Hindawi
Mathematical Problems in Engineering
Volume 2019, Article ID 1257386, 8 pages
https://doi.org/10.1155/2019/1257386

Research Article

A New Stability Analysis of Uncertain Delay Differential Equations

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Received 19 October 2018; Revised 1 December 2018; Accepted 25 December 2018; Published 6 January 2019

Academic Editor: Zhiwei Gao

This paper first provides a concept of almost sure stability for uncertain delay differential equations and analyzes this new sort of stability. In addition, this paper derives three sufficient conditions for uncertain delay differential equations being stable almost surely. Finally, the relationship between almost sure stability and stability in measure for uncertain delay differential equations is discussed.

1. Introduction

In order to deal with nondeterministic phenomenon in a dynamic system, Ito [1] proposed stochastic differential equation, which was driven by Wiener process. From then on, stochastic differential equation was employed to study dynamic systems with perturbation and applied in the fields of finance, control, and aerospace engineering. In the process of researching social system, the data used to describe the dynamic system may come from the domain experts. At this time, the expert data cannot be regard as a random variable. How to deal with these expert data in such dynamic system is an immediate problem. To tackle this problem, Liu [2] established uncertainty theory and proposed uncertain variable to describe the expert data. In addition, Liu [3] also proposed the concept of uncertain process to describe the evolution of an uncertain phenomenon. As a comparison of Wiener process, Liu process was designed by Liu [4]. Based on Liu process, uncertain calculus [4] is proposed to solve the integral and differential of an uncertain process.

Driven by a Liu process, uncertain differential equation was proposed [3] to deal with dynamic systems under uncertain environment. In the aspect of theory, Chen and Liu [5] and Gao [6] proved two existence and uniqueness theorems on uncertain differential equations, respectively. Since it was difficult to obtain the analytic solutions of vast majority of uncertain differential equations, some numerical methods were proposed, such as Milne method [7], Adams-Simpson method [8], Euler method [9], and Hamming method [10]. In regard to stability analysis, stability in measure of uncertain differential equations was put forward by Liu [4] and stability in measure of linear uncertain differential equations was discussed by Yao, Gao, and Gao [11]. In addition, other types of stability were studied, such as almost sure stability [12], stability in moment [13], exponential stability [14], and stability in inverse distribution [15]. So far, some researchers employed uncertain differential equations to model the financial market. Uncertain stock model [16, 17], uncertain interest rate model [18, 19], and uncertain currency model [20, 21] have become the focus of attention for many scholars. In addition, uncertain differential equations have also been introduced into string vibration [22], differential game [23], optimal control [24], and so on.

By using an uncertain differential equation, we can establish a mathematical model to describe the dynamic system under an uncertain environment, where the velocity of such dynamic system only depends on the state of the system at a given instant of time. However, the velocity of the dynamic

system depends not only on the current state but also upon the previous states in some real phenomena. In such case, it is inappropriate to insist on modeling by using uncertain differential equation. An extended type called uncertain delay differential equation happens to describe the dynamic system just mentioned. Uncertain delay differential equations have been widely used in the field of engineering, especially in automatic control system; in the field of natural science, such as ecosystem, infectious diseases, and population dynamics; and in the field of social science to describe economic phenomena, commercial sales and transportation scheduling [25–28]. Back to the theoretical research on uncertain delay differential equations, Barbacioru [29] and Ge and Zhu [30] explored two existence and uniqueness theorems on uncertain delay differential equations, respectively. In the aspect of stability of uncertain delay differential equations, Wang and Ning [31] defined the stability in measure, stability in mean, and stability in moment and proved the corresponding stability theorems. In addition, Jia and Sheng [32] also discussed the stability in distribution recently.

In this paper, we propose a new stability called almost sure stability for an uncertain delay differential equation and give its sufficient condition. The structure of this paper is organized as follows. Section 2 introduces some basic knowledge of uncertain delay differential equations. Section 3 gives the definition of almost sure stability for an uncertain delay differential equation. Then we present three sufficient conditions for an uncertain delay differential equation and linear uncertain delay differential equations being stable almost surely in Section 4. After that, Section 5 analyzes the relationship between almost sure stability and stability in measure. The last section makes a brief conclusion.

2. Uncertain Delay Differential Equation

This part makes a brief introduction of uncertain delay differential equation based on uncertain variable and uncertain process, where uncertain variable and uncertain process can be seen in Appendixes A and B.

