Research Article

Power-Serieswise McCoy Modules

Xinquan Zhang

School of Mathematics and Statistics, Hefei Normal University, Hefei 230601, China

Correspondence should be addressed to Xinquan Zhang; zhhfnu@126.com

Received 20 April 2018; Revised 15 July 2018; Accepted 27 August 2018; Published 9 October 2018

Academic Editor: Libor Pekař

Copyright © 2018 Xinquan Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Extending the notion of McCoy modules and power-serieswise Armendariz modules, we introduce the concept of power-serieswise McCoy modules. Various basic properties of these modules are discussed and many illustrative examples are provided. Moreover, power-serieswise McCoy modules are used to obtain results on some polynomial extensions and matrix extensions of a module.

1. Introduction

Throughout the paper, all rings considered are associative with identity and modules are unitary right modules. Let $\mathbb{N}$ be the set of natural integers. The symbol E_{ij} stands for the matrix unit. For a set X of a module M_R , $r_R(X)$ stands for the right annihilator of X in the ring R . We write $R[x]$, $R[[x]]$, $R[x^{-1}; x]$, and $R[[x^{-1}; x]]$ for the polynomial ring, the power-serieswise ring, the Laurent polynomial ring, and Laurent power-serieswise polynomial ring over R , respectively. For a module M_R , we consider

$$\begin{aligned} M[x] &= \left\{ \sum_{i=0}^s m_i x^i \mid s \geq 0, m_i \in M \right\}, \\ M[[x]] &= \left\{ \sum_{i=0}^{\infty} m_i x^i \mid m_i \in M \right\}, \\ M[x^{-1}; x] &= \left\{ \sum_{i=-s}^t m_i x^i \mid s \geq 0, t \geq 0, m_i \in M \right\}, \\ M[[x^{-1}; x]] &= \left\{ \sum_{i=-s}^{\infty} m_i x^i \mid s \geq 0, m_i \in M \right\}. \end{aligned} \quad (1)$$

Each of these is an abelian group under an obvious addition operation. Moreover, $M[[x]]$ becomes an $R[[x]]$ module under the following product operation: for $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$, $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x]]$,

$$m(x)f(x) = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} m_i a_j \right) x^k. \quad (2)$$

Similarly, $M[x]$, $M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ are $R[x]$ -module, $R[x, x^{-1}]$ -module, and $R[[x, x^{-1}]]$ -module, respectively.

Chhawchharia, Rege [1], and Nielsen [2] independently called a ring R *right McCoy* (respectively, *left McCoy*) if whenever $f(x)g(x) = 0$ for $f(x), g(x) \in R[x] \setminus \{0\}$, there exists $r \in R \setminus \{0\}$ (respectively, $s \in R \setminus \{0\}$) such that $f(x)r = 0$ (respectively, $sg(x) = 0$). A ring is *McCoy* if it is both left and right McCoy. The term “McCoy ring” was coined because McCoy [3] had shown that every commutative ring satisfies the above-mentioned condition. Armendariz rings (i.e., if $f(x) = \sum_{i=0}^s a_i x^i$ and $g(x) = \sum_{j=0}^t b_j x^j$ in $R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i and j) are well known example of McCoy ring. For more details of rings with McCoy conditions, one may refer to [4–6], etc. The McCoy properties of rings were extended to power-serieswise rings in [7]. A ring R is called a *right power-serieswise McCoy ring* if whenever $f(x), g(x) \in R[[x]] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, then there exists a nonzero element $r \in R$ such that $f(x)r = 0$. A left power-serieswise McCoy can be defined similarly. A left and right power-serieswise McCoy is called *power-serieswise McCoy*. Recall that a ring R is *reduced* [8] if it has no nonzero nilpotent elements. In [7], it was claimed that all reduced rings were power-serieswise McCoy and so are McCoy.

Due to [9], a module M_R is *Armendariz* if $n_i b_j = 0$ for all i, j whenever $n(x) = \sum_{i=0}^s n_i x^i \in M[x]$ and $g(x) =$

$\sum_{j=0}^t b_j x^j \in R[x]$ satisfy $n(x)g(x) = 0$. As both generalization of McCoy rings and Armendariz modules, McCoy modules were introduced in [10]. A module M_R is called *McCoy* if $m(x)g(x) = 0$ where $m(x) = \sum_{i=0}^s m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^t b_j x^j \in R[x] \setminus \{0\}$, then there exists $r \in R \setminus \{0\}$ such that $m(x)r = 0$. In [11], Lee and Zhou extended the concept of Armendariz modules to power-serieswise modules and introduced the concept of power-serieswise Armendariz modules. A module M_R is said to be *power-serieswise Armendariz* if $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$ with $m(x)g(x) = 0$, then $m_i b_j = 0$ for all i and j .

Motivated by the above, we introduce the notion of power-serieswise McCoy modules as a straightforward extensions of McCoy modules and power-serieswise McCoy rings. We show that the class of power-serieswise Armendariz modules are properly contained in the class of power-serieswise McCoy modules. Basic properties of this class of modules are investigated, and examples are provided. In addition, we study the relationship between a module and its polynomial module (respectively, matrix module) by means of power-serieswise McCoy properties.

