Research Article

Bipartite Consensus of High-Order Edge Dynamics on Coopetition Multiagent Systems

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The bipartite consensus of high-order edge dynamics is investigated for coopetition multiagent systems, in which the cooperative and competitive relationships among agents are characterized by positive weight and negative weight, respectively. By mapping the initial graph to a line graph, the distributed control protocol is proposed for the strongly connected, digon sign-symmetric structurally balanced line graph; and then we give sufficient conditions for the third-order multi-agent system to achieve both the bipartite consensus of edge dynamics and the final value of bipartite consensus. By transforming the coefficients of characteristic polynomial from complex domain to real number domain, the sufficient conditions for the bipartite consensus of high-order edge dynamics are also proposed, and the final values of the high-order edge dynamics on multiagent systems are obtained.

1. Introduction

In recent years, researchers from different fields have done many works for the networked system thanks to the wide applications both to formation control, such as mobile robots, unmanned air vehicles (UAVs), and autonomous underwater vehicles (AUVs), and to distributed control, such as distributed forecasting, monitoring, and diagnostics [1–6]. For multiagent network systems, controllability and consensus problems are the focus of researchers [7–15]. In this paper, we investigated the bipartite consensus problems of multiagent systems.

The consensus problem is of particular concern because most actual systems can be described by the node dynamic multiagent system. Most of the results on consensus were gained by cooperative systems for the case of nonnegative edge weight, see, e.g., [16–22]. In particular, the consensus of second-order multiagent systems with absolute velocity damping under directed topologies was investigated by transforming it to the first-order multiagent system [18]. Some sufficient and necessary conditions were provided for the consensus of multiagent systems with sampler and zero-order holder in [19], as well as for the consensus of hybrid multiagent systems [22, 23]. Recently, the consensus result of

first-order multiagent systems with time delay was improved in [17].

Since the relationship between agents can be cooperative and competitive in reality, the real world can be described more precisely by coopetition systems. The sign graphs are employed to represent the cooperative and competitive relationship between agents. The application of this model is generally inspired by, e.g., the social science [24], electrical power network [25, 26], and biological network [27]. For the coopetition system, the bipartite consensus was proposed, which means that all the agents of a subsystem can reach a common state. Many results on bipartite consensus have also been obtained. For example, the bipartite consensus was considered under connected and structurally balanced topology [28–31]. Sufficient conditions making the second-order multiagent system with time delay asymptotically achieve the bipartite consensus were proposed in [28]. For high-order multiagent systems, Valcher proposed necessary and sufficient conditions for consensus and bipartite consensus under structural balance graphs [29]. In [30], Ma designed a control protocol for high-order power integrator network multiagent systems by backstepping technique. After introducing potential function it was proved that the multiagent system under the protocol could reach bipartite consensus.

For the process of reaching consensus, Qu introduced a new superposition system to the original system and discussed how to get the fastest convergence rate without destroying the connectivity [32]. In [33], Liu found the optimal topology, by which multiagent systems achieved bipartite consensus with lower control cost than other topologies.

In some real systems, e.g., internet, biological network, and electrical power network, the links between agents might be dynamic as the agents. So the consensus problems of edge dynamic are necessary to be studied. By introducing adjacency matrix and defining Laplacian matrix, the consensus problem of the node system was considered by the information of nodes and edges [34]. In [35], the edge dynamics were denoted by the node dynamics on cyclic pursuit graph. Then the conditions of reaching consensus and consensus value were proposed. For directed multiagent systems, Wang transformed initial digraph to its line graph and established both continuous-time and discrete-time protocols. The systems under both of the protocols could asymptotically reach consensus and the consensus values were also given [36]. Meanwhile, for undirected graph, the sufficient conditions referring to the number of edges were gained and the feedback gain matrix was simplified [37]. In [38], Wang proposed distributed edge quasi-consensus protocol to achieve edge quasi-consensus and the sign of states of all edges can be kept for system with nonnegative initial edge states. In [39], consensus of edge dynamics on the strongly connected digraph was investigated after mapping the initial digraph to its line graph. Tian obtained the conditions of reaching bipartite consensus of edge dynamics on cooperation multiagent systems and calculated the final values of edge dynamics on the structurally balanced digraph by using the Jordan canonical form method given in [40].

The results above show that the relationships between agents still need to be further considered for higher-order multiagent systems, which can be cooperative and competitive. Our aim in this paper is to find conditions to make the edge dynamics on cooperation multiagent systems asymptotically reach high-order bipartite consensus. First of all, we use digraph to represent the multiagent system with positive and negative edges indicating cooperative and competitive relationship between the agents, respectively. Then the initial digraph is mapped to its line graph, and distributed consensus protocols are designed for the edge dynamics multiagent system. In this paper, we assume that the digraph is structurally balanced. After introducing gauge transformation, we transform the multiagent system to a new one whose adjacency matrix has only nonnegative elements. This means that the problems of bipartite consensus are transformed to the ordinary consensus problems. For the third-order edge dynamics multiagent system, sufficient conditions are obtained for the new system to asymptotically reach edge consensus, which means that the original multiagent system can asymptotically reach bipartite consensus of edge dynamics. Finally, for general high-order multiagent systems, we give the result of eigenvalues of Laplacian matrix and accordingly obtain the condition for the new system to asymptotically reach consensus of edge dynamics. Thus, the multiagent system can asymptotically reach high-order

bipartite consensus of edges. For the third-order and high-order bipartite consensus of edge dynamics, the final values are discussed regarding distinct feedback gains.

The rest of structure of the paper is as follows. Section 2 explains the preliminary knowledge of graph theory and line graphs. Section 3 gives the main research results. Section 4 gives some simulation results. Section 5 concludes this article.