Definition 1 (Barbacioru [29]). Let C_t be a Liu process, and f and g are two real-valued functions.

$$\mathrm{d}X_t = f\left(t, X_t, X_{t-\tau}\right)\mathrm{d}t + g\left(t, X_t, X_{t-\tau}\right)\mathrm{d}C_t \tag{1}$$

is called an *uncertain delay differential equation*, where $\tau > 0$ is called *time delay*.

Theorem 2 (Ge and Zhu [30]). *Uncertain delay differential equation (1) with initial states has a unique solution if the coefficients $f(t, x, y)$ and $g(t, x, y)$ satisfy*

$$\left|f\left(t, x, y\right)\right| + \left|g\left(t, x, y\right)\right| \le L\left(1 + |x| + |y|\right), \quad \forall x, y \in \mathfrak{R},\ t \ge 0 \tag{2}$$

and

$$\left|f\left(t, x_1, y\right) - f\left(t, x_2, y\right)\right| + \left|g\left(t, x_1, y\right) - g\left(t, x_2, y\right)\right| \le L\left|x_1 - x_2\right|, \quad \forall x_1, x_2, y \in \mathfrak{R},\ t \ge 0 \tag{3}$$

for some positive constant L.

Definition 3 (Wang and Ning [31]). Uncertain delay differential equation (1) is said to be stable in measure if for any two solutions X_t and Y_t with different initial states x_j and y_j for any $j \in [-\tau, 0]$, respectively, we have

$$\lim_{\sup_{j\in[-\tau,0]}|x_j - y_j| \longrightarrow 0} \mathscr{M}\left\{\left|X_t - Y_t\right| < \epsilon\right\} = 1, \quad \forall t > 0 \tag{4}$$

for any $\epsilon > 0$, where $\mathscr{M}$ is uncertain measure (see Appendix A).

Definition 4 (Liu [4]). Let X_t, C_t be an uncertain process and a Liu process, respectively. For any partition of the closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \sup_{1\le i\le k}\left|t_{i+1} - t_i\right|. \tag{5}$$

Then the *uncertain integral* of X_t with respect to C_t is defined by

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\longrightarrow 0}\sum_{i=1}^{k} X_{t_i}\cdot\left(C_{t_{i+1}} - C_{t_i}\right) \tag{6}$$

provided that the limit exists almost surely and is finite.

Theorem 5 (Chen and Liu [5]). *Supposing that C_t is a Liu process and X_t is an integrable uncertain process on $[a, b]$ with respect to t, then*

$$\left|\int_a^b X_t\left(\gamma\right)\mathrm{d}C_t\left(\gamma\right)\right| \le K\left(\gamma\right)\int_a^b \left|X_t\left(\gamma\right)\right|\mathrm{d}t \tag{7}$$

holds, where $K(\gamma)$ is the Lipschitz constant of $C_t(\gamma)$.

3. Almost Sure Stability

Uncertain delay differential equation (1) is equivalent to the uncertain delay integral equation

$$\begin{aligned} X_t = X_{t_0} &+ \int_{t_0}^{t} f\left(s, X_s, X_{s-\tau}\right)\mathrm{d}s \\ &+ \int_{t_0}^{t} g\left(s, X_s, X_{s-\tau}\right)\mathrm{d}C_s. \end{aligned} \tag{8}$$

For the sake of simplicity, we set the initial time t_0 to zero. Then, the above equation can be simplified as

$$\begin{aligned} X_t = X_0 &+ \int_{0}^{t} f\left(s, X_s, X_{s-\tau}\right)\mathrm{d}s \\ &+ \int_{0}^{t} g\left(s, X_s, X_{s-\tau}\right)\mathrm{d}C_s. \end{aligned} \tag{9}$$

Now let us present a definition of almost sure stability for uncertain delay differential equation (1).

Definition 6. Supposing that X_t and Y_t are two solutions of uncertain delay differential equation (1) with different initial states x_j and y_j for any $j \in [-\tau, 0]$, respectively, uncertain

delay differential equation (1) is said to be stable almost surely if

$$\mathcal{M}\left\{\gamma \in \Gamma \mid \lim_{\sup_{j\in[-\tau,0]}|x_j-y_j|\longrightarrow 0} \left|X_t(\gamma) - Y_t(\gamma)\right| = 0\right\} = 1, \quad \forall t > 0. \tag{10}$$

