In this paper, we introduce two new classes of acceptability indexes, named quasi-concave acceptability indexes and coherent acceptability indexes, for portfolio vectors. We establish the one-to-one correspondence between quasi-concave (coherent, resp.) acceptability indexes and convex (coherent, resp.) risk measures for portfolio vectors. We derive the representation results for coherent and convex risk measures. Finally, based on these results, we derive the representation results for quasi-concave acceptability indexes and coherent acceptability indexes for portfolio vectors. These new acceptability indexes can be considered as a kind of multivariate extension of univariate coherent and quasi-concave acceptability indexes introduced by Cherny and Madan (2009) and Rosazza Gianin and Sgarra (2013), respectively.

1. Introduction

Recently, several authors have focused their attention on acceptability indexes. Coherent acceptability indexes have been defined by Cherny and Madan [1] as performance measures of terminal cash flows seen as random variables, by proposing some basic axioms to be satisfied by every sound financial acceptability indexes. Further, Rosazza Gianin and Sgarra [2] introduced the broader class, named quasi-concave acceptability indexes, by dropping one of coherency axioms. More recently, Bielecki et al. [3] and Biagini and Bion-Nadal [4] extended the definitions of coherent and quasi-concave acceptability indexes to a dynamic setting, respectively. For more works about acceptability indexes, see Bielecki et al. [5] and the references therein.

In all the above-mentioned works, a financial position is described by a random variable or a one-dimensional stochastic process. However, in order to evaluate the degree of quality of a multivariate portfolio, among which there may exist some possible dependence between the components, it is natural to consider a random vector rather than random variables composed by all components. This observation motivates us to study acceptability indexes for portfolio vectors.

In the present paper, we will study multivariate coherent and quasi-concave acceptability indexes. Meanwhile, in order to investigate the correspondence between multivariate acceptability indexes and multivariate risk measures, we will make some adaptation to one of the axioms of multivariate risk measures introduced by Burgert and Rüscheidorf [6] and will establish the one-to-one correspondence between quasi-concave (resp. coherent) acceptability indexes and convex (resp. coherent) risk measures for portfolio vectors. Representation results for (the adapted) convex and coherent risk measures will be given. Finally, we will provide representation results for these new acceptability indexes.

It should be mentioned that there also exist many papers about multidimensional risk measures. For scalar multivariate risk measures, see Burgert and Rüscheidorf [6], Rüscheidorf [7], Ekeland and Schachermayer [8], Ekeland et al. [9], Rüscheidorf [10], Wei and Hu [11], Chen et al. [12], and the references therein. For set-valued multivariate risk measures, see Jouini et al. [13], Hamel [14], Hamel and Heyde [15], Hamel et al. [16], Hamel et al. [17], Labuschagne and Offwood-Le Roux [18], Farkas et al. [19], Molchanov and Cascos [20], Chen and Hu [21], and the references therein.

The rest of the paper is organized as follows. In Section 2, we briefly introduce some preliminaries. The main results
are stated in Section 3, and their proofs are postponed to Appendix.

2. Preliminaries

In this section, we will briefly introduce the preliminaries. Let $(\Omega, \mathcal{F})$ be a fixed measurable space and $(\Omega, \mathcal{F}, P)$ a fixed probability space. We denote by $\mathcal{G} = L^\infty(\Omega, \mathcal{F})$ the space of bounded $\mathcal{F}$-measurable random variables. Denote $\mathcal{H} = L^p(\Omega, \mathcal{F}, P)$ for $1 \leq p \leq +\infty$, where, when $p = +\infty$, $\mathcal{H}$ denotes the space of essentially bounded $\mathcal{F}$-measurable random variables and, when $1 \leq p < +\infty$, $\mathcal{H}$ denotes the space of random variables with finite $p$-order moment for $X, Y \in \mathcal{H}$, we will identify $X$ with $Y$ if $P(X = Y) = 1$. The space $\mathcal{G}$ (or $\mathcal{H}$) represents financial risk positions. Positive values of $X \in \mathcal{G}$ or $\mathcal{H}$ correspond to gains, while negative values correspond to losses. For $X \in \mathcal{G}$, define $\|X\|_\infty = \sup_{\omega \in \Omega} |X(\omega)|$, and then $(\mathcal{G}, \|\cdot\|_\infty)$ is a Banach space. For $H \in \mathcal{H}$, define $\|H\|_p = \text{esssup} |H|$, if $p = +\infty$; $\{E(|H|^p)\}^{1/p}$, if $1 \leq p < +\infty$, where $E(|H|^p)$ means the integral of $|H|^p$ with respect to the probability $P$, and then $(\mathcal{H}, \|\cdot\|_p)$ is a Banach space.

