}), \qquad (1)$$

where $\{\eta_{k,j}, k \geq 1, j \geq 1\}$ is a sequence of 2-index i.i.d. $N(0, 1)$ random variables, $\{f_{k,j}(\mathbf{t}), k \geq 1, j \geq 1\}$ forms an orthogonal sequence in $L^2([0, 1]^2)$, and $\{\lambda_{k,j}, k \geq 1, j \geq 1\}$ is the set of eigenvalues of the integral operator $T_X f(\mathbf{t}) = \int_{[0,1]^2} K_X(\mathbf{t}, \mathbf{s}) f(\mathbf{s}) d\mathbf{s}$. For the random variables X and Y, $X \overset{\text{law}}{=} Y$ means that X and Y have the same law; then a natural consequence of the KL expansions is the distributional identity

$$\int_{[0,1]^2} X^2(\mathbf{t})\, d\mathbf{t} \overset{\text{law}}{=} \sum_{k,j=1}^{\infty} \lambda_{k,j} \eta_{k,j}^2. \qquad (2)$$

As we know, a tied-down Brownian bridge (see [1]) is defined by

$$\begin{aligned} B_*(t_1, t_2) &= W(t_1, t_2) - t_1 W(1, t_2) - t_2 W(t_1, 1) \\ &\quad + t_1 t_2 W(1, 1), \end{aligned} \qquad (3)$$

whose covariance $(s_1 \wedge t_1 - s_1 t_1)(s_2 \wedge t_2 - s_2 t_2)$, $s_1, s_2, t_1, t_2 \in I = [0, 1]$ is tensor (separate) product and its Karhunen-Loève expansion is well known; see [2–4]. A bivariate Brownian bridge is defined by

$$\begin{aligned} B(t_1, t_2) &= W(t_1, t_2)|_{W(1,1)=0} \\ &= W(t_1, t_2) - t_1 t_2 W(1, 1), \end{aligned} \qquad (4)$$

whose covariance $(t_1 \wedge s_1)(t_2 \wedge s_2) - t_1 t_2 s_1 s_2$, $s_1, s_2, t_1, t_2 \in I = [0, 1]$, a nontensor (nonseparate) product, is mentioned in the literature, such as [3, 5–10] and references therein. In this paper, we provide the Karhunen-Loève expansion for the bivariate Brownian bridge, which is our main goal and the highlight of our work.

In particular, Deheuvels et al. in [6] give the explicit Laplace transform of the bivariate Brownian bridge; however, there are few references for the explicit eigenvalues and the associated simple eigenfunctions expression of the nontensor Gaussian process, which is one of our motivations. Somayasa in [9] studied the approximation methods for computing the quantiles about the bivariate Brownian bridge in our paper, which is considered as the residual partial sums limit process associated with a constant model; if we know the eigenvalues of the process, then we can compute the explicit quantiles analytically. Meanwhile, the explicit and simple eigenvalues can be used in other aspects of statistics, which is our another motivation.

The rest of the paper is organized as follows. In Section 2, we give the first part eigenvalues in (23) of the bivariate Brownian motion through solving the differential equation converted from the associated Fredholm integral equation of the covariance function of the process. The other part eigenvalues in (23) can be found via Laplace transform.

2. Karhunen-Loève Expansion

There are some lemmas needed to prove Theorem 4. Before we prove the lemmas, we introduce some notations.