2. Power-Serieswise McCoy Modules

Definition 1. Let M be a module over a ring R and $M[[x]]$ the corresponding polynomial module over $R[[x]]$. M_R is called *power-serieswise McCoy* if whenever $m(x)g(x) = 0$ where $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]] \setminus \{0\}$, there exists a nonzero element $r \in R$ such that $m(x)r = 0$.

It is clear that power-serieswise McCoy modules are McCoy and R_R is power-serieswise McCoy iff R is a right power-serieswise McCoy ring. Every submodule of a power-serieswise McCoy module is power-serieswise McCoy. In particular, if I is a right ideal of a power-serieswise McCoy ring R , then I_R is a power-serieswise McCoy module.

The following result is motivated by [10, Proposition 2.3].

Proposition 2. Let I be any nonzero ideal of a ring R , then $(R/I)_R$ is a power-serieswise McCoy module.

Proof. Give any $\overline{f(x)} \in (R/I)[[x]]$. Take a nonzero element $r \in I(\subseteq R)$. Since I is an ideal of R , $f(x)r \in I[[x]]$. That is, $\overline{f(x)}r = 0$. $\square$

The following example reveals that McCoy module need not be power-serieswise McCoy.

Example 3 (see [7, Example 2.6]). Let F be a field, $A = F\langle a, b, c \rangle$ the free algebra of polynomials in noncommuting indeterminates a, b, c over F , and I the ideal of $F+A$ generated by cc, ac, crc with $r \in A$. Let $R = (F + A)/I$. Identify f with $f + I$ for simplicity. Consider the equality

$$(a - abx)(c + bcx + b^2cx^2 + \cdots + b^ncx^n + \cdots) = ac \quad (3)$$

= 0

with $a - abx, c + bcx + b^2cx^2 + \cdots + b^ncx^n + \cdots \in R[[x]] \setminus \{0\}$. For any nonzero $r \in R$, if $(a - abx)r = 0$, then $ar = 0$ and $abr = 0$. The first equality implies $r \in Fc$, but $abc \neq 0$, a contradiction. Thus, R_R is not power-serieswise McCoy. However, the ring R is McCoy since it is Armendariz by [12, Example 2.1].

Theorem 4. Power-serieswise Armendariz modules are power-serieswise McCoy.

Proof. Let M_R be a power-serieswise Armendariz module and $m(x)g(x) = 0$, where $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$; there exists j_0 such that $b_{j_0} \in R \setminus \{0\}$. Hence, $m_i b_{j_0} = 0$ for each i which deduces that $m(x)b_{j_0} = 0$. Therefore, M_R is a power-serieswise McCoy module. $\square$

Let $e^2 = e \in R$. A module M_R is *p.p.-module* [13] if for any $m \in M$, $r_R(m) = eR$. A module M is called *abelian* [13] if for any $m \in M$ and any $a \in R$, any idempotent $e \in R$, $mae = mea$. In view of [13, Corollary 2.9] and Theorem 4, we have the following result.

Corollary 5. $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]] \setminus \{0\}$. Then $m_i b_j = 0$ for all i, j . Since $g(x) \neq 0$, if M_R is an abelian and p.p.-module, then M is power-serieswise McCoy.

The converse of Theorem 4 does not hold generally.

Example 6. Let S be a reduced ring and let

$$R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in S \right\}. \quad (4)$$

Then R_4 is not an Armendariz ring by [8, Example 3], so is not power-serieswise Armendariz. Since S is reduced, R_4 is a power-serieswise McCoy ring by [7, Corollary 2.9].

A module M_R is called *reduced* [11] if for any $m \in M$ and any $a \in R$, $ma = 0$ implies that $mR \cap Ma = 0$. All reduced modules are Armendariz and hence McCoy.

Remark 7. In [13], a ring R is called *strong regular* if for any $x \in R$, there exists $y \in R$ such that $x = x^2 y$. By [14, Theorem 2.16], all modules over strong regular rings are reduced, so is power-serieswise Armendariz by [11, Lemma 1.5], and hence power-serieswise McCoy.

A module M_R is *semicommutative* [9] if whenever $m \in M$ and $a \in R$ satisfy $ma = 0$, then $mra = 0$ for every $r \in R$. A module M_R is said to be *finitely cogenerated* [15] if for every set $\{A_i \mid i \in I\}$ of submodules A_i of M with $\bigcap_{i \in I} A_i = 0$, there is a finite subset $\{A_i \mid i \in I_0\}$ (i.e., $I_0 \subseteq I$ and I_0 is finite) with $\bigcap_{i \in I_0} A_i = 0$.

Cui and Chen [10] proved that a semicommutative module over a reduced ring is McCoy; we have the following result.

Proposition 8. Let R be a reduced ring and R_R a finitely cogenerated module. Then a semicommutative module M_R is power-serieswise McCoy.