A map $\mu : \mathcal{F} \to \mathbb{R}$ is called a finitely additive set function if for any finite collection $A_1, \ldots, A_n \in \mathcal{F}$ of mutually disjoint sets, $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$, and if $\mu(\emptyset) = 0$. The total variation of a finitely additive set function $\mu$ is defined as $\|\mu\|_\text{var} = \sup \sum_{i=1}^n |\mu(A_i)| : A_1, \ldots, A_n$ disjoint sets in $\mathcal{F}, n \in \mathbb{N}$. By $ba(\Omega, \mathcal{F})$, we denote the set of all finitely additive measures $\mu$ with $\|\mu\|_\text{var} < +\infty$. $E_\mu(X)$ denotes the integral of $X \in \mathcal{G}$ with respect to $\mu \in ba(\Omega, \mathcal{F})$. We denote by $\mathcal{M}_{s,f}(\Omega, \mathcal{F}) := \{\mu : ba(\Omega, \mathcal{F}), \mu(\Omega) \leq 1\}$, by $\mathcal{M}_{s,f}(\Omega, \mathcal{F}, P) := \{\mu \in \mathcal{M}_{s,f}(\Omega, \mathcal{F}) : \mu = \mathcal{F}\mu\}$, and by $\mathcal{M}_{s,f}(\Omega, \mathcal{F}, P)$, resp. the subclass of $ba(\Omega, \mathcal{F})$($\mathcal{M}_{s,f}(\Omega, \mathcal{F})$, $\mathcal{M}_{s,f}(\Omega, \mathcal{F}, P)$, resp.) which are absolutely continuous with respect to $P$.

Let $N \geq 1$ be a fixed positive integer, which represents the number of assets. For $1 \leq i \leq N$, let $(\Omega_i, \mathcal{F}_i)$ be a fixed measurable space and $(\Omega, \mathcal{F}, P)$ a fixed probability space. We denote by $\mathcal{G}_i = L^\infty(\Omega_i, \mathcal{F}_i)$ and $\mathcal{H}_i = L^p(\Omega_i, \mathcal{F}_i, P_i)$, $1 \leq p \leq +\infty$, $1 \leq i \leq N$. $\mathcal{H}^N$ denotes the space $\mathcal{G}_1 \times \cdots \times \mathcal{G}_N$, where $\mathcal{G}_i$ is either $\mathcal{G}$ or $\mathcal{H}$ for $1 \leq i \leq N$. Define a norm in the space $\mathcal{H}^N$ by $\|M\| = \|X_1\| + \cdots + \|X_N\|$ for $M = \{X_1, \ldots, X_N\} \in \mathcal{H}^N$, where $\|X_i\|$ equals $\|X_i\|_\infty$, if $X_i \in \mathcal{G}$, and $\|X_i\|_p$ if $X_i \in \mathcal{H}$, for $1 \leq i \leq N$. Then, $(\mathcal{H}^N, \|\cdot\|)$ is a Banach space.

Next, we introduce a notation $\mathcal{M}_{s,f}(\mathcal{X}_i)$ for $1 \leq i \leq N$. When $\mathcal{X}_i$ is $\mathcal{G}_i$, $\mathcal{M}_{s,f}(\mathcal{X}_i)$ stands for $\mathcal{M}_{s,f}(\Omega_i, F_i)$, when $\mathcal{X}_i$ is $\mathcal{H}_i$.

1. If $p_i = +\infty$, then $\mathcal{M}_{s,f}(\mathcal{X}_i)$ means the set $\mathcal{M}_{s,f}(\Omega_i, F_i, P_i)$.
2. If $p_i = 1$, then $\mathcal{M}_{s,f}(\mathcal{X}_i)$ stands for the set $\{Q \in \mathcal{M}_1(\Omega, F, P) : (dQ)(dP) \in L^\infty(\Omega, F, P)\}$.
3. If $1 < p_i < +\infty$, then $\mathcal{M}_{s,f}(\mathcal{X}_i)$ means the set $\{Q \in \mathcal{M}_{p_i}(\Omega_i, F_i, P_i) : (dQ)(dP) \in L^{p_i}(\Omega_i, F_i, P_i)\}$, where $p_i$ is the conjugate index of $p_i$.