2. Preliminaries

2.1. Theory of Graph. Let $\mathcal{V} = \{1, 2, \dots, n\}$ and $\mathcal{E} = \{(i, j) \mid \text{if } i \text{ can receive information from } j\}$ be the node set and edge set, respectively. We denote the signed digraph by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the adjacency matrix of $\mathcal{G}$, where $a_{ij} \neq 0$ for $(i, j) \in \mathcal{E}$. In this paper, we only discuss the digraph without self-loop, which means $a_{ii} = 0, \forall i \in 1, 2, \dots, n$. Furthermore, if the connection from j to i is cooperative, $a_{ij} = 1$; if the connection from j to i is competitive, $a_{ij} = -1$. In digraph, we use dotted lines and solid lines to represent competition connection and cooperative connection, respectively. We assume that the signed digraph is digon sign-symmetric, that is, $a_{ij}a_{ji} \geq 0$. We denote the set of neighbors and the in-degree of node i by $\mathcal{N}(i) = \{j \mid a_{ij} \neq 0\}$ and $d_{in}(i) = \sum_{j=1}^n |a_{ij}|$, respectively. $\mathcal{C}$ is the diagonal degree matrix with $c_{ii} = d_{in}(i)$ and $L = \mathcal{C} - \mathcal{A}$ is the Laplacian matrix of $\mathcal{G}$. Meanwhile, the adjacency matrix of undirected graph is symmetric, so $\mathcal{G}(\mathcal{A}_u)$ is considered to be an undirected graph derived from the digraph $\mathcal{G}$ with $\mathcal{A}_u = (\mathcal{A} + \mathcal{A}^T)/2$.

A series of interrelated edges in $\mathcal{E}$,

$$\mathcal{P} = \{(i_2, i_1), (i_3, i_2), \dots, (i_p, i_{p-1})\} \subset \mathcal{E}, \quad (1)$$

is called a directed path of $\mathcal{G}$, where nodes $i_1, i_2, \dots, i_p$ are different, and the length of the directed path $\mathcal{P}$ is $p - 1$. If there are at least one path from i to j and another one from j to $i, \forall i, j \in \mathcal{V}, i \neq j$, the digraph is strongly connected. A path becomes a cycle if the end point of $\mathcal{P}$ coincides with the starting point (i.e., $i_1 = i_p$). This means that a node in strongly connected digraph is in at least one cycle. If a cycle contains even negative edges, i.e., $a_{i_2 i_1} \cdots a_{i_p i_{p-1}} > 0$, it is positive. If $a_{i_2 i_1} \cdots a_{i_p i_{p-1}} < 0$, the cycle is negative.

2.2. Structural Balance. We define $\mathcal{V}_1, \mathcal{V}_2$ as two sets of nodes. A digraph is structurally balanced if all of its nodes can be divided into $\mathcal{V}_1, \mathcal{V}_2$, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $a_{ij} \geq 0, \forall i, j \in \mathcal{V}_p, (p \in \{1, 2\}), a_{ij} \leq 0, \forall i \in \mathcal{V}_p, j \in \mathcal{V}_q, (p, q \in \{1, 2\}, p \neq q)$. Otherwise, the digraph is structurally unbalanced. According to the property of structurally balanced digraph, we introduce a set of gauge transformations $\mathcal{D} = \{D \mid D = \text{diag}\{d_1, \dots, d_n\}, d_i = \pm 1, i = 1, 2, \dots, n\}$ and then have the following lemma.

Lemma 1 (see [41]). *A strongly connected, digon sign-symmetric digraph $\mathcal{G}(\mathcal{A})$ is structurally balanced if and only if any of the following equivalent conditions hold:*

- (1) $\mathcal{G}(\mathcal{A}_u)$ is structurally balanced;
- (2) all directed cycles of $\mathcal{G}(\mathcal{A})$ are positive;

FIGURE 1: Structurally balanced/unbalanced digraph (a)/(c) and their line graph (b)/(d).

- (3) $\exists D \in \mathcal{D}$ such that $D\mathcal{A}D$ has all nonnegative entries;
- (4) 0 is an eigenvalue of L .

From Lemma 1, a strongly connected, digon sign-symmetric digraph $\mathcal{G}$ is structurally balanced if and only if Laplacian matrix L has a simple eigenvalue 0 (i.e., $\text{rank}(L) = n - 1$).

Lemma 2 (see [42]). Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be a weighted digraph with Laplacian L . Denote the maximum node in-degree of the digraph $\mathcal{G}$ by $d_{\max}(\mathcal{G}) = \max_i d_{\text{in}}(i)$. Then, all the eigenvalues of L are located in the following disk:

$$D(\mathcal{G}) = \{z \in \mathbb{C} : |z - d_{\max}(\mathcal{G})| \leq d_{\max}(\mathcal{G})\}, \quad (2)$$

which is centered at $z = d_{\max}(\mathcal{G}) + 0j$ in the complex plane. In other words, except for the single eigenvalue 0, all of the other eigenvalues of L have positive real parts.

Lemma 3 (see [42]). Assume $\mathcal{G}$ is a strongly connected digraph with Laplacian matrix L satisfying $Lw_r = 0$, $w_l^T L = 0$, and $w_l^T w_r = 1$. Then, $\lim_{t \rightarrow \infty} e^{-Lt} = w_r w_l^T$.

2.3. Line Graph. Before mapping the initial digraph to its line graph, the out-neighbors of edge (i, j) are defined as $(k, i) \in \mathcal{E}$, $k \in \mathcal{V}$. In the process of transforming the initial digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ with n nodes and m edges to its line graph $\mathcal{L}(\mathcal{G})$, we follow the following rules [43]:

- (1) convert a directed edge (i, j) of $\mathcal{G}$ to a node (i, j) of $\mathcal{L}(\mathcal{G})$;
- (2) generate edges from node (i, j) to its out-neighbors $(k, i) \in \mathcal{E}$, $k \in \mathcal{V}$;
- (3) if edge (i, j) in initial digraph is positive, the generated edges are positive;
- (4) if edge (i, j) in initial digraph is negative, the generated edges are negative.

Similar to initial digraph, we denote the Laplacian matrix of line graph by $L' = \mathcal{E}' - \mathcal{A}'$, where $\mathcal{A}' = [a'_{ij}]^{m \times m}$ and $\mathcal{E}' = \text{diag}([d_{\text{in}}(i)]_{i=1}^m)$ are the adjacency matrix and in-degree matrix of line graph, respectively. The elements of L' are

$$l'_{st} = \begin{cases} -a_{st}, & s \neq t, \\ \sum_{t \in \mathcal{N}(s)} |a'_{st}|, & s = t. \end{cases} \quad (3)$$

Lemma 4 (see [43]). If a digraph $\mathcal{G}$ contains more than one node and is strongly connected, then its line graph $\mathcal{L}(\mathcal{G})$ is also strongly connected.

Lemma 5 (see [30]). For a strongly connected, digon sign-symmetric digraph $\mathcal{G}$, its line graph $\mathcal{L}(\mathcal{G})$ is structurally balanced if and only if $\mathcal{G}$ is structurally balanced.