Proof. Let $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]] \setminus \{0\}$ with $m(x)g(x) = 0$. Assume that $b_0 \neq 0$. (If not, set $g(x) = g'(x)x^k$ with a minimal k such that $b_k \neq 0$, and we have $m(x)g'(x) = 0$.) This implies the following system of equations:

$$m_0 b_0 = 0; \quad (0)$$

$$m_0 b_1 + m_1 b_0 = 0; \quad (1)$$

$$m_0 b_2 + m_1 b_1 + m_2 b_0 = 0; \quad (2)$$

$$\dots \quad \dots$$

$$m_0 b_k + m_1 b_{k-1} + \dots + m_k b_0 = 0; \quad (k)$$

$$\dots \quad \dots$$

Since M_R is semicommutative, we have $m_0 b_i b_0 = 0$ from Eq.(0). Multiplying (1) on the right by b_0 yields $m_1 b_0^2 = 0$. Similarly, multiplying (2) on the right by b_0^2 gives $m_2 b_0^3 = 0$ because $m_0 b_2 b_0^2 = m_1 b_1 b_0^2 = 0$. Continuing this procedure, one has $m_i b_0^{i+1} = 0$ for all i . Since R is reduced, $b_0 \neq 0$ implies $b_0^i \neq 0$ for each i . Thus $b_0^i R \neq 0$ and $b_0 R \supseteq b_0^2 R \supseteq \dots$ imply $\bigcap_{i \in \mathbb{N}} b_0^i R \neq 0$. Since R_R is a finitely cogenerated module, $\bigcap_{i \in I_0} b_0^i R \neq 0$ for some finite $I_0 \subseteq \mathbb{N}$. Take $r \in \bigcap_{i \in I_0} b_0^i R$. Then $m(x)r = 0$. Hence M_R is power-serieswise McCoy. $\square$

A module is *uniform* [15] if any two nonzero submodules have a nonzero intersection. We have a similar result as in [10].

Theorem 9. Let $\{M_i\}_{i \in \Lambda}$ be a family of McCoy R -modules for an index set Λ . Then

(1) if R_R is uniform, then a direct sum $M = \prod_{i \in \Lambda} M_i$ is power-serieswise McCoy;

(2) if Λ is an infinite set and R_R is uniform and finitely cogenerated, then a direct product $M = \prod_{i \in \Lambda} M_i$ is power-serieswise McCoy.

Proof. If $m(x)g(x) = 0$ with $m(x) = \sum_{k=0}^{\infty} (m_k)_{i \in \Lambda} x^k \in M[[x]]$ and $g(x) \in R[[x]] \setminus \{0\}$, let $m_i(x) = \sum_{k=0}^{\infty} m_{ik} x^k \in M_i[[x]]$. Since M_i is power-serieswise McCoy and $m_i(x)g(x) = 0$, there exists $r_i \in R \setminus \{0\}$ such that $m_i(x)r_i = 0$ for each $i \in \Lambda$.

We now prove (1). Note that the set $\Lambda_1 = \{i \in \Lambda \mid m_i(x) \neq 0\}$ is finite. Put $U = \bigcap_{i \in \Lambda_1} r_i R$. Since R_R is uniform, we have $U \neq 0$. Take any $r \in U \setminus \{0\}$. Then $m_i(x)r = 0$ for each i , whence $m(x)r = 0$. Thus, $M = \prod_{i \in \Lambda} M_i$ is power-serieswise McCoy.

To show (2), the assumption R_R is uniform implies that $\bigcap_{i \in \Lambda_0} r_i R \neq 0$ for any finite subset $\Lambda_0 \subseteq \Lambda$. As R_R is finite cogenerated, it follows that $\bigcap_{i \in \Lambda} r_i R \neq 0$. Similar to the proof of (1), we are done. $\square$

Let M_R be a module. Put

$$T(M) = \{x \in M \mid r_R(x) \neq 0\}. \quad (5)$$

If R is a commutative domain, then $T(M)$ is a submodule of M called the *torsion submodule* of M (see [15]).

Let M_i be a module over a ring R_i where $i \in \Lambda$. Set $m(x) = \sum_{p=0}^{\infty} (m_{ip})_{i \in \Lambda} x^p \in (\prod_{i \in \Lambda} M_i)[[x]]$, $g(x) = \sum_{q=0}^{\infty} ((b_{iq})_{i \in \Lambda}) x^q \in (\prod_{i \in \Lambda} R_i)[[x]]$. It is clear that $(\prod_{i \in \Lambda} M_i)[[x]] \rightarrow \prod_{i \in \Lambda} (M_i[[x]])$ is an abelian group isomorphism via the usual map. So $m(x) \in (\prod_{i \in \Lambda} M_i)[[x]]$ can be viewed as $(\sum_{p=0}^{\infty} m_{ip} x^p)_{i \in \Lambda} \in \prod_{i \in \Lambda} (M_i[[x]])$. Similarly, $(\prod_{i \in \Lambda} R_i)[[x]] \cong \prod_{i \in \Lambda} (R_i[[x]])$. Then $g(x) \in (\prod_{i \in \Lambda} R_i)[[x]]$ can be viewed as $(\sum_{q=0}^{\infty} b_{iq} x^q)_{i \in \Lambda} \in \prod_{i \in \Lambda} (R_i[[x]])$. So we can define

$$m(x)g(x) = \left(\sum_{p=0}^{\infty} m_{ip} x^p \right)_{i \in \Lambda} \left(\sum_{q=0}^{\infty} b_{iq} x^q \right)_{i \in \Lambda}. \quad (6)$$

Theorem 10. Let M_i be a module over a ring R_i where $i \in \Lambda$. If $M_i (i \in \Lambda)$ is a power-serieswise McCoy R_i -module, then $\prod_{i \in \Lambda} M_i$ is a power-serieswise McCoy $\prod_{i \in \Lambda} R_i$ -module. The converse holds if $T(M_i) \neq M_i$ for all $i \in \Lambda$.