For $M_1 = (X_1, \ldots, X_N)$, $M_2 = (Y_1, \ldots, Y_N) \in \mathcal{X}^N$, $M_1 + M_2$ means $(X_1 + Y_1, \ldots, X_N + Y_N)$, and $M_1 \leq M_2$ means $X_i \leq Y_i, 1 \leq i \leq N$. $\alpha$ stands for $(\alpha_{X_1}, \ldots, \alpha_{X_N})$ for $M = (X_1, \ldots, X_N) \in \mathcal{X}^N$ and $\alpha \in \mathbb{R}$. We set $0 = (0, \ldots, 0), 1 = (1, \ldots, 1)$, and for any fixed $i$ between 1 and $N$, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 occupies the $i$th position. For $1 \leq i \leq N$, $X_i$ means $(0, \ldots, 0, X, 0, \ldots, 0)$, where $X$ occupies the $i$th position for any random variable $X$.

On a general level, a multivariate acceptability index (or a performance measure for portfolio vectors) is any map $\alpha : \mathcal{X}^N \to [0, +\infty)$. Given a portfolio vector $M \in \mathcal{X}^N$, $\alpha(M)$ measures the degree of quality of $M$.

Definition 1. A quasi-concave acceptability index is a map $\alpha : \mathcal{X}^N \to [0, +\infty)$ satisfying the following two axioms:

(A1) Quasi-concavity: for any real $c \in [0,1]$ and any $M_1, M_2 \in \mathcal{X}^N$, then

$$\alpha(cM_1 + (1-c)M_2) \geq \min\{\alpha(M_1), \alpha(M_2)\}. \tag{1}$$

(A2) Monotonicity: $M_1 \geq M_2$ implies $\alpha(M_1) \geq \alpha(M_2)$ for any $M_1, M_2 \in \mathcal{X}^N$.

Furthermore, a quasi-concave acceptability index $\alpha$ is called a coherent acceptability index if it also satisfies:

(A3) Scaling invariance: $\alpha(cM) = \alpha(M)$ for any $M \in \mathcal{X}^N$ and $c > 0$.

The above axioms have natural financial interpretation. (A1), quasi-concavity, means that a diversified portfolio performs at higher level than its components. (A2), monotonicity, states that if $M_1$ dominates $M_2$, then $M_1$ is acceptable at least at the same level as $M_2$ is. (A3), scaling invariance, means that cash flows with the same direction of trade have the same level of acceptance. For more explanations about these axioms, see Cherny and Madan [1], Rosazza Gianin and Sgarra [2], Biagini and Bian-Nadal [4], Bielecki, Cialenco, and Zhang [3], and Bielecki et al. [5].

As mentioned in Cherny and Madan [1] and Rosazza Gianin and Sgarra [2], there is a strong relationship between univariate acceptability indexes and univariate risk measures. For multivariate acceptability indexes, it is natural to link them with multivariate risk measures. Theorem 1 shows that this is indeed the case. The delicate issue however is what family of multivariate risk measures should be used. It turns out that to produce a quasi-concave or coherent acceptability index for portfolio vectors, one needs to make an adaptation to the definition of coherent and convex risk measures for portfolio vectors introduced by Burgert and Rüschendorf [6]; see also Rüschendorf [10] and Wei and Hu [11]. Definition 2 is such an adaptation.

Definition 2. A map $\rho : \mathcal{X}^N \to \mathbb{R}$ is called a convex risk measure, if it satisfies the following three axioms:

(R1) Monotonicity: $M_1 \geq M_2$ implies $\rho(M_1) \leq \rho(M_2)$ for any $M_1, M_2 \in \mathcal{X}^N$.
Definition 3. We call a map \( f : \mathcal{X}^N \to \mathbb{R} \) has the Fatou property, if \( \{ M_n : n \geq 1 \} \) is a bounded sequence in \( \mathcal{X}^N \) which converges to \( M \in \mathcal{X}^N \), then
\[
 f(M) \leq \liminf_{n \to \infty} f(M_n). \tag{2}
\]

Definition 4. A family of risk measures \( \{ \rho_x \}_{x \in (0, +\infty)} \) is called increasing if \( \rho_x(M) \geq \rho_y(M) \) for all \( x \geq y > 0 \) and \( M \in \mathcal{X}^N \).

3. Main Results

In this section, we will state the representation results for quasi-concave and coherent acceptability indexes for portfolio vectors defined in Definition 1. We need first to give some propositions and theorems, whose proofs will be given in the Appendix.