Set

$$I_m = \{(k_{m,l}, j_{m,l}) : (2k_{m,l}+1)(2j_{m,l}+1) = 2m + 1, k_{m,1} < \cdots < k_{m,l} < \cdots < k_{m,|I_m|}, 1 \le l \le |I_m|, m, l \in \mathbf{N}\}, \quad (5)$$

and let I be the disjoint union of I_m; that is,

$$I = \{(k, j) : k \ge 0, j \ge 0\} = \bigcup_{m=1}^{\infty} I_m, \quad (6)$$

and then set

$$I_m \times I_m = \bigcup_{m=1}^{\infty} \{(k_{m,l}, j_{m,l}, k_{m,l'}, j_{m,l'}) : (2k_{m,l}+1) \cdot (2j_{m,l}+1) = 2m+1, (2k_{m,l'}+1)(2j_{m,l'}+1) = 2m + 1, k_{m,1} < \cdots < k_{m,l} < k_{m,l+1} < \cdots < k_{m,l'-1} < k_{m,l'} < \cdots < k_{m,|I_m|}, 1 \le l < l' \le |I_m|, m, l, l' \in \mathbf{N}\}. \quad (7)$$

It is proved in Lemma 1 that $|I_m| - 1$ is the number of pairs of (k, j), which is also the dimension number of the space spanned by the eigenfunctions associated with eigenvalues (41).

We use (k, j) to denote $(k_{m,l}, j_{m,l})$ and utilize (p, q) to represent $(k_{m,l'}, j_{m,l'})$ for convenience. Then we can simplify the eigenfunctions to the following:

$$f_{m,k_{m,l},k_{m,l'}}(t_1, t_2) = \sqrt{2}\left(\sin\frac{(2k_{m,l}+1)}{2}\pi t_1 \sin\frac{(2j_{m,l}+1)}{2}\pi t_2 - \sin\frac{(2k_{m,l'}+1)}{2}\pi t_1 \sin\frac{(2j_{m,l'}+1)}{2}\pi t_2\right). \quad (8)$$

We note that if $l = l'$, then $f(t_1, t_2) = 0$, and thus $l \ne l'$. It is easy to check that the eigenfunctions are orthogonal between different m, so we only need to find the multiplicity of the eigenfunctions for the same m. In Lemma 1, we use the formula

$$\dim\{\operatorname{span}\{f_{m,2}, \ldots, f_{m,3}, \ldots, f_{m,|I_m|}\}\} = |I_m| - 1 \quad (9)$$

instead of

$$\dim\{\operatorname{span}\{f_{m,k_{m,1},k_{m,2}}, \ldots, f_{m,k_{m,1},k_{m,l}}, \ldots, f_{m,k_{m,1},k_{m,|I_m|}}\}\} = |I_m| - 1 \quad (10)$$

for convenience.

Now we are in the position to provide the useful lemmas.

Lemma 1. *Let*

$$(2k_{m,i}+1)(2j_{m,i}+1) = (2p+1)(2q+1) = 2m+1 \quad (11)$$

and the eigenfunctions be defined by

$$f_{m,i} = \sqrt{2}\left(\sin\frac{2k_{m,1}+1}{2}\pi t_1 \sin\frac{2j_{m,1}+1}{2}\pi t_2 - \sin\frac{2k_{m,i}+1}{2}\pi t_1 \sin\frac{2j_{m,i}+1}{2}\pi t_2\right) \quad (12)$$

with the associated eigenvalues $16/(2m+1)^2\pi^4$*; then, for* $i = 1, 2, \ldots, |I_m|$, $m = 1, 2, \ldots$,

$$\dim\{\operatorname{span}\{f_{m,1}, \ldots, f_{m,2}, \ldots, f_{m,|I_m|-1}\}\} = |I_m| - 1. \quad (13)$$

Proof. Assume $\sum_{i=1}^{|I_m|-1} c_i f_{m,i} = 0$; then we only need to check $c_i = 0$, $i = 0, 1, \ldots, |I_m| - 1$.