Proof. Suppose that M_i is a power-serieswise McCoy R_i -module for any $i \in \Lambda$. Let $m(x)g(x) = 0$ where $m(x) = \sum_{p=0}^{\infty} (m_{ip})_{i \in \Lambda} x^p \in (\prod_{i \in \Lambda} M_i)[[x]]$ and $g(x) = \sum_{q=0}^{\infty} (b_{iq})_{i \in \Lambda} x^q \in (\prod_{i \in \Lambda} R_i)[[x]] \setminus \{0\}$. Set $m_i(x) = \sum_{p=0}^{\infty} m_{ip} x^p$, $g_i(x) = \sum_{q=0}^{\infty} b_{iq} x^q$. Then $m_i(x) \in M_i[[x]]$, $g_i(x) \in R_i[[x]]$. By the discussion above, we have

$$\begin{aligned} 0 &= m(x)g(x) = (m_i(x))_{i \in \Lambda} (g_i(x))_{i \in \Lambda} \\ &= (m_i(x)g_i(x))_{i \in \Lambda}. \end{aligned} \quad (7)$$

So $m_i(x)g_i(x) = 0$ for all $i \in \Lambda$. Since $g(x) \neq 0$, there exists $k \in \Lambda$ such that $g_k(x) \neq 0$. The assumption that M_k is a power-serieswise McCoy R_k -module implies that there exists $r_k \in R_k \setminus \{0\}$ such that $m_k(x)r_k = 0$. Take

$$r = (r_j)_{j \in \Lambda} = \begin{cases} r_k, & j = k; \\ 0, & \text{elsewise.} \end{cases} \quad (8)$$

Then $r \in \prod_{i \in \Lambda} R_i \setminus \{0\}$ and $m(x)r = (m_i(x)r_i)_{i \in \Lambda} = 0$.

Conversely, fix any $k \in \Lambda$. Let $m_k(x) = \sum_{p=0}^{\infty} m_{kp} x^p \in M_k[[x]]$, $g_k(x) = \sum_{q=0}^{\infty} b_{kq} x^q \in R_k[[x]] \setminus \{0\}$ with $m_k(x)g_k(x) = 0$ where $m_{kp} \in M_k$, $b_{kq} \in R_k$. Write

$$\begin{aligned} m(x) &= (m_s(x))_{s \in \Lambda} \\ &= \begin{cases} m_k(x), & s = k; \\ m_s, & s \neq k \text{ and } m_s \in M_s \setminus T(M_s), \end{cases} \end{aligned} \quad (9)$$

and

$$g(x) = (g_t(x))_{t \in \Lambda} = \begin{cases} g_k(x), & t = k; \\ 0, & t \neq k. \end{cases} \quad (10)$$

Then

$$m(x)g(x) = (m_s(x)g_s(x))_{s \in \Lambda} = 0. \quad (11)$$

Since $\prod_{i \in \Lambda} M_i$ is power-serieswise McCoy $\prod_{i \in \Lambda} R_i$ -module, there exists $r = (r_i)_{i \in \Lambda} \in \prod_{i \in \Lambda} R_i \setminus \{0\}$ such that $m(x)r = 0$. That is,

$$m(x)r = \begin{cases} m_s r_s, & s \neq k; \\ m_k(x)r_k, & s = k. \end{cases} \quad (12)$$

If $s \neq k$, then $m_s r_s = 0$ and $m_s \in M_s \setminus T(M_s)$ imply $r_s = 0$. However, $r \neq 0$ yields $r_k \neq 0$. Hence, M_k is a right power-serieswise McCoy R_k -module. Since $k \in \Lambda$ is arbitrary, we have done. $\square$

Proposition 11. *Let R be a commutative domain and M an R -module. The module M_R is power-serieswise McCoy if and only if its torsion submodule $T(M)$ is power-serieswise McCoy.*

Proof. Let $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$ satisfy $m(x)g(x) = 0$. Then

$$\begin{aligned} m_0 b_0 &= 0, m_0 b_1 + m_1 b_0 \\ &= 0, \dots, m_0 b_n + m_1 b_{n-1} + \dots + m_n b_0 = 0, \dots \end{aligned} \quad (13)$$