Example 7. Consider an uncertain delay differential equation

$$\mathrm{d}X_t = \mu X_{t-\tau}\mathrm{d}t + \sigma \mathrm{d}C_t. \tag{11}$$

The analytical solution of uncertain delay differential equation (11) with two initial states $\phi(t)$ and $\psi(t)$ $(t \in [-\tau, 0])$ is

$$X_t = \begin{cases} \phi(t), & t \in [-\tau, 0] \\ \phi(0) + \mu \int_0^t \phi(s-\tau)\,\mathrm{d}s + \sigma C_t, & t \in (0, \tau] \\ X_\tau + \mu \int_\tau^t X_{s-\tau}\mathrm{d}s + \sigma\left(C_t - C_\tau\right), & t \in (\tau, 2\tau] \\ \cdots\cdots \end{cases} \tag{12}$$

and

$$Y_t = \begin{cases} \psi(t), & t \in [-\tau, 0] \\ \psi(0) + \mu \int_0^t \psi(s-\tau)\,\mathrm{d}s + \sigma C_t, & t \in (0, \tau] \\ Y_\tau + \mu \int_\tau^t Y_{s-\tau}\mathrm{d}s + \sigma\left(C_t - C_\tau\right), & t \in (\tau, 2\tau] \\ \cdots\cdots \end{cases} \tag{13}$$

respectively.

Then

$$\left|X_t - Y_t\right| = \begin{cases} \left|\phi(t) - \psi(t)\right|, & t \in [-\tau, 0] \\ \left|\phi(0) - \psi(0)\right| + \mu \int_0^t \left|\phi(s-\tau) - \psi(s-\tau)\right| \mathrm{d}s, & t \in (0, \tau] \\ \left|X_\tau - Y_\tau\right| + \mu \int_\tau^t \left|X_{s-\tau} - Y_{s-\tau}\right| \mathrm{d}s, & t \in (\tau, 2\tau] \\ \cdots\cdots \end{cases} \tag{14}$$

and, therefore, we have

$$\mathcal{M}\left\{\gamma \in \Gamma \mid \lim_{\sup_{w\in[-\tau,0]}|\phi(w)-\psi(w)|\longrightarrow 0} \left|X_t(\gamma) - Y_t(\gamma)\right| = 0\right\} = 1, \quad \forall t > 0. \tag{15}$$

By using Definition 6, we have that uncertain delay differential equation (11) is stable almost surely.

Example 8. Consider an uncertain delay differential equation

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma X_{t-\tau}\mathrm{d}C_t. \tag{16}$$

The analytical solution of uncertain delay differential equation (16) with two initial states $\phi(t)$ and $\psi(t)$ $(t \in [-\tau, 0])$ is

$$X_t = \begin{cases} \phi(t), & t \in [-\tau, 0] \\ \phi(0) + \mu t + \sigma \int_0^t \phi(s-\tau)\,\mathrm{d}C_s, & t \in (0, \tau] \\ X_\tau + \mu(t-\tau) + \sigma \int_\tau^t X_{s-\tau}\mathrm{d}C_s, & t \in (\tau, 2\tau] \\ \cdots\cdots \end{cases} \tag{17}$$

and

$$Y_t = \begin{cases} \psi(t), & t \in [-\tau, 0] \\ \psi(0) + \mu t + \sigma \int_0^t \psi(s-\tau)\,\mathrm{d}C_s, & t \in (0, \tau] \\ Y_\tau + \mu(t-\tau) + \sigma \int_\tau^t Y_{s-\tau}\mathrm{d}C_s, & t \in (\tau, 2\tau] \\ \cdots\cdots \end{cases} \tag{18}$$

respectively.

Then

$$\left|X_t - Y_t\right| = \begin{cases} \left|\phi(t) - \psi(t)\right|, & t \in [-\tau, 0] \\ \left|\phi(0) - \psi(0)\right| + \sigma \int_0^t \left|\phi(s-\tau) - \psi(s-\tau)\right| \mathrm{d}C_s, & t \in (0, \tau] \\ \left|X_\tau - Y_\tau\right| + \sigma \int_\tau^t \left|X_{s-\tau} - Y_{s-\tau}\right| \mathrm{d}C_s, & t \in (\tau, 2\tau] \\ \cdots\cdots \end{cases} \tag{19}$$

and, therefore, we have

$$\mathcal{M}\left\{\gamma \in \Gamma \mid \lim_{\sup_{w\in[-\tau,0]}|\phi(w)-\psi(w)|\longrightarrow 0} \left|X_t(\gamma) - Y_t(\gamma)\right| = 0\right\} = 1, \quad \forall t > 0. \tag{20}$$

This means that uncertain delay differential equation (16) is stable almost surely by using Definition 6.

4. Stability Theorem

A sufficient condition for uncertain delay differential equation (1) being stable almost surely is discussed and shown by the following theorem.