In fact

$$c_1 f_{m,1} + c_2 f_{m,2} + \cdots + c_{|I_m|-1} f_{m,|I_m|-1} = 0 \quad (14)$$

implies

$$\sum_{i=1}^{|I_m|-1} c_i \sin\frac{2k_{m,1}+1}{2}\pi t_1 \sin\frac{2j_{m,1}+1}{2}\pi t_2 - \sum_{i=2}^{|I_m|} c_{i-1} \sin\frac{2k_{m,i}+1}{2}\pi t_1 \sin\frac{2j_{m,i}+1}{2}\pi t_2 = 0. \quad (15)$$

By multiplying $\sin((2k_{m,l}+1)/2)\pi t_1$, $2 \le l \le |I_m|$ on both sides of (15) and integrate from 0 to 1 with respect to t_1, we obtain

$$\sum_{i=1}^{|I_m|-1} c_i \sin\frac{2j_{m,1}+1}{2}\pi t_2 \int_0^1 \sin\frac{2k_{m,l}+1}{2}\pi t_1 \cdot \sin\frac{2k_{m,1}+1}{2}\pi t_1 dt_1 - \sum_{i=2}^{|I_m|} c_{i-1} \sin\frac{2j_{m,i}+1}{2} \cdot \pi t_2 \int_0^1 \sin\frac{2k_{m,l}+1}{2}\pi t_1 \sin\frac{2k_{m,i}+1}{2}\pi t_1 dt_1 = 0, \quad (16)$$

which gives $c_1 = c_2 = \cdots = c_{|I_m|-1} = 0$. Thus we claim the conclusion. □

The following lemma is from [11].

Lemma 2. *For* $s > 1$,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \quad (17)$$

where the above product takes over all prime numbers p *and* $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

Lemma 3. *Let $d(n)$ be the number of divisors of positive integer n; then*

$$\sum_{m=0}^{\infty} \frac{d(2m+1)}{(2m+1)^2} = \frac{\pi^4}{2^6}. \tag{18}$$

Proof. By the representation of Dirichlet series $\sum_{n=1}^{\infty}(d(n)/n^2)$ and prime number decomposition, we obtain

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^2} = \prod_{p} \left(\sum_{k=0}^{\infty} \frac{d(p^k)}{p^{2k}} \right), \tag{19}$$

and the product of (19) takes over all primes

$$\begin{aligned} \prod_{p} \sum_{k=0}^{\infty} \frac{d(p^k)}{p^{2k}} &= \prod_{p} \sum_{k=0}^{\infty} (k+1) p^{-2k} = \prod_{p} \left(1 - p^{-2}\right)^{-2} \\ &= \zeta(2)^2, \end{aligned} \tag{20}$$

and the last equal sign is given by Lemma 2.

On the other hand, the integer n in the left hand side of (19) can be expressed by the product form of 2 and all the other odds; thus we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{d(n)}{n^2} &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(k+1)\, d(2m+1)}{2^{2k}(2m+1)^2} \\ &= \sum_{k=0}^{\infty} \frac{(k+1)}{2^{2k}} \sum_{m=0}^{\infty} \frac{d(2m+1)}{(2m+1)^2} \\ &= \frac{16}{9} \sum_{m=0}^{\infty} \frac{d(2m+1)}{(2m+1)^2}. \end{aligned} \tag{21}$$

Notice that $\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$; by (20) and (21), the lemma is proved. □

Some important eigenvalues can be provided by the following analysis from Fredholm integral equation, but the entire eigenvalues can be found by the Laplace transform in Theorem 4.

We use Mercer's theorem to compute the following integral equation of the covariance

$$\begin{aligned} K_B(t_1, t_2; s_1, s_2) &= (t_1 \wedge s_1)(t_2 \wedge s_2) - t_1 t_2 s_1 s_2, \\ &\qquad s_1, s_2, t_1, t_2 \in I = [0,1]^2, \end{aligned} \tag{22}$$

of the bivariate Brownian bridge; that is,

$$\begin{aligned} T_B f(t_1, t_2) &= \int_{I^2} K_B(t_1, t_2; s_1, s_2) f(s_1, s_2)\, ds_1 ds_2 \\ &= \lambda f(t_1, t_2) \end{aligned} \tag{23}$$