We may assume $b_0 \neq 0$ since $g(x) \neq 0$. Now multiplying $m_0 b_1 + m_1 b_0 = 0$ by b_0 on the right yields $m_1 b_0^2 = 0$; thus b_0^2 annihilates both m_0 and m_1 . Similarly, multiplying $m_0 b_2 + m_1 b_1 + m_2 b_0 = 0$ on the right by b_0^2 yields $m_2 b_0^3 = 0$. Continuing this procedure, one has $m_i b_0^{i+1} = 0$ for all i . So $m_i \in T(M)$, that is $m(x) \in T(M)[[x]]$. As $T(M)$ is power-serieswise McCoy, there exists $r \in R \setminus \{0\}$ such that $m(x)r = 0$. So M_R is power-serieswise McCoy. The other implication is trivial. $\square$

Recall that a *classical right quotient ring* for a ring R is a ring Q which contains R as a subring in such a way that every regular element (that is, nonzero-divisor) of R is invertible in Q and $Q = \{a\mu^{-1} \mid a, \mu \in R, \mu \text{ regular}\}$.

Proposition 12. *Suppose that there exists the classical right quotient ring Q of a ring R and M is a Q -module. Then M_R is power-serieswise McCoy if and only if M_Q is power-serieswise McCoy.*

Proof. Suppose that M_R is a power-serieswise McCoy module. Let $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in Q[[x]] \setminus \{0\}$ with $m(x)g(x) = 0$. Since Q is the classical right quotient ring of R , by [16, Proposition 2.1.16] we may assume that $b_j = b'_j \mu^{-1}$ with $b'_j \in R$ and some regular element $\mu \in R$. Set $g'(x) = \sum_{j=0}^{\infty} b'_j x^j$, then $g'(x) \in R[[x]] \setminus \{0\}$ and $0 = m(x)g(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} m_i b'_j x^{i+j} \mu^{-1} = m(x)g'(x)\mu^{-1}$. So $m(x)g'(x) = 0$. Since M_R is power-serieswise McCoy, there exists $r \in R \setminus \{0\} (\subseteq Q \setminus \{0\})$ such that $m(x)r = 0$. Therefore, M_Q is power-serieswise McCoy.

Conversely, note that R is the subring of Q , so M_R is power-serieswise McCoy. $\square$

3. Polynomial Modules

In this section, we study the extensions of power-serieswise McCoy module M , such as $M[x]$ and $M[x; x^{-1}]$. And the properties of special matrix extensions of a power-serieswise McCoy module are also considered.

Let M be a right R -module. If S is a multiplicatively closed subset of central regular elements of R , set

$$\begin{aligned} S^{-1}M &= \left\{ \frac{m}{s} \mid m \in M, s \in S \right\}, \\ S^{-1}R &= \left\{ \frac{r}{t} \mid r \in R, t \in S \right\}, \end{aligned} \quad (14)$$

then $S^{-1}M$ has an $S^{-1}R$ -module structure. M is called *S-torsion free* if whenever $m \in M \setminus \{0\}$ and $s \in S$, we have $ms \neq 0$ (see [17]).

Proposition 13. *Let M be an S -torsion free right R -module. Then M_R is power-serieswise McCoy if and only if $S^{-1}M$ is power-serieswise McCoy as an $S^{-1}R$ -module.*

Proof. Suppose that M_R is a power-serieswise McCoy module. Let $m'(x) = \sum_{i=0}^{\infty} (m_i/s)x^i \in S^{-1}M[[x]]$ and $g(x) = \sum_{j=0}^{\infty} (b_j/t)x^j \in S^{-1}R[[x]] \setminus \{0\}$ satisfy $m'(x)g(x) = 0$. Thus $m(x)g(x) = 0$ where $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]] \setminus \{0\}$. Since M is power-serieswise McCoy, there exists $r \in R \setminus \{0\} (\subseteq S^{-1}R \setminus \{0\})$ such that $m(x)r = 0$, so $m'(x)r = 0$. Hence $S^{-1}M$ is power-serieswise McCoy.

Conversely, let $n(x) = \sum_{i=0}^{\infty} n_i x^i \in M[[x]]$, $h(x) = \sum_{j=0}^{\infty} c_j x^j \in R[[x]] \setminus \{0\}$ satisfy $n(x)h(x) = 0$. Since $S^{-1}M$ is power-serieswise McCoy, there exists $r/t \in S^{-1}R \setminus \{0\}$ such that $n(x)(r/t) = 0$, thus $n(x)r = 0$. So M_R is power-serieswise McCoy. $\square$

Corollary 14. *A ring R is power-serieswise McCoy if and only if so is $S^{-1}R$.*

Theorem 15. *Let M_R be a module. Then the following statements are equivalent:*

- (1) M is a power-serieswise McCoy R -module.
- (2) $M[x]$ is a power-serieswise McCoy $R[x]$ -module.
- (3) $M[[x]]$ is a McCoy $R[[x]]$ -module.
- (4) $M[x; x^{-1}]$ is a power-serieswise McCoy $R[x; x^{-1}]$ -module.
- (5) $M[[x; x^{-1}]]$ is a McCoy $R[[x; x^{-1}]]$ -module.

Proof. Some of the proof is similar to that of [10, Theorem 3.2].