In Figure 1, (a) is a strongly connected and digon sign-symmetric digraph. From the rules, we map the digraph to its line graph. For example, the edges in (a) are converted into nodes in (b). There is an edge from nodes (1,2) to (6,1) and the edge is positive because the edge (1,2) in (a) is positive. Following this procedure for all nodes, Figure 1(b) is gained. From Lemma 4, the line graph (b) is also strongly connected and digon sign-symmetric. Nodes in (a) can be divided into $\mathcal{V}_1 = \{1, 2, 3\}$, $\mathcal{V}_2 = \{4, 5, 6\}$ and all the negative edges are between different sets. The nodes in (b) (i.e., the edges in (a)) can be divided into $\mathcal{E}_1 = \{(1, 2), (2, 3), (4, 2), (5, 2), (6, 1)\}$ and $\mathcal{E}_2 = \{(2, 6), (3, 4), (4, 5), (5, 6)\}$. Thus (a) is structurally balanced. The adjacency matrix $\mathcal{A}$ of (a) and the Laplacian matrix L' of (b) are

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (4)$$

$$L' = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Changing the interaction between nodes in (a), we have the structurally unbalanced digraph (c). Following the rules,

we gain the line graph (d) in Figure 1. There is at least one negative cycle, so the line graph (d) is structurally unbalanced. From Lemma 5, we get the same conclusion.

By Lemma 1, there are eigenvalue 0 of L' and matrix $D = \text{diag}(1, 1, -1, -1, 1, 1, -1)$ such that $\bar{L}' = DL'D$ has positive diagonal elements and nonpositive nondiagonal elements. From Lemma 2, the other eigenvalues of L' except the simple eigenvalue 0 are all located in the right half complex plane. Moreover, due to the structure of D , $D = D^{-1}$.

3. Main Results

3.1. Bipartite Consensus of Third-Order Edge Dynamics. Let $\xi_{ij}, \theta_{ij}, \psi_{ij}$ represent, respectively, the position, velocity, and acceleration of edge (i, j) at time t . The third-order continuous-time edge dynamics model is

$$\begin{aligned}\dot{\xi}_{ij} &= \theta_{ij}, \\ \dot{\theta}_{ij} &= \psi_{ij}, \\ \dot{\psi}_{ij} &= u_{ij}.\end{aligned}\quad (5)$$

For system (5), we design the following control protocol

$$\begin{aligned}u_{ij} &= -k_1\theta_{ij} - k_2\psi_{ij} \\ &+ \sum_{r \in N(j)} |\mathcal{A}_{jr}| \left(\text{sgn}(\mathcal{A}_{jr}) \xi_{jr}(t) - \xi_{ij}(t) \right) \\ &+ \sum_{r \in N(j)} |\mathcal{A}_{jr}| \left[\beta_1 \left(\text{sgn}(\mathcal{A}_{jr}) \theta_{jr}(t) - \theta_{ij}(t) \right) \right] \\ &+ \sum_{r \in N(j)} |\mathcal{A}_{jr}| \left[\beta_2 \left(\text{sgn}(\mathcal{A}_{jr}) \psi_{jr}(t) - \psi_{ij}(t) \right) \right],\end{aligned}\quad (6)$$

$\forall (i, j) \in \mathcal{E},$

where $k_1, k_2, \beta_1, \beta_2 \geq 0$ are feedback gains.

Lemma 6 (see [44]). *Given a third-order complex coefficient polynomial, in the form of*

$$p_3(s) = s^3 + (\zeta_2 + j\eta_2)s^2 + (\zeta_1 + j\eta_1)s + \zeta_0 + j\eta_0, \quad (7)$$

then $p_3(s)$ is stable if and only if $\zeta_2 > 0$, $\zeta_2\eta_2\eta_1 + \zeta_2^2\zeta_1 - \eta_1^2 - \zeta_2\zeta_0 > 0$ and

$$\begin{aligned}\zeta_2 \cdot \det \begin{pmatrix} \eta_2 & -\zeta_1 & -\eta_0 & 0 \\ \zeta_2 & \eta_1 & -\zeta_0 & 0 \\ 1 & \eta_2 & -\zeta_1 & -\eta_0 \\ 0 & \zeta_2 & \eta_1 & -\zeta_0 \end{pmatrix} \\ - \det \begin{pmatrix} \eta_1 & -\zeta_0 & 0 & 0 \\ \zeta_2 & \eta_1 & -\zeta_0 & 0 \\ 1 & \eta_2 & -\zeta_1 & -\eta_0 \\ 0 & \zeta_2 & \eta_1 & -\zeta_0 \end{pmatrix} > 0.\end{aligned}\quad (8)$$

Let $\xi = (\xi_{ij}), \theta = (\theta_{ij}), \psi = (\psi_{ij}) \in R^{m \times 1}$, $i = 1, 2, \dots, n$; $j \in \mathcal{N}_i$; $m = \sum_{i=1}^n d_{in}(i)$. Then system (5) can be simplified as $\begin{bmatrix} \dot{\xi}^T & \dot{\theta}^T & \dot{\psi}^T \end{bmatrix}^T = M \begin{bmatrix} \xi^T & \theta^T & \psi^T \end{bmatrix}^T$, where

$$M = \begin{bmatrix} 0 & I_m & 0 \\ 0 & 0 & I_m \\ -L' & -k_1 I_m - \beta_1 L' & -k_2 I_m - \beta_2 L' \end{bmatrix}, \quad (9)$$

and matrix L' is defined as above. Let $\bar{\xi} = D\xi, \bar{\theta} = D\theta, \bar{\psi} = D\psi$, $D \in \mathcal{D}$, we have

$$\begin{bmatrix} \dot{\bar{\xi}}^T & \dot{\bar{\theta}}^T & \dot{\bar{\psi}}^T \end{bmatrix}^T = \bar{M} \begin{bmatrix} \bar{\xi}^T & \bar{\theta}^T & \bar{\psi}^T \end{bmatrix}^T \quad (10)$$

where

$$\bar{M} = \begin{bmatrix} 0 & I_m & 0 \\ 0 & 0 & I_m \\ -\bar{L}' & -k_1 I_m - \beta_1 \bar{L}' & -k_2 I_m - \beta_2 \bar{L}' \end{bmatrix}, \quad (11)$$