which is simplified to be

$$\begin{aligned} \lambda f(t_1, t_2) &= (1 - t_1 t_2) \int_0^{t_2} \int_0^{t_1} s_1 s_2 f(s_1, s_2)\, ds_1 ds_2 \\ &\quad + t_1 \int_0^{t_2} \int_{t_1}^{1} s_2 (1 - t_2 s_1) f(s_1, s_2)\, ds_1 ds_2 \\ &\quad + t_2 \int_{t_2}^{1} \int_0^{t_1} s_1 (1 - t_1 s_2) f(s_1, s_2)\, ds_1 ds_2 \\ &\quad + t_1 t_2 \int_{t_2}^{1} \int_{t_1}^{1} (1 - s_1 s_2) f(s_1, s_2)\, ds_1 ds_2. \end{aligned} \tag{24}$$

Taking derivative of (24) with respect to t_1, we obtain

$$\begin{aligned} &\lambda \frac{\partial}{\partial t_1} f(t_1, t_2) \\ &\quad = -t_2 \int_0^{t_2} \int_0^{t_1} s_1 s_2 f(s_1, s_2)\, ds_1 ds_2 \\ &\qquad + \int_0^{t_2} \int_{t_1}^{1} s_2 (1 - t_2 s_1) f(s_1, s_2)\, ds_1 ds_2 \\ &\qquad + t_2 \int_{t_2}^{1} \int_0^{t_1} s_1 (-s_2) f(s_1, s_2)\, ds_1 ds_2 \\ &\qquad + t_2 \int_{t_2}^{1} \int_{t_1}^{1} (1 - s_1 s_2) f(s_1, s_2)\, ds_1 ds_2. \end{aligned} \tag{25}$$

Taking derivative of (25) with respect to t_1 again, we obtain

$$\begin{aligned} &- \int_0^{t_2} s_2 f(t_1, s_2)\, ds_2 - t_2 \int_{t_2}^{1} f(t_1, s_2)\, ds_2 \\ &\quad = \lambda \frac{\partial^2}{\partial t_1^2} f(t_1, t_2). \end{aligned} \tag{26}$$

By differentiating both sides of (26) with respect to t_2, we have

$$- \int_{t_2}^{1} f(t_1, s_2)\, ds_2 = \lambda \frac{\partial^3}{\partial t_1^2 t_2} f(t_1, t_2). \tag{27}$$

By differentiating both sides of (27) with respect to t_2 again, we get

$$f(t_1, t_2) = \lambda \frac{\partial^4}{\partial t_1^2 t_2^2} f(t_1, t_2). \tag{28}$$

From (24) to (27), we know the following facts:

$$\begin{aligned} f(t_1, 0) &= 0, \\ f(0, t_2) &= 0, \end{aligned} \tag{29}$$

$$f(1, 1) = 0, \tag{30}$$

$$\begin{aligned} \frac{\partial^3}{\partial t_2^2 \partial t_1} f(1, t_2) &= 0, \\ \frac{\partial^3}{\partial t_1^2 \partial t_2} f(t_1, 1) &= 0. \end{aligned} \tag{31}$$

Define the following functions:

$$F_n(t_1) = c_{1n} \sin \sqrt{a_n} t_1 + c'_{1n} \cos \sqrt{a_n} t_1,$$
$$G_n(t_2) = c_{2n} \sin \frac{t_2}{\sqrt{\lambda a_n}} + c'_{2n} \cos \frac{t_2}{\sqrt{\lambda a_n}}, \tag{32}$$

where c_{in}, c'_{in}, $i = 1, 2$ and, for $n \in \mathbf{N}$, $a_n > 0$ are constants (depending on λ) to be determined by (29) to (31). Then it is easy to see that $F_n(t_1)G_n(t_2)$ are solutions of (28) and thus