(1) $\implies$ (2) Suppose that M_R is power-serieswise McCoy. Let $m(y) = \sum_{i=0}^{\infty} m_i(x)y^i \in M[x][[y]]$ and $g(y) = \sum_{j=0}^{\infty} b_j(x)y^j \in R[x][[y]] \setminus \{0\}$ with $m(y)g(y) = 0$, where

$m_i(x) \in M[x]$, $b_j(x) \in R[x]$. The degree of zero polynomial is taken to be 0. Let

$$\begin{aligned} k_n &= \deg(m_0(x)) + \deg(m_1(x)) + \cdots \\ &+ \deg(m_n(x)) + \deg(b_0(x)) + \deg(b_1(x)) \\ &+ \cdots + \deg(b_n(x)) + 1. \end{aligned} \quad (15)$$

If we set $m(x) = \sum_{i=0}^{\infty} m_i(x)x^{ik_i}$, $b(x) = \sum_{j=0}^{\infty} b_j(x)x^{jk_j}$, then $b(x) \neq 0$, and the set of coefficients of $m_i(x)^{j_k}$'s (respectively, $b_j(x)^{i_k}$'s) equals the set of coefficients of $m(x)$ (respectively, $b(x)$). Since the equation $m(y)g(y) = 0$ deduced that $m(x)b(x) = 0$, there exists $r \in R \setminus \{0\} (\subseteq R[x] \setminus \{0\})$ such that $m(x)r = 0$. Thus $m(y)r = 0$. Hence, $M[x]$ is power-serieswise McCoy.

(2) $\implies$ (1) Assume that $M[x]$ is power-serieswise McCoy and $m(x)g(x) = 0$, where $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]] \setminus \{0\}$. Set $m(y) = \sum_{i=0}^{\infty} m_i y^i$ and $g(y) = \sum_{j=0}^{\infty} b_j y^j$. Then $m(y) \in M[x][[y]]$, $g(y) \in R[x][[y]] \setminus \{0\}$ and $m(y)g(y) = 0$. Since $M[x]_{R[x]}$ is power-serieswise McCoy, there exists $r(x) \in R[x] \setminus \{0\}$ such that $m(y)r(x) = 0$. Let c be a nonzero coefficient of $r(x)$ since $r(x) \neq 0$, then $m(y)c = 0$, that is $m(x)c = 0$. Hence M_R is power-serieswise McCoy.

(2) $\iff$ (3) Suppose that $M[x]$ is power-serieswise McCoy. Let $m(y) = \sum_{i=0}^p m_i(x)y^i \in M[[x]][y]$ and $g(y) = \sum_{j=0}^q b_j(x)y^j \in R[[x]][y] \setminus \{0\}$ with $m(y)g(y) = 0$, where $m_i(x) = \sum_{s=0}^{\infty} m_{si}x^s \in M[[x]]$, $b_j(x) = \sum_{t=0}^{\infty} b_{tj}x^t \in R[[x]]$ for $i = 0, 1, \dots, p$ and $j = 0, 1, \dots, q$. Note that

$$\begin{aligned} m(y) &= m_0(x) + m_1(x)y + m_2(x)y^2 + \cdots \\ &+ m_p(x)y^p \\ &= \sum_{s=0}^{\infty} m_{s0}x^s + \sum_{s=0}^{\infty} m_{s1}x^s y + \sum_{s=0}^{\infty} m_{s2}x^s y^2 + \cdots \\ &+ \sum_{s=0}^{\infty} m_{sp}x^s y^p \\ &= \sum_{k=0}^p m_{0k}y^k + \sum_{k=0}^p m_{1k}y^k x + \sum_{k=0}^p m_{2k}y^k x^2 + \cdots \\ &+ \sum_{k=0}^p m_{wk}y^k x^w + \cdots = \sum_{w=0}^{\infty} \left(\sum_{k=0}^p m_{wk}y^k \right) x^w. \end{aligned} \quad (16)$$

Write $m(x) = \sum_{w=0}^{\infty} n_w(y)x^w$ where $n_w(y) = \sum_{k=0}^p m_{wk}y^k \in M[y]$. Then $m(x) \in M[y][[x]]$. Similarly,

$$\begin{aligned} g(y) &= b_0(x) + b_1(x)y + b_2(x)y^2 + \cdots + b_q(x)y^q \\ &= \sum_{t=0}^{\infty} b_{t0}x^t + \sum_{t=0}^{\infty} b_{t1}x^t y + \sum_{t=0}^{\infty} b_{t2}x^t y^2 + \cdots \\ &+ \sum_{t=0}^{\infty} b_{tq}x^t y^q \end{aligned}$$

$$\begin{aligned} &= \sum_{\mu=0}^q b_{0\mu}y^{\mu} + \sum_{\mu=0}^q b_{1\mu}y^{\mu}x + \sum_{\mu=0}^q b_{2\mu}y^{\mu}x^2 + \cdots \\ &+ \sum_{\mu=0}^q b_{v\mu}y^{\mu}x^v + \cdots = \sum_{v=0}^{\infty} \left(\sum_{\mu=0}^q b_{v\mu}y^{\mu} \right) x^v. \end{aligned} \quad (17)$$