with $\bar{L}' = DL'D$. Notice that

$$\begin{aligned}\det [\lambda I_{3m} - \bar{M}] \\ = \det \left(\begin{bmatrix} \lambda I_m & -I_m & 0 \\ 0 & \lambda I_m & -I_m \\ \bar{L}' & k_1 I_m + \beta_1 \bar{L}' & (\lambda + k_2) I_m + \beta_2 \bar{L}' \end{bmatrix} \right) \\ = \det \left[(\lambda^3 + k_2 \lambda^2 + k_1 \lambda) I_m + (\beta_2 \lambda^2 + \beta_1 \lambda + 1) \bar{L}' \right] \\ = \prod_{i=1}^m [\lambda^3 + (k_2 - \beta_2 \mu_i) \lambda^2 + (k_1 - \beta_1 \mu_i) \lambda - \mu_i]\end{aligned}\quad (12)$$

where μ_i , $i = 1, 2, \dots, m$, are, respectively, the eigenvalues of $-\bar{L}'$. There is an eigenvalue $\mu_i = 0$ of $-\bar{L}'$. Thus matrix $\bar{M}$ has at least one 0 eigenvalue.

Assumption. For $l \in [1, h-1]$, where h is the order of system, if $k_l \neq 0$, then $k_i \neq 0$, $i \geq l$.

Theorem 7. *For a continuous-time edge dynamic multiagent system (5), let the corresponding digraph $\mathcal{G}$ be strongly connected, digon sign-symmetric, and structurally balanced. When the assumption holds, system (5) can asymptotically reach the bipartite consensus of edge dynamics under protocol (6) for $\mu_i \neq 0$, if the following inequalities are satisfied*

$$k_2 - \beta_2 \text{Re}(\mu_i) > 0, \quad (13)$$

$$\begin{aligned}(\beta_1 \beta_2 (\text{Im}(\mu_i))^2 + \text{Re}(\mu_i) \\ + (k_2 - \beta_2 \text{Re}(\mu_i))(k_1 - \beta_1 \text{Re}(\mu_i))) \cdot (k_2\end{aligned}\quad (14)$$

$$- \beta_2 \text{Re}(\mu_i) - \beta_1^2 (\text{Im}(\mu_i))^2 > 0,$$

and

$$\begin{aligned}
& (k_2 - \beta_2 \operatorname{Re}(\mu_i)) \det \begin{pmatrix} -\beta_2 \operatorname{Im}(\mu_i) & -(k_1 - \beta_1 \operatorname{Re}(\mu_i)) & \operatorname{Im}(\mu_i) & 0 \\ (k_2 - \beta_2 \operatorname{Re}(\mu_i)) & -\beta_1 \operatorname{Im}(\mu_i) & \operatorname{Re}(\mu_i) & 0 \\ 1 & -\beta_2 \operatorname{Im}(\mu_i) & -(k_1 - \beta_1 \operatorname{Re}(\mu_i)) & \operatorname{Im}(\mu_i) \\ 0 & k_2 - \beta_2 \operatorname{Re}(\mu_i) & -\beta_1 \operatorname{Im}(\mu_i) & \operatorname{Re}(\mu_i) \end{pmatrix} \\
& - \det \begin{pmatrix} -\beta_1 \operatorname{Im}(\mu_i) & \operatorname{Re}(\mu_i) & 0 & 0 \\ (k_2 - \beta_2 \operatorname{Re}(\mu_i)) & -\beta_1 \operatorname{Im}(\mu_i) & \operatorname{Re}(\mu_i) & 0 \\ 1 & -\beta_2 \operatorname{Im}(\mu_i) & -(k_1 - \beta_1 \operatorname{Re}(\mu_i)) & \operatorname{Im}(\mu_i) \\ 0 & k_2 - \beta_2 \operatorname{Re}(\mu_i) & -\beta_1 \operatorname{Im}(\mu_i) & \operatorname{Re}(\mu_i) \end{pmatrix} > 0.
\end{aligned} \tag{15}$$

Moreover, if $k_1 = k_2 = 0$, the final values of bipartite consensus are

$$\begin{aligned}
\lim_{t \rightarrow \infty} \xi(t) &= w_l^T D \xi(0) D w_r + t w_l^T D \theta(0) D w_r \\
&+ \frac{1}{2} t^2 w_l^T D \psi(0) D w_r, \\
\lim_{t \rightarrow \infty} \theta(t) &= w_l^T D \theta(0) D w_r + t w_l^T D \psi(0) D w_r, \\
\lim_{t \rightarrow \infty} \psi(t) &= w_l^T D \psi(0) D w_r.
\end{aligned} \tag{16}$$

where w_l and w_r are the left and right eigenvectors associated with the single eigenvalue 0 of the Laplacian matrix $\bar{L}'$, respectively, and $w_l^T w_r = 1$. If $k_1 = 0, k_2 \neq 0$, the final values of bipartite consensus are

$$\begin{aligned}
\lim_{t \rightarrow \infty} \xi(t) &= w_l^T D \xi(0) D w_r \\
&+ \left(\frac{1}{k_2} + t \right) w_l^T D \theta(0) D w_r \\
&+ \frac{t}{k_2} w_l^T D \psi(0) D w_r,
\end{aligned} \tag{17}$$

$$\lim_{t \rightarrow \infty} \theta(t) = w_l^T D \theta(0) D w_r + \frac{1}{k_2} w_l^T D \psi(0) D w_r$$

$$\lim_{t \rightarrow \infty} \psi(t) = 0.$$

If $k_1 \neq 0$, the consensus values are as follows, respectively,

$$\begin{aligned}
\lim_{t \rightarrow \infty} \xi(t) &= w_l^T D \xi(0) D w_r + \frac{k_2}{k_1} w_l^T D \theta(0) D w_r \\
&+ \frac{1}{k_1} w_l^T D \psi(0) D w_r, \\
\lim_{t \rightarrow \infty} \theta(t) &= 0, \\
\lim_{t \rightarrow \infty} \psi(t) &= 0.
\end{aligned} \tag{18}$$

Proof. For $\mu_i = 0$, when the assumption holds, the roots of polynomial $\lambda^2 + k_2 \lambda + k_1$ are 0 or complex numbers with

negative real parts. For $\mu_i \neq 0$, when conditions (13), (14), and (15) are satisfied, the roots of polynomial all have negative real parts, that is, all eigenvalues except 0 of $\bar{M}$ have negative real parts. Therefore, we have the following cases.