$$f(t_1, t_2) = \sum_{n\geq 1} F_n(t_1) G_n(t_2) \tag{33}$$

are a set of solutions of (28). It is easy to check that when there is only one item in the series in (33) there is the trivial solution of (28), and when the items are more than two, the set of eigenfunctions is the same as the case of the two items. So the eigenfunction question is reduced to consider the sum of two items $F_1(t_1)G_1(t_2)$ and $F_2(t_1)G_2(t_2)$ in (33) ($F_n(t_1) = 0$ for $n \geq 3$), where the four functions $F_1(t_1)$, $F_2(t_1)$, $G_1(t_2)$, and $G_2(t_2)$ satisfy

$$\frac{d^2 F_1(t_1)}{dt_1^2} = -a_1 F_1(t_1),$$
$$\frac{d^2 G_1(t_2)}{dt_2^2} = \frac{F_2(t_2)}{-\lambda a_1},$$
$$\frac{d^2 F_2(t_1)}{dt_1^2} = -a_2 G_1(t_1), \tag{34}$$
$$\frac{d^2 G_2(t_2)}{dt_2^2} = \frac{G_2(t_2)}{-\lambda a_2},$$

for $a_1 \neq a_2$, $a_1 > 0$, $a_2 > 0$. Actually, if $a_1 = a_2$, then $F_1(t_1) = F_2(t_1)$, $G_1(t_2) = G_2(t_2)$, and thus $f(t_1, t_2) = 2F_1(t_1)G_1(t_2)$, which is not the solution of the original integral equation (23). Hence $a_1 \neq a_2$.

Then the eigenfunction can be expressed by

$$\begin{aligned} f(t_1, t_2) &= F_1(t_1) G_1(t_2) + F_2(t_1) G_2(t_2) \\ &= (c_{1n} \sin \sqrt{a_1} t_1 + c_2 \cos \sqrt{a_1} t_1) \\ &\quad \cdot \left(c_{12} \sin \frac{1}{\sqrt{\lambda a_1}} t_2 + c'_{12} \cos \frac{1}{\sqrt{\lambda a_1}} t_2 \right) \\ &\quad + (c_{21} \sin \sqrt{a_2} t_1 + c'_{21} \cos \sqrt{a_2} t_1) \\ &\quad \cdot \left(c_{22} \sin \frac{1}{\sqrt{\lambda a_2}} t_2 + c'_{22} \cos \frac{1}{\sqrt{\lambda a_2}} t_2 \right). \end{aligned} \tag{35}$$

Firstly, we deal with (35). Using (29), for any $t_1 \in [0, 1]$ and $t_2 \in [0, 1]$, we have

$$\begin{aligned} &c'_{12} (c_{1n} \sin \sqrt{a_1} t_1 + c'_{11} \cos \sqrt{a_1} t_1) \\ &\quad + c'_{22} (c_{21} \sin \sqrt{a_2} t_1 + c'_{21} \cos \sqrt{a_2} t_1) = 0, \\ &c'_{11} \left(c_{12} \sin \frac{1}{\sqrt{\lambda a_1}} t_2 + c'_{12} \cos \frac{1}{\sqrt{\lambda a_1}} t_2 \right) \\ &\quad + c'_{21} \left(c_{22} \sin \frac{1}{\sqrt{\lambda a_2}} t_2 + c'_{22} \cos \frac{1}{\sqrt{\lambda a_2}} t_2 \right) = 0, \end{aligned} \tag{36}$$

which give

$$c'_{12} = 0,$$
$$c'_{22} = 0,$$
$$c'_{11} = 0, \tag{37}$$
$$c'_{21} = 0.$$

Then we can rewrite (35) as

$$\begin{aligned} f(t_1, t_2) &= F_1(t_1) G_1(t_2) + F_2(t_1) G_2(t_2) \\ &= c_{11} c_{12} \sin \sqrt{a_1} t_1 \sin \frac{1}{\sqrt{\lambda a_1}} t_2 \\ &\quad + c_{21} c_{22} \sin \sqrt{a_2} t_1 \sin \frac{1}{\sqrt{\lambda a_2}} t_2. \end{aligned} \tag{38}$$