The following proposition shows that a quasi-concave (coherent, resp.) acceptability index can induce an increasing family of convex (coherent, resp.) risk measures, and vice versa, an increasing family of convex (coherent, resp.) risk measures can also induce a quasi-concave (coherent, resp.) acceptability index.

Proposition 1

1. Assume that \( \alpha : \mathcal{X}^N \to [0, +\infty) \) is a quasi-concave (coherent, resp.) acceptability index. Then, the set of functions \( \rho_x, x \in (0, +\infty) \), defined by
\[
 \rho_x(M) = \inf \{ m \in \mathbb{R} : \alpha(M + m) \geq x \}, \quad M \in \mathcal{X}^N, \tag{3}
\]
is an increasing family of convex (coherent, resp.) risk measures.

2. Assume that \( \{ \rho_x \}_{x \in (0, +\infty)} \) is an increasing family of convex (coherent, resp.) risk measures. Then, the function \( \alpha \) defined by
\[
 \alpha(M) = \sup \{ x \in (0, +\infty) : \rho_x(M) \leq 0 \}, \quad M \in \mathcal{X}^N, \tag{4}
\]
is a quasi-concave (coherent, resp.) acceptability index.

Remark 1. If \( \alpha \) is a quasi-concave (coherent, resp.) acceptability index with the Fatou property, then \( \{ \rho_x \}_{x \in (0, +\infty)} \) defined by (3) also has the Fatou property. And vice versa, if \( \{ \rho_x \}_{x \in (0, +\infty)} \) is a quasi-concave (coherent, resp.) acceptability index with the Fatou property, then \( \alpha \) defined by (4) also has the Fatou property.

The following theorem shows that a quasi-concave (coherent, resp.) acceptability index can be represented by a family of convex (coherent, resp.) risk measures, and vice versa, convex (coherent, resp.) risk measures can be represented by a quasi-concave (coherent, resp.) acceptability index.

Theorem 1

1. If \( \alpha \) is a quasi-concave (coherent, resp.) acceptability index, then there exists an increasing family of convex (coherent, resp.) risk measures \( \{ \rho_x \}_{x \in (0, +\infty)} \) such that
\[
 \alpha(M) = \sup \{ x \in (0, +\infty) : \rho_x(M) \leq 0 \}. \tag{5}
\]

2. If \( \{ \rho_x \}_{x \in (0, +\infty)} \) is an increasing family of convex (coherent, resp.) risk measures, then there exists a quasi-concave (coherent, resp.) acceptability index \( \alpha \) such that
\[
 \rho_x(M) = \inf \{ m \in \mathbb{R} : \alpha(M + m) \geq x \}, \tag{6}
\]
here we assume that \( \inf \emptyset = \infty \) and \( \sup \emptyset = 0 \).

Next, we will state the representation results for coherent and convex risk measures defined in Definition 2.

Theorem 2 (representation result for convex risk measures). A function \( \rho : \mathcal{X}^N \to \mathbb{R} \) is a convex risk measure if and only if there exists a function \( F : \prod_{i=1}^N \mathcal{M}_{s,f}(\mathcal{X}_i) \to \mathbb{R} \cup \{ +\infty \} \) with
\[
 \inf_{(Q_1, \ldots, Q_N) \in \mathcal{M}_{s,f}(\mathcal{X}_1) \times \cdots \times \mathcal{M}_{s,f}(\mathcal{X}_N) \cup \{ +\infty \}} F(Q_1, \ldots, Q_N) \tag{7}
\]
such that
\[
 \rho(M) = \sup_{(Q_1, \ldots, Q_N) \in \mathcal{M}_{s,f}(\mathcal{X}_1) \times \cdots \times \mathcal{M}_{s,f}(\mathcal{X}_N)} \left\{ E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N) - F(Q_1, \ldots, Q_N) \right\}, \tag{8}
\]
for any $M = (X_1, \ldots, X_N) \in \mathcal{X}^N$, where $F$ can be chosen as
\begin{equation}
F_{\text{max}}(Q_1, \ldots, Q_N) = \sup_{(x_1, \ldots, x_N) \in A_p} \{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N)\},
\end{equation}
where
\begin{equation}
A_p := \{M \in \mathcal{X}^N : \rho(M) \leq 0\}.
\end{equation}

Corollary 1 (representation result for coherent risk measures). A function $\rho : \mathcal{X}^N \to \mathbb{R}$ is a coherent risk measure if and only if there exists a subset
\begin{equation}
\mathcal{D} \subset \left\{ Q = (Q_1, \ldots, Q_N) \in \prod_{i=1}^N \mathcal{M}_s(f_i(\mathcal{X}_i)) : \sum_{i=1}^N Q_i(1) = 1 \right\},
\end{equation}
such that
\begin{equation}
\rho(M) = \sup_{(Q_1, \ldots, Q_N) \in \mathcal{D}} \{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N)\},
\end{equation}
for any $M = (X_1, \ldots, X_N) \in \mathcal{X}^N$.