Theorem 9. *Supposing that uncertain delay differential equation (1) has a unique solution for each given initial state. Then uncertain delay differential equation (1) is stable almost surely if the coefficients $f(t, x, y)$ and $g(t, x, y)$ satisfy*

$$\left|f(t, x_1, y) - f(t, x_2, y)\right| + \left|g(t, x_1, y) - g(t, x_2, y)\right| \le L_t \left|x_1 - x_2\right|, \quad \forall x_1, x_2, y \in \mathfrak{R},\ t \ge 0 \tag{21}$$

where $L_t \ge 0$ and $\int_0^{+\infty} L_t \mathrm{d}t < +\infty$.

Proof. Let X_t and Y_t denote two solutions of uncertain delay differential equation (1) with different initial states $\phi(t)$ and $\psi(t)$ $(t \in [-\tau, 0])$, respectively. That is,

$$\begin{aligned} &\mathrm{d}X_t = f(t, X_t, X_{t-\tau})\,\mathrm{d}t + g(t, X_t, X_{t-\tau})\,\mathrm{d}C_t, \\ &\qquad\qquad t \in (0, +\infty) \\ &X_t = \phi(t), \quad t \in [-\tau, 0] \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\mathrm{d}Y_t = f(t, Y_t, Y_{t-\tau})\,\mathrm{d}t + g(t, Y_t, Y_{t-\tau})\,\mathrm{d}C_t, \\ &\qquad\qquad t \in (0, +\infty) \\ &Y_t = \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{23}$$

Then for any Lipschitz continuous sample $C_t(\gamma)$, we have

$$\begin{aligned} X_t(\gamma) &= X_0 + \int_0^t f(s, X_s(\gamma), X_{s-\tau}(\gamma))\,\mathrm{d}s \\ &\quad + \int_0^t g(s, X_s(\gamma), X_{s-\tau}(\gamma))\,\mathrm{d}C_s(\gamma), \\ &\qquad\qquad t \in (0, +\infty) \\ X_t &= \phi(t), \quad t \in [-\tau, 0] \end{aligned} \tag{24}$$

and

$$\begin{aligned} Y_t(\gamma) &= Y_0 + \int_0^t f(s, Y_s(\gamma), Y_{s-\tau}(\gamma))\,\mathrm{d}s \\ &\quad + \int_0^t g(s, Y_s(\gamma), Y_{s-\tau}(\gamma))\,\mathrm{d}C_s(\gamma), \\ &\qquad\qquad t \in (0, +\infty) \\ Y_t &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{25}$$

By using formula (21) and Theorem 5, the inequality

$$\begin{aligned} |X_t(\gamma) - Y_t(\gamma)| &= \Big| X_0 - Y_0 \\ &\quad + \int_0^t f(s, X_s(\gamma), X_{s-\tau}(\gamma)) \\ &\quad - f(s, Y_s(\gamma), Y_{s-\tau}(\gamma))\,\mathrm{d}s \\ &\quad + \int_0^t g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \\ &\quad - g(s, Y_s(\gamma), Y_{s-\tau}(\gamma))\,\mathrm{d}C_s(\gamma) \Big| \le |X_0 \\ &\quad - Y_0| + \Big| \int_0^t f(s, X_s(\gamma), X_{s-\tau}(\gamma)) \\ &\quad - f(s, Y_s(\gamma), Y_{s-\tau}(\gamma))\,\mathrm{d}s \Big| \\ &\quad + \Big| \int_0^t g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \\ &\quad - g(s, Y_s(\gamma), Y_{s-\tau}(\gamma))\,\mathrm{d}C_s(\gamma) \Big| \le |X_0 \\ &\quad - Y_0| + \int_0^t L_s |X_s(\gamma) - Y_s(\gamma)|\,\mathrm{d}s + K(\gamma) \\ &\quad \cdot \int_0^t L_s |X_s(\gamma) - Y_s(\gamma)|\,\mathrm{d}s = |X_0 - Y_0| + (1 \\ &\quad + K(\gamma)) \int_0^t L_s |X_s(\gamma) - Y_s(\gamma)|\,\mathrm{d}s \end{aligned} \tag{26}$$

holds, where $K(\gamma)$ is the Lipschitz constant of $C_t(\gamma)$.