Making use of (31), for any $t_1 \in [0, 1]$ and $t_2 \in [0, 1]$, we obtain

$$\begin{aligned} &\sqrt{\frac{a_1}{\lambda}} c_{11} c_{12} \sin \sqrt{a_1} t_1 \cos \frac{1}{\sqrt{\lambda a_1}} \\ &\quad + c_{21} c_{22} \sqrt{\frac{a_2}{\lambda}} \sin \sqrt{a_2} t_1 \cos \frac{1}{\sqrt{\lambda a_2}} = 0 \\ &c_{11} c_{12} \left(\frac{1}{\sqrt{\lambda a_1}} \right)^2 \sqrt{a_1} \cos \sqrt{a_1} \sin \frac{1}{\sqrt{\lambda a_1}} t_2 \\ &\quad + c_{21} c_{22} \left(\frac{1}{\sqrt{\lambda a_2}} \right)^2 \sqrt{a_2} \cos \sqrt{a_2} \sin \frac{1}{\sqrt{\lambda a_2}} t_2 = 0, \end{aligned} \tag{39}$$

which give

$$\sqrt{a_1} = \left(k + \frac{1}{2}\right)\pi,$$
$$\sqrt{a_2} = \left(p + \frac{1}{2}\right)\pi,$$
$$\frac{1}{\sqrt{\lambda a_1}} = \left(j + \frac{1}{2}\right)\pi, \tag{40}$$
$$\frac{1}{\sqrt{\lambda a_2}} = \left(q + \frac{1}{2}\right)\pi.$$

Thus the eigenvalues are

$$\lambda_{k,j} = \frac{16}{(2k+1)^2 (2j+1)^2 \pi^4}, \quad k, j \in \mathbb{N} \times \mathbb{N} \setminus \{0, 0\}. \tag{41}$$

Based on fact (30), the condition $(2k+1)(2j+1) = (2p+1)(2q+1)$, and the normalization condition, we can obtain $c_{11}c_{12} = -c_{21}c_{22} = \sqrt{2}$ and then the simple eigenfunctions are, for $k, j, p, q \in \mathbb{N} \times \mathbb{N} \setminus \{0,0\}$,

$$f(t_1,t_2) = \sqrt{2}\left(\sin\left(k+\frac{1}{2}\right)\pi t_1 \sin\left(j+\frac{1}{2}\right)\pi t_2 - \sin\left(p+\frac{1}{2}\right)\pi t_1 \sin\left(q+\frac{1}{2}\right)\pi t_2\right) \tag{42}$$

with associated eigenvalues (41). The multiplicity of the eigenfunctions (42) associated with (41) is $|I_m| - 1$, which has been shown by Lemma 1.

Although $\sin(1/2\pi t_1)\sin(1/2\pi t_2)$ is the solution of (28), it does not satisfy (23), so the index becomes $k, j \in \mathbb{N}\times\mathbb{N}\setminus\{0,0\}$ (or only $p, q \in \mathbb{N}\times\mathbb{N}\setminus\{0,0\}$). Hence we have to search the solution form of (23). We find the interesting fact that $\sqrt{2}\sin(k+1/2)\pi t_1 \sin(j+1/2)\pi t_2$ and $\sin(p+1/2)\pi t_1 \sin(q+1/2)\pi t_2$ satisfy differential equation (28), but they are not the solutions of original integral equation (23). A natural idea is to combine them to relation (42), and then we should check whether (42) are the solutions of both (28) and (23) or not. We realize that (42) satisfy (28), while we do not make sure it is also doable for (23). By routine calculation, if (42) satisfy original integral equation (23), we must make $(2k+1)(2j+1) = (2p+1)(2q+1)$. Therefore the index for the eigenfunction (42) should also be $k, j \in \mathbb{N}\times\mathbb{N}\setminus\{0,0\}$ (or only $p, q \in \mathbb{N}\times\mathbb{N}\setminus\{0,0\}$).