Remark 2. A function $\rho : \mathcal{X}^N \to \mathbb{R}$ is a convex risk measure with the Fatou property if and only if there exists a function $F : \prod_{i=1}^N \mathcal{M}_s(f_i(\mathcal{X}_i)) \to \mathbb{R} \cup \{+\infty\}$ with
\begin{equation}
\inf_{(Q_1, \ldots, Q_N) \in \mathcal{D}} F(Q_1, \ldots, Q_N) = \rho(M),
\end{equation}
such that
\begin{equation}
\rho(M) = \inf_{(Q_1, \ldots, Q_N) \in \mathcal{D}} \{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N)\},
\end{equation}
for any $M = (X_1, \ldots, X_N) \in \mathcal{X}^N$.

Remark 3. A function $\rho : \mathcal{X}^N \to \mathbb{R}$ is a coherent risk measure with the Fatou property if and only if there exists a subset
\begin{equation}
\mathcal{D} \subset \left\{ Q = (Q_1, \ldots, Q_N) \in \prod_{i=1}^N \mathcal{M}_s(f_i(\mathcal{X}_i)) : \sum_{i=1}^N Q_i(1) = 1 \right\},
\end{equation}
such that
\begin{equation}
\rho(M) = \sup_{(Q_1, \ldots, Q_N) \in \mathcal{D}} \{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N)\},
\end{equation}
for any $M = (X_1, \ldots, X_N) \in \mathcal{X}^N$.

Theorem 3 (representation result for quasi-concave acceptability indexes). A map $\alpha : \mathcal{X}^N \to [0, +\infty]$ is a quasi-concave acceptability index if and only if there exists a decreasing family $(F_x)_{x \in (0, +\infty)}$ of functionals such that
\begin{equation}
\alpha(M) = \sup_{x \in (0, +\infty)} \inf_{(Q_1, \ldots, Q_N) \in \mathcal{D}} \{E_{Q_1}(X_1) + \cdots + E_{Q_N}(X_N) + F_x(Q_1, \ldots, Q_N)\} \geq 0,
\end{equation}
and here we assume that $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$.

Corollary 2 (representation result for coherent acceptability indexes). A map $\alpha : \mathcal{X}^N \to [0, +\infty]$ is a coherent acceptability index if and only if there exists a family of subsets $(\mathcal{D}_x)_{x \in (0, +\infty)}$ of
\begin{equation}
\left\{ Q = (Q_1, \ldots, Q_N) \in \prod_{i=1}^N \mathcal{M}_s(f_i(\mathcal{X}_i)) : \sum_{i=1}^N Q_i(1) = 1 \right\}
\end{equation}
with $\mathcal{D}_x \subseteq \mathcal{D}_y$ for $0 < x < y$, such that
\begin{equation}
\alpha(M) = \sup_{x \in (0, +\infty)} \inf_{(Q_1, \ldots, Q_N) \in \mathcal{D}_x} \{E_{Q_1}(X_1) + \cdots + E_{Q_N}(X_N) + F_x(Q_1, \ldots, Q_N)\} \geq 0.
\end{equation}
\[ \alpha(M) = \sup \left\{ x \in (0, +\infty) : \inf_{(Q_1, \ldots, Q_N) \in \mathcal{M}} \left\{ E_{Q_1}(X_1) + \cdots + E_{Q_N}(X_N) \right\} \geq 0 \right\}, \quad (21) \]

and here we assume that \( \inf \emptyset = \infty \) and \( \sup \emptyset = 0 \).

**Remark 4**

(a) A map \( \alpha : \mathcal{X}^N \rightarrow [0, +\infty) \) is a quasi-concave acceptability index with the Fatou property if and only if

\[
\alpha(M) = \sup \left\{ x \in (0, +\infty) : \inf_{(Q_1, \ldots, Q_N) \in \mathcal{M}, (X_1, \ldots, X_N) \in \mathcal{X}_N} \left\{ E_{Q_1}(X_1) + \cdots + E_{Q_N}(X_N) + F_x(Q_1, \ldots, Q_N) \right\} \geq 0 \right\},
\]

for any \( M \in \mathcal{X}^N \), where \( F \) can be chosen as

\[
F_x(Q_1, \ldots, Q_N) = \sup_{M \in \mathcal{X}^N, \alpha(M) \geq x} \left\{ E_{Q_1}(X_1) + \cdots + E_{Q_N}(X_N) \right\},
\]

and here we assume that \( \inf \emptyset = \infty \) and \( \sup \emptyset = 0 \).