- (i) If $k_1 = k_2 = 0$, we see from Lemma 6 that matrix $\bar{M}$ has exactly three zero eigenvalues and all the other eigenvalues have negative real parts. From the stability theorem and the results in [29], system (10) can asymptotically reach the consensus of edge dynamics, and $\lim_{t \rightarrow \infty} \bar{\xi}(t) = w_r w_l^T \bar{\xi}(0) + t w_r w_l^T \bar{\theta}(0) + (1/2) t^2 w_r w_l^T \bar{\psi}(0)$, $\lim_{t \rightarrow \infty} \bar{\theta}(t) = w_r w_l^T \bar{\theta}(0) + t w_r w_l^T \bar{\psi}(0)$, and $\lim_{t \rightarrow \infty} \bar{\psi}(t) = w_r w_l^T \bar{\psi}(0)$, where w_r and w_l are the right and left eigenvectors associated with $\mu_i = 0$ of Laplacian matrix $\bar{L}' = D L' D$, respectively, and $w_l^T w_r = 1$. Thus, from $D = D^{-1}$, system (5) can asymptotically reach the bipartite consensus of edge dynamics under protocol (6), and

$$\begin{aligned}
\lim_{t \rightarrow \infty} \xi(t) &= D \left(w_r w_l^T D \xi(0) + t w_r w_l^T D \theta(0) \right. \\
&+ \left. \frac{1}{2} t^2 w_r w_l^T D \psi(0) \right) = w_l^T D \xi(0) D w_r \\
&+ t w_l^T D \theta(0) D w_r + \frac{1}{2} t^2 w_l^T D \psi(0) D w_r,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\lim_{t \rightarrow \infty} \theta(t) &= D \left(w_r w_l^T D \theta(0) + t w_r w_l^T D \psi(0) \right) \\
&= w_l^T D \theta(0) D w_r + t w_l^T D \psi(0) D w_r,
\end{aligned}$$

$$\lim_{t \rightarrow \infty} \psi(t) = D w_r w_l^T D \psi(0) = w_l^T D \psi(0) D w_r.$$

- (ii) If $k_1 = 0, k_2 \neq 0$, similar to the analysis above, matrix $\bar{M}$ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. Then (10) can asymptotically reach the consensus of edge dynamics, and the final values of consensus are $\lim_{t \rightarrow \infty} \bar{\xi}(t) = w_r w_l^T \bar{\xi}(0) + (1/k_2 + t) w_r w_l^T \bar{\theta}(0) + (t/k_2) w_r w_l^T \bar{\psi}(0)$, $\lim_{t \rightarrow \infty} \bar{\theta}(t) = w_r w_l^T \bar{\theta}(0) + (1/k_2) w_r w_l^T \bar{\psi}(0)$, $\lim_{t \rightarrow \infty} \bar{\psi}(t) = 0$, where w_r and w_l are defined as

above. Thus, (5) can asymptotically reach the bipartite consensus of edge dynamics under protocol (6), and

$$\begin{aligned} \lim_{t \rightarrow \infty} \xi(t) &= D \left(w_r w_l^T D \xi(0) + \left(\frac{1}{k_2} + t \right) w_r w_l^T D \theta(0) \right. \\ &\quad \left. + \frac{t}{k_2} w_r w_l^T D \psi(0) \right) = w_l^T D \xi(0) D w_r + \left(\frac{1}{k_2} + t \right) \\ &\quad \cdot w_l^T D \theta(0) D w_r + \frac{t}{k_2} w_l^T D \psi(0) D w_r, \end{aligned} \quad (20)$$

$$\begin{aligned} \lim_{t \rightarrow \infty} \theta(t) &= D \left(w_r w_l^T D \theta(0) + \frac{1}{k_2} w_r w_l^T D \psi(0) \right) \\ &= w_l^T D \theta(0) D w_r + \frac{1}{k_2} w_l^T D \psi(0) D w_r, \end{aligned}$$

$$\lim_{t \rightarrow \infty} \psi(t) = 0$$

(iii) If $k_1 \neq 0$, similar to the analysis in part one, matrix $\bar{M}$ has exactly one zero eigenvalue and all the other eigenvalues have negative real parts. Then (10) can asymptotically reach the consensus of edge dynamics, and the final values of consensus are $\lim_{t \rightarrow \infty} \bar{\xi}(t) = w_r w_l^T \bar{\xi}(0) + (k_2/k_1) w_r w_l^T \bar{\theta}(0) + (1/k_1) w_r w_l^T \bar{\psi}(0)$, $\lim_{t \rightarrow \infty} \bar{\theta}(t) = 0$, $\lim_{t \rightarrow \infty} \bar{\psi}(t) = 0$, where w_r and w_l^T are defined as above. Therefore (5) can asymptotically reach the bipartite consensus of edge dynamics under protocol (6), and

$$\begin{aligned} \lim_{t \rightarrow \infty} \xi(t) &= D \left(w_r w_l^T D \xi(0) + \frac{k_2}{k_1} w_r w_l^T D \theta(0) \right. \\ &\quad \left. + \frac{1}{k_1} w_r w_l^T D \psi(0) \right) = w_l^T D \xi(0) D w_r + \frac{k_2}{k_1} \\ &\quad \cdot w_l^T D \theta(0) D w_r + \frac{1}{k_1} w_l^T D \psi(0) D w_r, \end{aligned} \quad (21)$$

$$\lim_{t \rightarrow \infty} \theta(t) = 0,$$

$$\lim_{t \rightarrow \infty} \psi(t) = 0.$$

(iv) If signed digraph $\mathcal{G}$ is structurally unbalanced, its line graph $\mathcal{L}(\mathcal{G})$ is also structurally unbalanced, which means that all the eigenvalues of Laplacian matrix $-L'$ have negative real parts. Therefore, all the eigenvalues of matrix M have negative real parts under conditions (13), (14), and (15). Thus, system (5) is asymptotically stable under protocol (6). Accordingly, we have $\lim_{t \rightarrow \infty} \xi(t) = 0$, $\lim_{t \rightarrow \infty} \theta(t) = 0$, $\lim_{t \rightarrow \infty} \psi(t) = 0$.