The well-known trace-variance formula gives

$$\int_0^1\int_0^1 K_B(t_1,t_2;t_1,t_2)\,dt_1dt_2 = \int_0^1\int_0^1 \left(t_1t_2 - (t_1t_2)^2\right)dt_1dt_2 = \frac{5}{36}. \tag{43}$$

However, on the other hand, with the help of Lemmas 1 and 3 and the fact that $\sum_{n=1}^{\infty}(1/n^2) = \pi^2/6$, we have the following comparison relation:

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{16}{(2m+1)^2\pi^4}\left(|I_m|-1\right) &= \frac{16}{\pi^4}\left(\sum_{m=1}^{\infty}\frac{|I_m|}{(2m+1)^2} - \sum_{m=1}^{\infty}\frac{1}{(2m+1)^2}\right)\\ &= \frac{16}{\pi^4}\left(\frac{\pi^4}{64} - 1 - \left(\frac{\pi^2}{6} - \frac{\pi^2}{24} - 1\right)\right) = \left(\frac{1}{4} - \frac{2}{\pi^2}\right)\\ &< \frac{5}{36}.\end{aligned} \tag{44}$$

Therefore, there are still other eigenvalues lost so far. Fortunately, Laplace transform of the bivariate Brownian bridge provided them.

Based on the analysis of the above, now we give the main result, namely, the following Theorem 4 in this paper.

Theorem 4. *The spectrum of the KL expansion for the bivariate Brownian bridge $\{B(t_1,t_2), t_1, t_2 \in [0,1]^2\}$ is given by (41) and (52). In particular,*

$$\begin{aligned}\int_0^1\int_0^1 B(t_1,t_2)^2\,dt_1dt_2 &\overset{\text{law}}{=} \int_0^1\int_0^1\left(W(t_1,t_2) - \int_0^1\int_0^1 W(s_1,s_2)\,ds_1ds_2\right)^2 dt_1dt_2\\ &\overset{\text{law}}{=} \sum_{k,j\ge1}\frac{16}{(2k+1)^2(2j+1)^2\pi^4}\eta_{k,j}^2 + \sum_{k,j\ge1}\lambda_{k,j}^*{\eta^*}_{k,j}^2,\end{aligned} \tag{45}$$

where $\{\lambda_{k,j}^*, k\ge1, j\ge1\}$ are the reciprocal of the roots of (52) and ${\lambda_{k,j}^*}^{-1} \in [u_{k,j}^-, u_{k,j}^+]$, $u_{k,j}^{\pm} = (1/16)[(2j+1)(2k\pm1)\pi^2]^2$, and $\{\eta_{k,j}, k\ge1, j\ge1\}$ and $\{{\eta^*}_{k,j}, k\ge1, j\ge1\}$ are two independent sequences of 2-index i.i.d. $N(0,1)$ random variables. Furthermore, for $k\ge k_0$, $j\ge j_0$, k_0 and j_0 are large enough, and we have

$${\lambda_{k,j}^*}^{-1} = \left[\frac{(2j+1)k\pi^2}{2}\right]^2 + O\left(k^{-2}j^{-2}\right), \tag{46}$$

$$\sum_{k\ge k_0, j\ge j_0}\left|\frac{\left[(2j+1)k\pi^2/2\right]^2}{{\lambda_{k,j}^*}^{-1}} - 1\right| < \infty. \tag{47}$$

Here the first identity in Theorem 4 can be found in Corollary 3.1 in [6].