(b) A map \( \alpha : \mathcal{X}^N \rightarrow [0, +\infty) \) is a coherent acceptability index with the Fatou property if and only if there exists a family of subsets \( \mathcal{D} \) of \( \mathcal{M} \) such that \( Q = (Q_1, \ldots, Q_N) \in \prod_{i=1}^N \mathcal{M}(X_i) : \sum_{i=1}^N Q_i(1) = 1 \) for any \( x \in (0, +\infty) \).

Next we will show that \( \rho_x \) satisfies (R1), monotonicity. Let \( M_1, M_2 \in \mathcal{X}^N \) with \( M_1 \leq M_2 \), and then from the monotonicity of \( \alpha \), we have

\[
\rho_x(M_1) = \inf \{ m \in \mathbb{R} : \alpha(M_1 + m) \geq x \} \\
\geq \inf \{ m \in \mathbb{R} : \alpha(M_2 + m) \geq x \} \\
= \rho_x(M_2).
\]

Now we will show \( \rho_x \) satisfies (R2), translation invariance. For any \( M \in \mathcal{X}^N \) and \( h \in \mathbb{R} \), we have that

\[
\rho_x(M + h) = \sup \{ m \in \mathbb{R} : \alpha(M + (h + m)) \geq x \} \\
= \sup \{ m - h \in \mathbb{R} : \alpha(M + m) \geq x \} \\
= \rho_x(M) - h.
\]

Finally, we will show that \( \rho_x \) satisfies (R3), the convexity. For any \( m_1 \in \{ m \in \mathbb{R} : \alpha(M_1 + m) \geq x \} \), \( m_2 \in \{ m \in \mathbb{R} : \alpha(M_2 + m) \geq x \} \), and \( \lambda \in [0, 1] \), by the quasi-concavity of \( \alpha \), we have

\[ \rho_x(M + \lambda h) \leq \rho_x(M + h) - h \]

for any \( M \in \mathcal{X}^N \) and \( h \in \mathbb{R} \), we have

\[
\rho_x(M + h) = \sup \{ m \in \mathbb{R} : \alpha(M + (h + m)) \geq x \} \\
= \sup \{ m - h \in \mathbb{R} : \alpha(M + m) \geq x \} \\
= \rho_x(M) - h.
\]
Assume that $\alpha$ is a increasing family of convex risk measures. In particular, if $\alpha$ is an increasing family of convex risk measures.

By taking infimum in (A.4), first with respect to $m_1$, and then with respect to $m_2$, we have

$$\lambda m_1 + (1-\lambda)m_2 \geq \rho_x(\alpha M_1 + (1-\lambda)M_2),$$

(A.4)

By convexity of $\alpha$, this is an increasing family of convex risk measures. Hence, $\rho_x$ is a quasi-concave acceptability index.

In particular, if $\rho_x(\alpha)$ is an increasing family of coherent risk measures, then it is easy to check that $\alpha$ defined by (4) is scale invariant, which implies that $\alpha$ is a coherent acceptability index.

**Proof of Theorem 1**

1. Assume that $\alpha$ is a quasi-concave (coherent, resp.) acceptability index. For every $x \in (0, +\infty)$, define $\rho_x$ as follows:

$$\rho_x(M) := \inf\{m \in R : \alpha(M + m) \geq x\},$$

(A.8)

for any $M \in X^N$. By Proposition 1, $(\rho_x)_{x \in (0, +\infty)}$ is an increasing family of convex (coherent, resp.) risk measures. We will show that

$$\alpha(M) = \sup\{x \in (0, +\infty) : \rho_x(M) \leq 0\},$$

(A.9)

for any $M \in X^N$. It is easy to see that, for any $M \in X^N$ and $x \in (0, +\infty)$, $\alpha(M) \geq x$ if and only if $\rho_x(M) \leq 0$. Hence,

$$\alpha(M) = \sup\{x \in (0, +\infty) : \rho_x(M) \leq 0\}.$$  

(A.10)