□

3.2. Bipartite Consensus of High-Order Edge Dynamics. The h order continuous-time edge dynamics model is

$$\dot{\xi}_{ij}^{(0)} = \xi_{ij}^{(1)},$$

$$\dot{\xi}_{ij}^{(1)} = \xi_{ij}^{(2)},$$

$$\vdots,$$

$$\dot{\xi}_{ij}^{(h-1)} = u_{ij},$$

(22)

where $\xi_{ij}^{(l)}$, $l \in [0, h-1]$ are the states of edges at time t . For system 3.2, we design the high-order control protocol

$$\begin{aligned} u_{ij} &= - \sum_{l=1}^{h-1} k_l \xi_{ij}^{(l)} \\ &\quad + \sum_{r \in \mathcal{N}(j)} |\mathcal{A}_{jr}| \left(\text{sgn}(\mathcal{A}_{jr}) \xi_{jr}^{(0)}(t) - \xi_{ij}^{(0)}(t) \right) \\ &\quad + \sum_{l=1}^{h-1} \sum_{r \in \mathcal{N}(j)} [\beta_l \left(\text{sgn}(\mathcal{A}_{jr}) \xi_{jr}^{(l)}(t) - \xi_{ij}^{(l)}(t) \right)] \end{aligned} \quad (23)$$

where $k_1, k_2, \dots, k_{h-1} \geq 0$ and $\beta_1, \beta_2, \dots, \beta_{h-1} > 0$ are feedback gains. Denote $\xi^{(l)} = (\xi_{ij}^{(l)}(t)) \in R^{m \times 1}$, $j \in \mathcal{N}(i)$, $m = \sum_{i=1}^n d_{in}(i)$, and $\omega = [\xi^{(0)T} \ \xi^{(1)T} \ \dots \ \xi^{(h-1)T}]^T$. Then we rewrite 3.2 with (23) as

$$\dot{\omega} = M \omega, \quad (24)$$

where

$$M = \begin{bmatrix} 0 & I_m & 0 & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_m \\ -L' & -k_1 I_m - \beta_1 L' & -k_2 I_m - \beta_2 L' & \dots & -k_{h-1} I_m - \beta_{h-1} L' \end{bmatrix}, \quad (25)$$

and matrix L' is the Laplacian matrix of $\mathcal{L}(\mathcal{G})$. Let $\bar{\xi}^{(l)} = D \xi^{(l)}$, $l \in [0, h-1]$, $D \in D$, and the state vector $\bar{\omega} = [\bar{\xi}^{(0)T} \ \bar{\xi}^{(1)T} \ \dots \ \bar{\xi}^{(h-1)T}]^T$. We have

$$\dot{\bar{\omega}} = \bar{M} \bar{\omega}, \quad (26)$$

where

$$\bar{M} = \begin{bmatrix} 0 & I_m & 0 & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_m \\ -\bar{L}' & -k_1 \bar{L}' - \beta_1 \bar{L}' & -k_2 \bar{L}' - \beta_2 \bar{L}' & \dots & -k_{h-1} \bar{L}' - \beta_{h-1} \bar{L}' \end{bmatrix} \quad (27)$$

with $\bar{L}' = D L' D$.

Notice that

$$\begin{aligned}
\det(\lambda I_{hm} - \overline{M}) &= \det \left(\begin{bmatrix} \lambda I_m & -I_m & 0 & \cdots & 0 \\ 0 & \lambda I_m & -I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -I_m \\ \overline{L}' & k_1 I_m + \beta_1 \overline{L}' & k_2 I_m + \beta_2 \overline{L}' & \cdots & (\lambda + k_{h-1}) I_m + \beta_{h-1} \overline{L}' \end{bmatrix} \right) \\
&= \det \left[(\lambda^h + k_{h-1} \lambda^{h-1} + \cdots + k_1 \lambda) I_m + (\beta_{h-1} \lambda^{h-1} + \cdots + \beta_1 \lambda + 1) \overline{L}' \right] \\
&= \prod_{i=1}^m \left[\lambda^h + (k_{h-1} - \beta_{h-1} \mu_i) \lambda^{h-1} + \cdots + (k_2 - \beta_2 \mu_i) \lambda^2 + (k_1 - \beta_1 \mu_i) \lambda - \mu_i \right]
\end{aligned} \tag{28}$$

where $\mu_i, i = 1, 2, \dots, m$ are eigenvalues of $-L'$ and there is a simple eigenvalue $\mu_i = 0$. Then matrix $\overline{M}$ has at least one 0 eigenvalue.

By Lemma 2, all the eigenvalues of $-L'$ except the simple eigenvalue 0 have negative real parts, which shows that the eigenvalues of $-L'$ are 0, negative real numbers, and conjugate complex numbers with negative real parts. Therefore, we reorder the eigenvalues of $-L'$ to

$$0, \mu_2, \dots, \mu_l, \mu_{l+1}, \overline{\mu}_{l+1}, \dots, \mu_{l+k}, \overline{\mu}_{l+k}, \tag{29}$$

$l, k \geq 0, l + 2k = m.$

Theorem 8. Given a high-order complex coefficient polynomial

$$\begin{aligned}
f(x) &= (a_n + b_n i) x^n + (a_{n-1} + b_{n-1} i) x^{n-1} + \cdots \\
&\quad + (a_1 + b_1 i) x + a_0 + b_0 i,
\end{aligned} \tag{30}$$

and another complex coefficient polynomial $g(x)$ with the same order and conjugate coefficients as the above polynomial $f(x)$.

$$\begin{aligned}
g(x) &= (a_n - b_n i) x^n + (a_{n-1} - b_{n-1} i) x^{n-1} + \cdots \\
&\quad + (a_1 - b_1 i) x + a_0 - b_0 i.
\end{aligned} \tag{31}$$

Their product is

$$\begin{aligned}
h(x) &= f(x) \times g(x) \\
&= \sum_{l=1}^n \left(\sum_{\substack{i+j=2l-1 \\ i \neq j}} 2(a_i a_j + b_i b_j) \right) x^{2l-1} \\
&\quad + \sum_{l=0}^n \left(\sum_{\substack{i+j=2l \\ i \neq j}} 2(a_i a_j + b_i b_j) + a_l^2 + b_l^2 \right) x^{2l}
\end{aligned} \tag{32}$$

Proof. The result can be proved by the multiplication rule of complex coefficient polynomial and the fact that every coefficient of $h(x)$ is greater than 0. $\square$