By using the Gronwall's inequality [33], we have

$$\begin{aligned} &|X_t(\gamma) - Y_t(\gamma)| \\ &\quad \le |X_0 - Y_0| \exp\left((1 + K(\gamma)) \int_0^t L_s \mathrm{d}s \right) \\ &\quad \le |X_0 - Y_0| \exp\left((1 + K(\gamma)) \int_0^{+\infty} L_s \mathrm{d}s \right) \\ &\quad \le \sup_{j \in [-\tau, 0]} |X_j - Y_j| \exp\left((1 + K(\gamma)) \int_0^{+\infty} L_s \mathrm{d}s \right) \end{aligned} \tag{27}$$

for any $t > 0$.

Since

$$\int_0^{+\infty} L_t \mathrm{d}t < +\infty \tag{28}$$

and $K(\gamma)$ is finite, we have that $|X_t(\gamma) - Y_t(\gamma)| \longrightarrow 0$ as long as $\sup_{j \in [-\tau, 0]} |X_j - Y_j| \longrightarrow 0$, which implies that

$$\mathscr{M}\left\{ \lim_{\sup_{j \in [-\tau, 0]} |X_j - Y_j| \longrightarrow 0} |X_t - Y_t| = 0 \right\} = 1, \quad \forall t > 0. \tag{29}$$

This means that uncertain delay differential equation (1) is almost surely stable under formula (21). □

Example 10. Consider an uncertain delay differential equation

$$\mathrm{d}X_t = (\exp(-t) X_t + \mu X_{t-\tau})\,\mathrm{d}t + \sigma \mathrm{d}C_t. \tag{30}$$

Take $f(t, x, y) = \exp(-t)x + \mu y$ and $g(t, x, y) = \sigma$. Let N denote a common upper bound of $|\mu|, |\sigma|$ and $|\exp(-t)|$ with $t > 0$. The inequalities

$$\begin{aligned} &|f(t, x, y)| + |g(t, x, y)| \le N(1 + |x| + |y|), \\ &\qquad\qquad \forall x, y \in \mathfrak{R},\ t \ge 0, \end{aligned} \tag{31}$$

and

$$\begin{aligned} &|f(t, x_1, y) - f(t, x_2, y)| + |g(t, x_1, y) - g(t, x_2, y)| \\ &\quad \le \exp(-t) |x_1 - x_2|, \quad \forall x_1, x_2, y \in \mathfrak{R},\ t \ge 0 \end{aligned} \tag{32}$$

hold. According to Theorem 2, we obtain that uncertain delay differential equation (30) with initial states has a unique solution. In addition, by using inequality (32) and

$$\int_0^{+\infty} \exp(-t)\,\mathrm{d}t < +\infty, \tag{33}$$

uncertain delay differential equation (30) is stable almost surely by Theorem 9.

Example 11 (uncertain cell population growth model). The initial cell population growth model was provided by the following equation [34]:

$$\mathrm{d}N_t = (\rho_0 N_t + \rho_1 N_{t-\tau})\,\mathrm{d}t, \quad t \geq 0 \tag{34}$$

where N_t is the number of cells in cell population, $\rho_0 > 0$ is the instantaneous growth rate, and ρ_1 is the delayed growth rate.

If these biological systems operate in uncertain environment, the population N_t is an uncertain process and its growth is described by the uncertain delay differential equation

$$\mathrm{d}N_t = (\rho_0 N_t + \rho_1 N_{t-\tau})\,\mathrm{d}t + \sigma \mathrm{d}C_t, \quad t \geq 0 \tag{35}$$

where σ is a constant and C_t is a Liu process. Uncertain delay differential equation (35) is called uncertain cell population growth model.

Take $f(t,x,y) = \rho_0 x + \rho_1 y$ and $g(t,x,y) = \sigma$. Let M denote a common upper bound of $\rho_0, |\rho_1|$ and $|\sigma|$. The inequalities

$$\begin{aligned} &|f(t,x,y)| + |g(t,x,y)| \leq M(1 + |x| + |y|), \\ &\qquad \forall x, y \in \mathfrak{R},\ t \geq 0, \end{aligned} \tag{36}$$

and

$$\begin{aligned} &|f(t,x_1,y) - f(t,x_2,y)| + |g(t,x_1,y) - g(t,x_2,y)| \\ &\quad \leq \rho_0 |x_1 - x_2|, \quad \forall x_1, x_2, y \in \mathfrak{R},\ t \geq 0 \end{aligned} \tag{37}$$

hold. According to Theorem 2, we have that uncertain delay differential equation (35) with the initial states has a unique solution.

However, take $L_t = \rho_0$ and $\int_0^{+\infty} L_t \mathrm{d}t = +\infty$. According to Theorem 9, we cannot judge whether the solution of uncertain delay differential equation (35) is almost surely stable or not.