Write $g(x) = \sum_{v=0}^{\infty} c_v(y)x^v$ where $c_v(y) = \sum_{\mu=0}^q b_{v\mu}y^{\mu} \in R[y]$. Then $g(x) \in R[y][[x]]$. Obviously, $m(x)g(x) = 0$. By the assumption, $M[y]$ is a power-serieswise McCoy $R[y]$ -module. Thus there exists $g_1(y) = \sum_{i=0}^t a_i y^i \in R[y] \setminus \{0\}$ such that $m(x)g_1(y) = 0$. Set $g_2(x) = \sum_{i=0}^t a_i x^i$. Then $g_2(x) \in R[x] \setminus \{0\} (\subseteq R[[x]] \setminus \{0\})$ and $m(y)g_2(x) = 0$. Hence $M[[x]]$ is McCoy. The other complication can be proved similarly.

(2) $\iff$ (4) Let $S = \{1, x, x^2, \dots\}$. Then S is a multiplicatively closed subset of central regular elements of $R[x]$. Since $M[x; x^{-1}] = S^{-1}M[x]$, $R[x; x^{-1}] = S^{-1}R[x]$, we have done by Proposition 13.

(4) $\iff$ (5) It is similar to the proof of (2) $\iff$ (3). $\square$

Corollary 16. Let R be a ring. Then the following statements are equivalent:

- (1) R is a power-serieswise McCoy ring.
- (2) $R[x]$ is a power-serieswise McCoy ring.
- (3) $R[[x]]$ is a McCoy ring.
- (4) $R[x; x^{-1}]$ is a power-serieswise McCoy ring.
- (5) $R[[x; x^{-1}]]$ is a McCoy ring.

Let M_R be a module. For $n \in \mathbb{N} \setminus \{0\}$, R_n denotes the following subring of the ring of all matrices over R :

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}, \quad (18)$$

and M_n denotes the following additive group:

$$M_n = \left\{ \begin{pmatrix} m & m_{12} & \cdots & m_{1n} \\ 0 & m & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{pmatrix} \mid m, m_{ij} \in M \right\}. \quad (19)$$

Then M_n is an R_n -module under the usual matrix addition operation and scalar multiplication.

Proposition 17. The R -module M is power-serieswise McCoy if and only if the R_n -module M_n is power-serieswise McCoy.

Proof. Suppose that M_R is power-serieswise McCoy. Let $m(x) = \sum_{i=0}^{\infty} A_i x^i \in M_n[[x]]$, $g(x) = \sum_{j=0}^{\infty} B_j x^j \in R[[x]] \setminus \{0\}$ satisfy $m(x)g(x) = 0$, where

$$\begin{aligned} A_i &= \begin{pmatrix} a^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^i \end{pmatrix} \in M_n, \\ B_j &= \begin{pmatrix} b^j & b_{12}^j & \cdots & b_{1n}^j \\ 0 & b^j & \cdots & b_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^j \end{pmatrix} \in R_n. \end{aligned} \quad (20)$$

Under the usual matrix addition operation, we have

$$\begin{aligned} m(x) &= \sum_{i=0}^{\infty} A_i x^i = \sum_{i=0}^{\infty} \begin{pmatrix} a^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^i \end{pmatrix} x^i \\ &= \begin{pmatrix} \sum_{i=0}^{\infty} a^i x^i & \sum_{i=0}^{\infty} a_{12}^i x^i & \cdots & \sum_{i=0}^{\infty} a_{1n}^i x^i \\ 0 & \sum_{i=0}^{\infty} a^i x^i & \cdots & \sum_{i=0}^{\infty} a_{2n}^i x^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{i=0}^{\infty} a^i x^i \end{pmatrix} \\ &\triangleq \begin{pmatrix} a(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ 0 & a(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a(x) \end{pmatrix} \end{aligned} \quad (21)$$

and

$$g(x) = \sum_{j=0}^{\infty} B_j x^j = \sum_{j=0}^{\infty} \begin{pmatrix} b^j & b_{12}^j & \cdots & b_{1n}^j \\ 0 & b^j & \cdots & b_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^j \end{pmatrix} x^j$$

$$\begin{aligned} &= \begin{pmatrix} \sum_{j=0}^{\infty} b^j x^j & \sum_{j=0}^{\infty} b_{12}^j x^j & \cdots & \sum_{j=0}^{\infty} b_{1n}^j x^j \\ 0 & \sum_{j=0}^{\infty} b^j x^j & \cdots & \sum_{j=0}^{\infty} b_{2n}^j x^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j=0}^{\infty} b^j x^j \end{pmatrix} \\ &\triangleq \begin{pmatrix} b(x) & b_{12}(x) & \cdots & b_{1n}(x) \\ 0 & b(x) & \cdots & b_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b(x) \end{pmatrix}. \end{aligned} \quad (22)$$

Now we proceed with the following cases. $\square$

Case 1. Suppose that $a(x) = 0$. Take $D = E_{1n} \in R_n \setminus \{0\}$. Thus $m(x)D = 0$.

Case 2. Suppose that $a(x) \neq 0$ and $b(x) \neq 0$. Since M is power-serieswise McCoy, $a(x)b(x) = 0$ implies that there exists $r \in R \setminus \{0\}$ such that $a(x)r = 0$. Take $D = rE_{1n}$, then $m(x)D = 0$.