2. Assume that $\rho_x(\alpha)$ is an increasing family of convex (coherent, resp.) risk measures. Define the function $\alpha$ as follows:

$$\alpha(M) = \sup\{x \in (0, +\infty) : \rho_x(M) \leq 0\},$$

(A.11)

for all $M \in X^N$. By Proposition 1, $\alpha$ is a quasi-concave (coherent, resp.) acceptability index. Finally, it is easy to see that for any $M \in X^N$, $x \in (0, +\infty)$ and $m \in R$, $\rho_x(M) \geq m$ if and only if $\alpha(M + m) \geq x$. In fact, if $\alpha(M + m) \geq x$, then $\sup\{y \in (0, +\infty) : \rho_y(M + m) \leq 0\} \geq x$, i.e., $\sup\{y \in (0, +\infty) : \rho_y(M) \leq m\} \geq x$, which, together with the property that $(\rho_x)_{x \in (0, +\infty)}$ is increasing with respect to $x$, implies that $\rho_x(M) \leq m$. On the other hand, if $\rho_x(M) \leq m$, then $\rho_x(M + m) \leq 0$. Thus, $\sup\{y \in (0, +\infty) : \rho_y(M + m) \leq 0\} \geq x$, i.e., $\alpha(M + m) \geq x$. 

Hence, $\sup\{x \in (0, +\infty) : \rho_x(\alpha M_1 + (1-\lambda)M_2) \leq 0\} \geq x_0$, and thus, by the definition (4) of $\alpha$, we have $\alpha(\lambda M_1 + (1-\lambda)M_2) \geq x_0$. This yields the quasi-concavity of $\alpha$. The monotonicity is clear. Hence, $\alpha$ defined by (4) is a quasi-concave acceptability index.
Hence,
\[ \rho_\alpha(M) = \inf \{ m \in \mathbb{R} : \rho_\alpha(M) \geq m \} = \inf \{ m \in \mathbb{R} : \alpha(M + m) \geq x \} \] (A.12)

for any \( M = (X_1, \ldots, X_N) \in \mathcal{X}^N \).

To this end, given \( M = (X_1, \ldots, X_N) \in \mathcal{X}^N \), note that \( \tilde{M} = (\tilde{X}_1, \ldots, \tilde{X}_N) = M + \rho(M) 1 \in \mathcal{X}_p \). Thus, for any \( (Q_1, \ldots, Q_N) \in \mathcal{M}_{s,f}(\mathcal{X}_1) \times \cdots \times \mathcal{M}_{s,f}(\mathcal{X}_N) \) with \( \sum_{i=1}^N Q_i(1) = 1 \), we have that
\[ F_{\min}(Q_1, \ldots, Q_N) \geq E_{Q_1}(-\tilde{X}_1) + \cdots + E_{Q_N}(-\tilde{X}_N) \]
\[ = E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N) - \rho(M) \sum_{i=1}^N Q_i(1) \]
\[ = E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N) - \rho(M) \]
(A.14)

which implies (A.13).

For \( M = (X_1, \ldots, X_N) \) given, we will now construct \( Q_M := (Q_{1M}, \ldots, Q_{NM}) \in \mathcal{M}_{s,f}(\mathcal{X}_1) \times \cdots \times \mathcal{M}_{s,f}(\mathcal{X}_N) \) with \( \sum_{i=1}^N Q_i(1) = 1 \), such that
\[ \rho(M) \leq E_{Q_{1M}}(-X_1) + \cdots + E_{Q_{NM}}(-X_N) - F_{\min}(Q_{1M} \ldots, Q_{NM}), \] (A.15)

which, together with (A.13), proves representation (8).

By translation invariance, it suffices to prove (A.15) for \( M_0 = (X_0^1, \ldots, X_0^N) \in \mathcal{X}^N \) with \( \rho(M_0) = 0 \). Moreover, we may assume, without loss of generality, that \( \rho(0) = 0 \). Then, \( M_0 \) is not contained in the nonempty convex set
\[ \mathcal{B} := \{ M \in \mathcal{X}^N : \rho(M) < 0 \}. \] (A.16)

Since \( \mathcal{B} \) is a convex set, by the Hahn-Banach theorem, there exists a nonzero continuous linear functional \( \lambda \) (depending on \( M_0 \)) on \( \mathcal{X}^N \) such that
\[ \lambda(M_0) \leq \inf_{M \in \mathcal{B}} \lambda(M) = b. \] (A.17)

For \( 1 \leq i \leq N \), define
\[ \lambda_i : \mathcal{X}_i \rightarrow \mathbb{R}, \quad \lambda_i(X_i) = \lambda(X_i e_i). \] (A.18)

The proof of Theorem 1 is completed.