Denote $\lambda^h + (k_{h-1} - \beta_{h-1} \mu_i) \lambda^{h-1} + \cdots + (k_2 - \beta_2 \mu_i) \lambda^2 + (k_1 - \beta_1 \mu_i) \lambda - \mu_i$ by $P_h(\mu_i)$, and denote $P_h(\mu_i) \cdot P_{h'}(\mu_i)$ by $P_{2h}(\mu_i)$. Then from Theorem 8, (26) can be written as

$$\det(\lambda I_{hm} - \Gamma) = \prod_{i=1}^l P_h(\mu_i) \cdot \prod_{i=l+1}^{l+k} P_{2h}(\mu_i). \tag{33}$$

Lemma 9 (see [45]). Given a system with characteristic polynomial

$$\begin{aligned}
D(s) &= a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0, \\
a_i &> 0, \quad i \in [0, n],
\end{aligned} \tag{34}$$

then the system is stable if and only if the determinant consists of coefficients of characteristic polynomials

$$\Delta_n = \begin{vmatrix} a_{n-1} & a_{n-3} & \cdots & 0 & 0 & 0 \\ a_n & a_{n-2} & \cdots & 0 & 0 & 0 \\ 0 & a_{n-1} & \cdots & 0 & 0 & 0 \\ 0 & a_n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_3 & a_1 & 0 \\ 0 & 0 & \cdots & a_4 & a_2 & a_0 \end{vmatrix} \tag{35}$$

and its order principal minor determinant Δ_i ($i = 1, 2, \dots, n-1$) is greater than 0, namely,

$$\begin{aligned}
\Delta_1 &= a_{n-1} > 0, \\
\Delta_2 &= \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} > 0, \\
&\vdots, \\
\Delta_n &> 0.
\end{aligned} \tag{36}$$

Theorem 10. For a continuous-time edge dynamics multiagent system 3.2, let the corresponding signed digraph $\mathcal{G}$ be strongly connected, digon sign-symmetric and structurally balanced. When the Assumption holds, system 3.2 can asymptotically reach the bipartite consensus of edge dynamics under protocol (23), if the determinants and their order principal minor determinants of $P_h(\mu_i)$ and $P_{2h}(\mu_i)$ in (33) are all greater than 0 for $\mu_i \neq 0$, and $\Delta_j > 0$, $j \in [1, h-l]$ of $P_h(0)$ with $k_l = 0$, $k_{l+1} \neq 0$ for $\mu_i = 0$.

Proof. From Lemma 9 and the results in [40], if the determinants and their order principal minor determinants of $P_h(\mu_i)$ and $P_{2h}(\mu_i)$ in (33) are all greater than 0 for $\mu_i \neq 0$, all roots of the corresponding subpolynomial (28) have negative real parts. If $\Delta_j > 0$, $j \in [1, h-l]$ with $k_l = 0$, $k_{l+1} \neq 0$ for $\mu_i = 0$, the roots except 0 of the corresponding subpolynomial (28) have negative real parts. Thus system (26) can reach the consensus of edges. We have the following cases.

- (i) If $k_i = 0$, $i \in [1, h-1]$, matrix $\bar{M}$ has exactly h zero eigenvalues and all the other eigenvalues have negative real parts. By the results of [29], system (26) can asymptotically reach the consensus of edge dynamics, and $\lim_{t \rightarrow \infty} \bar{\xi}^{(i)} = \sum_{k=i}^{h-1} (t^{k-i}/(k-i)!) w_r w_l^T \bar{\xi}^{(k)}(0)$, $i \in [0, h-1]$, where w_r and w_l^T are the same as above. By the property of D , system 3.2 can asymptotically reach the bipartite consensus of edge dynamics under protocol (23), and the final values of bipartite consensus are

$$\begin{aligned} \lim_{t \rightarrow \infty} \xi^{(i)} &= \sum_{k=i}^{h-1} \frac{t^{k-i}}{(k-i)!} D w_r w_l^T D \xi^{(k)}(0) \\ &= \sum_{k=i}^{h-1} \frac{t^{k-i}}{(k-i)!} w_l^T D \xi^{(k)}(0) D w_r, \end{aligned} \quad (37)$$

$i \in [0, h-1].$

- (ii) If $k_i = 0$, $k_{i+1} \neq 0$, $i \in [1, h-2]$, matrix $\bar{M}$ has exactly $i+1$ zero eigenvalues and all the other eigenvalues have negative real parts. Therefore, system (26) can asymptotically reach the consensus, and $\lim_{t \rightarrow \infty} \bar{\xi}^{(i)} = w_r w_l^T \bar{\xi}^{(i)}(0) + \sum_{j=i+1}^{h-2} (k_{l+1}/k_{i+1}) w_r w_l^T \bar{\xi}^{(j)}(0) + (1/k_{i+1}) w_r w_l^T \bar{\xi}^{(h-1)}(0)$, $\lim_{t \rightarrow \infty} \bar{\xi}^{(l)} = 0$, $l > i$. Similar to the analysis in the first part, system 3.2 can asymptotically reach the bipartite consensus of edges dynamics under protocol (23), and

$$\begin{aligned} \lim_{t \rightarrow \infty} \xi^{(i)} &= w_l^T D \xi^{(i)}(0) D w_r \\ &+ \sum_{l=i+1}^{h-2} \frac{k_{l+1}}{k_{i+1}} w_l^T D \xi^{(l)}(0) D w_r \end{aligned}$$

$$+ \frac{1}{k_{i+1}} w_l^T D \xi^{(h-1)}(0) D w_r,$$

$$\lim_{t \rightarrow \infty} \xi^{(l)} = 0, \quad l > i \quad (38)$$

Furthermore, the limit of $\xi^{(l)}$ at $t \rightarrow \infty$ is $i-l$ order function of t when $l < i$.