Case 3. Suppose that $a(x) \neq 0$ and $b(x) = 0$. Since $g(x) \neq 0$, there must be a minimal index ℓ such that $b_{k\ell} \neq 0$ and $b_{s\ell} = 0$, $k < s < \ell$, where $b_{ij}(x) = \sum_{t=0}^{\infty} b_{ij}^t x^t$. Consider the (k, ℓ) -entry of matrix $m(x)g(x)$; we have

$$a(x)b_{k\ell}(x) = 0. \quad (23)$$

It follows from M_R is power-serieswise McCoy that there exists $r \in R \setminus \{0\}$ such that $a(x)r = 0$. Take $D = rE_{1n} \in R_n \setminus \{0\}$, then $m(x)D = 0$.

Therefore, M_n is a power-serieswise McCoy R_n -module for all cases.

Conversely, let $m(x) = \sum_{i=0}^{\infty} a_i x^i \in M[[x]]$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]] \setminus \{0\}$ satisfy $m(x)g(x) = 0$ where $a_i \in M, b_j \in R$. Take $\alpha(x) = m(x)E \in M_n[[x]]$, $\beta(x) = g(x)E \in R_n[[x]]$, where E is the unit matrix. It is easy to see that $\alpha(x)\beta(x) = 0$. Since M_n is a power-serieswise McCoy R_n -

module, there exists $D = \begin{pmatrix} d & d_{12} & \cdots & d_{1n} \\ 0 & d & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d \end{pmatrix} \in R_n \setminus \{0\}$, such that $\alpha(x)D = 0$. By $D \neq 0$, there exists $d_{st} \neq 0$ (if $s = t$, $d_{st} = d$) such that $m(x)d_{st} = 0$. Hence M_R is power-serieswise McCoy.

Corollary 18 (see [7, Corollary 2.9]). *R is a power-serieswise McCoy ring if and only if so is R_n where $n \geq 1$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] M. B. Rege and S. Chhawchharia, "Armendariz rings," *Proceedings of the Japan Academy, Series A: Mathematical Sciences*, vol. 73, no. 1, pp. 14–17, 1997.
- [2] P. P. Nielsen, "Semi-commutativity and the McCoy condition," *Journal of Algebra*, vol. 298, no. 1, pp. 134–141, 2006.
- [3] N. H. McCoy, "Remarks on Divisors of Zero," *The American Mathematical Monthly*, vol. 49, no. 5, pp. 286–295, 2018.
- [4] A. Alhevaz, A. Moussavi, and M. Habibi, "On rings having McCoy-like conditions," *Communications in Algebra*, vol. 40, no. 4, pp. 1195–1221, 2012.
- [5] Z. Lei, J. Chen, and Z. Ying, "A question on McCoy rings," *Bulletin of the Australian Mathematical Society*, vol. 76, no. 1, pp. 137–141, 2007.
- [6] Z.-l. Ying, J.-l. Chen, and Z. Lei, "Extensions of McCoy rings," *Northeastern Mathematical Journal*, vol. 24, no. 1, pp. 85–94, 2008.
- [7] S. Yang, X. Song, and Z. Liu, "Power-serieswise McCoy rings," *Algebra Colloquium*, vol. 18, no. 2, pp. 301–310, 2011.
- [8] N. K. Kim and Y. Lee, "Armendariz rings and reduced rings," *Journal of Algebra*, vol. 223, no. 2, pp. 477–488, 2000.
- [9] A. M. Buhphang and M. B. Rege, "Semi-commutative modules and Armendariz modules," *Arab Journal of Mathematical Sciences*, vol. 8, no. 1, pp. 53–65, 2002.
- [10] J. Cui and J. Chen, "On McCoy modules," *Bulletin of the Korean Mathematical Society*, vol. 48, no. 1, pp. 23–33, 2011.
- [11] T.-K. Lee and Y. Zhou, "Reduced modules," in *Rings, Modules, Algebras and Abelian Groups*, vol. 236 of *Lecture Notes in Pure and Applied Mathematics*, pp. 365–377, Marcel Dekker, New York, NY, USA, 2004.
- [12] N. K. Kim, K. H. Lee, and Y. Lee, "Power series rings satisfying a zero divisor property," *Communications in Algebra*, vol. 34, no. 6, pp. 2205–2218, 2006.
- [13] N. Agayev, G. Gungoroglu, A. Harmanaci et al., "Abelian modules," *Acta Mathematica Universitatis Comenianae*, vol. 78, no. 2, pp. 235–244, 2009.
- [14] M. B. Rege and A. M. Buhphang, "On reduced modules and rings," *International Electronic Journal of Algebra*, vol. 3, pp. 58–74, 2008.
- [15] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, vol. 13 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1974.
- [16] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley, New York, NY, USA, 1987.
- [17] A. Nasr-Isfahani and A. Moussavi, "On skew power serieswise Armendariz rings," *Communications in Algebra*, vol. 39, no. 9, pp. 3114–3132, 2011.