From Examples 10 and 11, we can see that Theorem 9 only gives the sufficient condition but not the necessary and sufficient condition for uncertain delay differential equation being almost surely stable. On the basis of Theorem 9, we immediately present a corollary about a sufficient condition for a linear uncertain delay differential equation being stable almost surely.

Corollary 12. *Supposing that u_{it}, v_{it}, and w_{it} $(i = 1,2)$ are real-valued functions, then the linear uncertain delay differential equation*

$$\begin{aligned} \mathrm{d}X_t &= (u_{1t} X_t + v_{1t} X_{t-\tau} + w_{1t})\,\mathrm{d}t \\ &\quad + (u_{2t} X_t + v_{2t} X_{t-\tau} + w_{2t})\,\mathrm{d}C_t \end{aligned} \tag{38}$$

is almost surely stable if u_{it}, v_{it}, and w_{it} $(i = 1,2)$ are bounded,

$$\int_0^{+\infty} |u_{1t}|\,\mathrm{d}t < +\infty \tag{39}$$

and

$$\int_0^{+\infty} |u_{2t}|\,\mathrm{d}t < +\infty. \tag{40}$$

Proof. Take $f(t,x,y) = u_{1t}x + v_{1t}y + w_{1t}$ and $g(t,x,y) = u_{2t}x + v_{2t}y + w_{2t}$. Let N denote a common upper bound of $|u_{it}|, |v_{it}|$ and $|w_{it}|$ $(i = 1,2)$. The inequalities

$$\begin{aligned} &|f(t,x,y)| + |g(t,x,y)| \leq 2N(1 + |x| + |y|), \\ &\qquad \forall x, y \in \mathfrak{R},\ t \geq 0, \end{aligned} \tag{41}$$

and

$$\begin{aligned} &|f(t,x_1,y) - f(t,x_2,y)| + |g(t,x_1,y) - g(t,x_2,y)| \\ &\quad = |u_{1t}||x_1 - x_2| + |u_{2t}||x_1 - x_2| \leq 2M|x_1 - x_2|, \\ &\qquad \forall x_1, x_2, y \in \mathfrak{R},\ t \geq 0 \end{aligned} \tag{42}$$

hold.

According to Theorem 2, we have that linear uncertain delay differential equation (38) with initial states has a unique solution. Since

$$\begin{aligned} &|f(t,x_1,y) - f(t,x_2,y)| + |g(t,x_1,y) - g(t,x_2,y)| \\ &\quad = |u_{1t}||x_1 - x_2| + |u_{2t}||x_1 - x_2| \\ &\quad \leq (|u_{1t}| + |u_{2t}|)|x_1 - x_2|, \\ &\qquad \forall x_1, x_2, y \in \mathfrak{R},\ t \geq 0, \end{aligned} \tag{43}$$

we take $L_t = |u_{1t}| + |u_{2t}|$ which is integrable on $[0, +\infty)$. Hence, we have

$$\int_0^{+\infty} |u_{1t}|\,\mathrm{d}t < +\infty \tag{44}$$

and

$$\int_0^{+\infty} |u_{2t}|\,\mathrm{d}t < +\infty. \tag{45}$$

By using Theorem 9, the linear uncertain delay differential equation (38) is almost surely stable. □

Example 13. Consider a linear uncertain delay differential equation (30):

$$\mathrm{d}X_t = (\exp(-t) X_t + \mu X_{t-\tau})\,\mathrm{d}t + \sigma \mathrm{d}C_t. \tag{46}$$

The real-valued functions $\exp(-t)$, μ, and σ are bounded on the interval $[0, +\infty)$, and

$$\int_0^{+\infty} \exp(-t)\,\mathrm{d}t = 1 < +\infty. \tag{47}$$

By using Corollary 12, we also have that uncertain delay differential equation (30) is stable almost surely.

Up to now, we have discussed the almost sure stability of linear uncertain delay differential equation (38). In what follows, let us consider another type of uncertain delay differential equation.