Proof. Based on the Laplace transform of the bivariate Brownian bridge $B(s,t)$ (see Proposition 4.1.(i) of page 521 in [6])

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}B(s,t)^2\,ds\,dt\right)\right] &= \left(D\left(-a^2\right)\right)^{-1/2} = \left\{\left[\prod_{j=0}^{\infty}\cosh\left(\frac{2a}{(2j+1)\pi}\right)\right]\frac{4}{u}\right.\\ &\left.\cdot\sum_{j=0}^{\infty}\left\{\tanh\left(\frac{2a}{(2j+1)\pi}\right)\left[(2j+1)\pi\right]^{-1}\right\}\right\}^{-1/2}\end{aligned} \tag{48}$$

and setting $-a^2 = u$ in (48), we get the Fredholm determinant of the covariance function of the process

$$\begin{aligned}D(u) &= \prod_{j=0}^{\infty}\cos\left[\frac{2u^{1/2}}{(2j+1)\pi}\right]\\ &\times 4u^{-1/2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)\pi}\left\{\tan\left(\frac{2u^{1/2}}{(2j+1)\pi}\right)\right\},\end{aligned} \tag{49}$$

And according to (23) and formula (5.10) of page 146 and Theorems 5.1, 5.2, and 5.3 from page 148 to 150 in [12], the entire eigenvalues $\lambda_{k,j}$ of the integral operator

$$\int_{[0,1]^2}\left[(s_1\wedge t_1)(s_2\wedge t_2) - s_1s_2t_1t_2\right]f(s_1,s_2)\,ds_1ds_2 \tag{50}$$

are the reciprocal of the roots of $D(u)$ in (49), which satisfy

$$\cos\left[\frac{2u^{1/2}}{(2j+1)\pi}\right]=0 \tag{51}$$

or

$$\sum_{j=0}^{\infty}\frac{1}{(2j+1)\pi}\left\{\tan\left(\frac{2u^{1/2}}{(2j+1)\pi}\right)\right\}=0. \tag{52}$$

Equation (51) also gives the eigenvalues of the form

$$\lambda_{k,j}=\frac{16}{(2k+1)^2(2j+1)^2\pi^4},\quad k,j\in\mathbb{N}\times\mathbb{N}\setminus\{0,0\}, \tag{53}$$

which are obtained in (41).

Let $F(u)$ stand for (52); then by routine calculation, we know that $F(u)$ is an increasing function on the interval $[u_{k,j}^{-},u_{k,j}^{+}]$, $u_{k,j}^{\pm}=(1/16)[(2j+1)(2k\pm1)\pi^2]^2$ and $F(u_{k,j}^{\pm})=\pm\infty$ by the following fact:

$$\begin{aligned}0&<F'(u)\\&=\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2\pi^2}\frac{1}{\sqrt{u}\cos^2(2u^{1/2}/(2j+1)\pi)}<\infty.\end{aligned} \tag{54}$$

Therefore, there is only one positive zero in $[u_{k,j}^{-},u_{k,j}^{+}]$, $u_{k,j}^{\pm}=(1/16)[(2j+1)(2k\pm1)\pi^2]^2$, which is denoted by $\xi_{k,j}$. Let $\xi_{k,j}={\lambda_{k,j}^{*}}^{-1}$ denote the solution of (52); then $\{\lambda_{k,j}^{*},k,j=1,2,\ldots\}$ are the other eigenvalues of the integral operator T_B.

For $k>k_0$, $j>j_0$, k_0 and j_0 are large enough. The inequality

$$\begin{aligned}\xi_{k,j}-\left[\frac{(2j+1)k\pi^2}{2}\right]^2&\le u_{k,j}^{+}-u_{k,j}^{-}\\&=2k\left[\frac{(2j+1)\pi^2}{2}\right]^2\end{aligned} \tag{55}$$

implies relation (46). It is also easy to check that (47) holds. □

Remark 5. Although the entire eigenvalues can not be solved by the Fredholm integral equation, it gives the important information, which is stated in Theorem 4.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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