- (iii) If $k_1 \neq 0$, matrix $\bar{M}$ only has exactly one zero eigenvalue and all the other eigenvalues have negative real parts. Thus, system (26) can asymptotically reach the consensus of edge dynamics, and $\lim_{t \rightarrow \infty} \bar{\xi}^{(0)} = w_r w_l^T \bar{\xi}^{(0)}(0) + \sum_{i=1}^{h-2} (k_{i+1}/k_1) w_r w_l^T \bar{\xi}^{(i)}(0) + (1/k_1) w_r w_l^T \bar{\xi}^{(h-1)}(0)$, $\lim_{t \rightarrow \infty} \bar{\xi}^{(i)} = 0$, $i \in [1, h-1]$. System 3.2 can asymptotically reach the bipartite consensus of edge dynamics under protocol (23), and the final values of bipartite consensus are

$$\begin{aligned} \lim_{t \rightarrow \infty} \xi^{(0)} &= w_l^T D \xi^{(0)}(0) D w_r \\ &+ \sum_{i=1}^{h-2} \frac{k_{i+1}}{k_1} w_l^T D \xi^{(i)}(0) D w_r \\ &+ \frac{1}{k_1} w_l^T D \xi^{(h-1)}(0) D w_r, \end{aligned} \quad (39)$$

$$\lim_{t \rightarrow \infty} \xi^{(i)} = 0,$$

$$i \in [1, h-1].$$

- (iv) If $\mathcal{L}(\mathcal{G})$ is structurally unbalanced, all eigenvalues of Laplacian matrix $-L'$ have negative real parts. All eigenvalues of matrix M have negative real parts, which means 3.2 can be asymptotically stable under protocol (23) and $\lim_{t \rightarrow \infty} \xi^{(i)} = 0$, $i \in [0, h-1]$. $\square$

Remark 11. Although the general conditions on higher-order systems are valid for third-order systems, the conditions derived for third-order systems have their own characteristics and-values. To be more specific, the judgment method of the third-order system is more direct than that of the general higher-order system. More importantly, the results of third-order systems are much more detailed. Thus, the results of third-order systems are useful for detailing the effect of feedback gains on the consensus end-value and for understanding the implications of the results of general higher-order systems. For example, one can see from the third-order system that $\xi_{ij}^{(0)}$ is the function of t in the presence of $k_l = 0$ with the order equal to the number of k_l equal to zero.

FIGURE 2: Position, velocity, and acceleration trajectories of l(a) with $k_1 = k_2 = 0$, $\beta_1 = \beta_2 = 3$.

4. Simulation

We choose digraph (a) with 6 nodes and 9 edges in Figure 1, in which the solid line between nodes represents cooperative relationship between agents and the dotted line represents competitive relationship between agents. The weights of solid line and dotted line are taken as 1 and -1, respectively. The line graph of (a) is (b) with 9 nodes. Let $\xi(0) = [1.6, -3.7, -1.8, 2.5, -2, -1, 3.6, 1, -3]^T$, $\theta(0) = [-3.7, -0.5, 2, -2.5, 1, 2.5, -1, -2, 0.6]^T$, $\psi(0) = [1.6, 2, -2, 1, -2.5, -3, 3.6, -1, 0.4]^T$. Let $D = \text{diag}(1, 1, -1, -1, 1, -1, 1, -1, 1)$, then we have

$$\bar{L}' = DL'D$$

$$= \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (40)$$

and 0 is a simple eigenvalue of $\bar{L}'$. The corresponding left eigenvector and right eigenvector are

$$\begin{aligned} w_l &= [-0.2328 \quad -0.6209 \quad -0.3105 \quad -0.3105 \quad -0.3105 \quad -0.1552 \quad -0.0776 \quad -0.1552 \quad -0.4657]^T, \\ w_r &= [-0.3789 \quad -0.3789 \quad -0.3789 \quad -0.3789 \quad -0.3789 \quad -0.3789 \quad -0.3789 \quad -0.3789 \quad -0.3789]^T, \end{aligned} \quad (41)$$

and w_l^T, w_r satisfy $w_l^T w_r = 1$.

Figure 2 shows that the absolute value of position and velocity of all the edges in the system converge to $\lim_{t \rightarrow \infty} |\xi_{ij}(t)| = 0.4234t^2 - 0.2205t - 1.4705$ and $\lim_{t \rightarrow \infty} |\theta_{ij}(t)| = 0.8468t - 0.2205$, respectively, with the absolute values of acceleration of edges converging to 0.8468.

If $k_1 = 0$, $k_2 = 1$, $\beta_1 = \beta_2 = 3$, Figure 3 shows that the absolute value of position trajectories of all the edges in the system converge to $\lim_{t \rightarrow \infty} |\xi_{ij}(t)| = 0.6263t - 1.691$, and the absolute values of velocity and acceleration trajectories of all edges are $\lim_{t \rightarrow \infty} |\theta_{ij}(t)| = 2.3601$, $\lim_{t \rightarrow \infty} \psi_{ij}(t) = 0$.

In Figure 4, for the third-order edge dynamic multiagent system (5), if $k_1 = k_2 = 1$, $\beta_1 = \beta_2 = 2$, we can get the following results by calculation

$$\begin{aligned} \lim_{t \rightarrow \infty} |\xi_{ij}| &= 0.8442, \\ \lim_{t \rightarrow \infty} \theta_{ij} &= 0, \end{aligned}$$

$$\lim_{t \rightarrow \infty} \psi_{ij} = 0,$$

$$(i, j) \in \mathcal{E}.$$

$$(42)$$

If the digraph is structurally unbalanced ((c) in Figure 1), system (5) is asymptotically stable. Figure 5 shows that if the digraph is structurally unbalanced, all states of edges converge to 0, that is to say, $\lim_{t \rightarrow \infty} \xi_{ij}(t) = 0$, $\lim_{t \rightarrow \infty} \theta_{ij}(t) = 0$, $\lim_{t \rightarrow \infty} \psi_{ij}(t) = 0$, $(i, j) \in \mathcal{E}$.

5. Conclusion

In this paper, we first considered the information of edges and mapped the digraph to a line graph. Then the protocols of third-order and ordinary high-order multiagent systems were designed on the line graph, respectively. We proposed the conditions such that multiagent systems can asymptotically reach the third-order bipartite consensus of edge dynamics. For the high-order multiagent system, we simplified the

FIGURE 3: Position, velocity, and acceleration trajectories of 1(a) with $k_1 = 0$, $k_2 = 1$, $\beta_1 = \beta_2 = 3$.

FIGURE 4: Position, velocity, and acceleration trajectories of 1(a) with $k_1 = 1$, $k_2 = 1$, $\beta_1 = \beta_2 = 3$.

FIGURE 5: Position, velocity, and acceleration trajectories of 1(c) with $k_1 = 0.5$, $k_2 = 1$, $\beta_1 = \beta_2 = 1$.

characteristic polynomial of the system on the line graph and gain the consensus conditions making system asymptotically reach the bipartite consensus of edge dynamics. By taking advantage of the properties of eigenvalues, we discussed the state values of all the edges for the third-order and ordinary high-order multiagent system. The high-order consensus of edges with other constraints has not been discussed, which is one of our research directions in the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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