Theorem 14. *Supposing that u_t, v_t, and w_t are real-valued functions, then the uncertain delay differential equation*

$$\mathrm{d}X_t = \left(u_t X_{t-\tau} + v_t\right)\mathrm{d}t + w_t \mathrm{d}C_t \tag{48}$$

is almost surely stable if u_t, v_t, and w_t are bounded, and

$$\int_0^{+\infty} \left|u_t\right| \mathrm{d}t < +\infty. \tag{49}$$

Proof. According to Theorem 2, we have that uncertain delay differential equation (48) has a unique solution with given initial states when u_t, v_t, and w_t are bounded. It is supposed that X_t and Y_t are the solutions of uncertain delay differential equation (48) with different initial states $\phi(t)$ and $\psi(t)$ ($t \in [-\tau, 0]$), respectively. That is,

$$\begin{aligned} \mathrm{d}X_t &= \left(u_t X_{t-\tau} + v_t\right)\mathrm{d}t + w_t \mathrm{d}C_t, \quad t \in (0, +\infty) \\ X_t &= \phi(t), \quad t \in [-\tau, 0] \end{aligned} \tag{50}$$

and

$$\begin{aligned} \mathrm{d}Y_t &= \left(u_t Y_{t-\tau} + v_t\right)\mathrm{d}t + w_t \mathrm{d}C_t, \quad t \in (0, +\infty) \\ Y_t &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{51}$$

Then, we have

$$\mathrm{d}\left(X_t - Y_t\right) = u_t\left(X_{t-\tau} - Y_{t-\tau}\right)\mathrm{d}t, \tag{52}$$

and

$$\begin{aligned} \left|X_t - Y_t\right| &= \left|X_0 - Y_0 + \int_0^t u_s\left(X_{s-\tau} - Y_{s-\tau}\right)\mathrm{d}s\right| \\ &= \left|X_0 - Y_0 + \int_{-\tau}^{t-\tau} u_{p+\tau}\left(X_p - Y_p\right)\mathrm{d}p\right| \\ &\le \left|X_0 - Y_0\right| + \left|\int_{-\tau}^{t-\tau} u_{p+\tau}\left(X_p - Y_p\right)\mathrm{d}p\right| \\ &\le \left|X_0 - Y_0\right| + \int_{-\tau}^{t-\tau} \left|u_{p+\tau}\right|\left|X_p - Y_p\right|\mathrm{d}p \\ &\le \left|X_0 - Y_0\right| + \int_{-\tau}^{t} \left|u_{p+\tau}\right|\left|X_p - Y_p\right|\mathrm{d}p. \end{aligned} \tag{53}$$

By using the Gronwall's inequality [33], we have

$$\begin{aligned} \left|X_t(\gamma) - Y_t(\gamma)\right| &\le \left|X_0 - Y_0\right| \exp\left(\int_{-\tau}^{t} \left|u_{p+\tau}\right|\mathrm{d}p\right) \\ &\le \left|X_0 - Y_0\right| \exp\left(\int_{-\tau}^{+\infty} \left|u_{p+\tau}\right|\mathrm{d}p\right) \\ &= \left|X_0 - Y_0\right| \exp\left(\int_{0}^{+\infty} \left|u_s\right|\mathrm{d}s\right) \\ &\le \sup_{j\in[-\tau,0]} \left|X_j - Y_j\right| \exp\left(\int_{0}^{+\infty} \left|u_s\right|\mathrm{d}s\right) \end{aligned} \tag{54}$$

for any $t > 0$ and $\gamma \in \Gamma$.

Hence, uncertain delay differential equation (48) is almost surely stable if

$$\exp\left(\int_0^{+\infty} \left|u_s\right|\mathrm{d}s\right) < +\infty. \tag{55}$$

The above inequality is equivalent to the following inequality

$$\int_0^{+\infty} \left|u_s\right|\mathrm{d}s < +\infty. \tag{56}$$

Therefore,

$$\mathcal{M}\left\{\lim_{\sup_{j\in[-\tau,0]}\left|X_j - Y_j\right| \longrightarrow 0} \left|X_t - Y_t\right| = 0\right\} = 1, \quad \forall t > 0, \tag{57}$$

holds, and uncertain delay differential equation (48) is almost surely stable following from Definition 6. □

Example 15. Consider an uncertain delay differential equation

$$\mathrm{d}X_t = \left(\exp(-t)\, X_{t-\tau} + \mu\right)\mathrm{d}t + \sigma \mathrm{d}C_t. \tag{58}$$

It follows from Theorem 2 that uncertain delay differential equation (58) has a unique solution with given initial states. In addition, the real-valued functions $\exp(-t)$, μ, and σ are bounded on the interval $[0, +\infty)$, and

$$\int_0^{+\infty} \exp(-t)\,\mathrm{d}t = 1 < +\infty. \tag{59}$$

According to Theorem 14, uncertain delay differential equation (58) is stable almost surely.

5. Comparison

The relationship between almost sure stability and stability in measure for uncertain delay differential equation (1) is shown as below.

Theorem 16. *If uncertain delay differential equation (1) is almost surely stable, uncertain delay differential equation (1) is stable in measure.*

Proof. Supposing that X_t and Y_t are two solutions of the uncertain delay differential equation (1) with different initial states x_j and y_j for any $j \in [-\tau, 0]$, respectively, according to Definition 6, we have

$$\begin{aligned} &\mathcal{M}\left\{\gamma \in \Gamma \mid \lim_{\sup_{j\in[-\tau,0]}\left|x_j - y_j\right| \longrightarrow 0} \left|X_t(\gamma) - Y_t(\gamma)\right| = 0\right\} \\ &\quad = 1, \quad \forall t > 0. \end{aligned} \tag{60}$$

That is, there exists a set Γ_0 in Γ with $\mathcal{M}\{\Gamma_0\} = 1$ such that, for any $\gamma \in \Gamma_0$,

$$\lim_{\sup_{j\in[-\tau,0]}\left|x_j - y_j\right| \longrightarrow 0} \left|X_t(\gamma) - Y_t(\gamma)\right| = 0. \tag{61}$$

According to (61), for any $\epsilon > 0$ and $\gamma \in \Gamma_0$, we have $|X_t(\gamma) - Y_t(\gamma)| \le \epsilon$ as long as $\sup_{j\in[-\tau,0]}|x_j - y_j| \longrightarrow 0$. It directly leads to

$$\lim_{\sup_{j\in[-\tau,0]}|x_j-y_j|\longrightarrow 0} \mathscr{M}\{\gamma \in \Gamma \mid |X_t(\gamma) - Y_t(\gamma)| \le \epsilon\} \ge 1. \tag{62}$$

However,

$$\lim_{\sup_{j\in[-\tau,0]}|x_j-y_j|\longrightarrow 0} \mathscr{M}\{\gamma \in \Gamma \mid |X_t(\gamma) - Y_t(\gamma)| \le \epsilon\} \le 1. \tag{63}$$

Hence, for any $\epsilon > 0$, we have

$$\lim_{\sup_{j\in[-\tau,0]}|x_j-y_j|\longrightarrow 0} \mathscr{M}\{|X_t - Y_t| \le \epsilon\} = 1, \quad \forall t > 0. \tag{64}$$

Thus almost sure stability implies the stability in measure. □

6. Conclusion

In this paper, we proposed a concept of almost sure stability of analytical solutions of uncertain delay differential equations. Meanwhile, we provided three sufficient conditions for uncertain delay differential equations being stable almost surely. At last, we analyzed the relationship between almost sure stability and stability in measure and found that almost sure stability could imply the stability in measure for uncertain delay differential equations. In future, we will focus on the application of uncertain delay differential equations in the area of biological systems and finance.

Appendix

A. Uncertain Variable

Definition A.1 (Liu [2, 4]). Let $\mathscr{L}$ be a σ-algebra on a nonempty set Γ. *Uncertain measure* $\mathscr{M}$ is a set function from $\mathscr{L}$ to $[0, 1]$ satisfying the following axioms:

Axiom 1. $\mathscr{M}\{\Gamma\} = 1$.

Axiom 2. $\mathscr{M}\{\Lambda\} + \mathscr{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathscr{L}$.

Axiom 3. For every countable sequence of $\{\Lambda_i\} \in \mathscr{L}$, we have

$$\mathscr{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathscr{M}\{\Lambda_i\}. \tag{A.1}$$

Axiom 4. Let $(\Gamma_k, \mathscr{L}_k, \mathscr{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathscr{M}$ is an uncertain measure satisfying

$$\mathscr{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathscr{M}_k\{\Lambda_k\} \tag{A.2}$$

where $\Lambda_k \in \mathscr{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition A.2 (Liu [2]). An *uncertain variable* ξ is a measurable function from an uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ to the set of real numbers.

B. Uncertain Process

Definition B.1 (Liu [3]). Let $(\Gamma, \mathscr{L}, \mathscr{M})$ be an uncertainty space and T be a totally ordered set. An *uncertain process* X_t is a function from $T \times (\Gamma, \mathscr{L}, \mathscr{M})$ to the set of real numbers such that

$$\{X_t \in B\} = \{\gamma \in \Gamma \mid X_t(\gamma) \in B\} \tag{B.1}$$

is an event for any Borel set B at each t.

Definition B.2 (Liu [4]). An uncertain process C_t $(t \ge 0)$ is called a *Liu process* if

(1) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;

(2) C_t is a stationary independent increment process;

(3) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

This work was funded by National Natural Science Foundation of China (11701338) and a Project of Shandong Province Higher Educational Science and Technology Program (J17KB124).

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