**Proof of Theorem 2.** The sufficiency is obvious. Next, we prove the necessity. First, we show that
\[ \lambda(M) = \lambda(X_1 e_1 + \cdots + X_N e_N) = \lambda_1(X_1) + \cdots + \lambda_N(X_N), \]
(A.19)

for any \( M = (X_1, \ldots, X_N) \in \mathcal{X}^N \). We further claim that \( \lambda \) and \( \lambda_i, 1 \leq i \leq N \), have the following two properties:

(1) \( \lambda(M) \geq 0 \) for any \( M \geq 0 \). Particularly, for \( 1 \leq i \leq N \), \( \lambda_i(X_i) \geq 0 \) for any \( \mathcal{X}_i \ni X_i \geq 0 \).

(2) \( \lambda(1) > 0 \).

First, we prove (1). For any \( c > 0 \) and any \( M \geq 0 \),
\[ c M + 1 \in \mathcal{B}. \] Hence,
\[ \lambda(M_0) \leq \lambda(cM + 1) = c \lambda(M) + \lambda(1), \] for any \( c > 0 \),
(A.20)

which could not be true if \( \lambda(M) < 0 \).

Next, we prove (2). Since \( \lambda \) is nonzero, there exists an \( M^* = (X_1^*, \ldots, X_N^*) \in \mathcal{X}^N \) such that \( \lambda(M^*) > 0 \). Taking the truncation argument into account, with no loss of generality, we assume that \( \|M^*\| < 1 \). Thus, \( \|X_i^*\| < 1 \), \( 1 \leq i \leq N \), where \( \|X_i^*\| \) means that \( \|X_i^*\| := \|X_i^*\|_{\infty} \) if \( X_i^* \in \mathcal{F}_i = \mathcal{F}_p \), and \( \|X_i^*\| = \|X_i^*\|_p \) if \( X_i^* \in \mathcal{F}_i = \mathcal{F}_f \). Thus, \( \lambda(1 - X_1^*, \ldots, 1 - X_N^*) \geq 0 \) by property (1). Hence, \( \lambda(1) = \lambda(M^* + 1 - M^*) = \lambda(M^*) + \lambda(1 - X_1^*, \ldots, 1 - X_N^*) > 0 \). Therefore, we can choose \( \lambda \) such that \( \lambda(1) = 1 \).

Consequently, \( \lambda_1, \ldots, \lambda_N \) are continuous linear functionals on \( \mathcal{X}_1, \ldots, \mathcal{X}_N \), respectively, and satisfy \( \sum_{i=1}^N \lambda_i(1) = 1 \).

By the preceding discussion and Theorem A.51 of Föllmer and Schied [22], we conclude that there exist \( Q_{\lambda M_0} \in \mathcal{M}_{s,f}(\mathcal{X}_i), 1 \leq i \leq N \), such that
\[ \sum_{i=1}^N Q_i(1) = 1, \]
\[ E_{Q_{\lambda M_0}}(X_i) = \lambda_i(X_i), \quad \text{for any } (X_1, \ldots, X_N) \in \mathcal{X}^N. \] (A.21)

Note that \( B \in A_p \), we know that
Proof of Theorem 3. Assume that \( \alpha \) is a quasi-concave acceptability index, and then by Theorem 1, we know that there exists an increasing family of convex risk measures \( \rho_{x} \) such that (5) holds. By Theorem 2, for each \( \rho_{x} \), there exists a function \( F_{x} \) such that (8) holds. Since \( \rho_{x} \) is increasing, \( F_{x} \) is decreasing, which, together with (5) and (8) yields (19). The proof of Theorem 3 is completed.

Proof of Corollary 1. Assume that \( \rho \) is a coherent risk measure, and then \( \mathscr{A} \) is a cone. Thus,

\[
F_{\min}(Q_{1}, \ldots, Q_{N}) = \sup_{(x_{1}, \ldots, x_{N}) \in \mathscr{A}} \left\{ E_{Q_{1}}(-X_{1}) + \cdots + E_{Q_{N}}(-X_{N}) \right\}
\]

which, together with (A.17), implies that

\[
E_{Q_{i}}(-X_{i}) + \cdots + E_{Q_{N}}(-X_{N}) - F_{\min}(Q_{1}, \ldots, Q_{N}) = b - \lambda(M_{0}) \geq 0 = \rho(M_{0}).
\]

Thus, \( Q_{i}, \ldots, Q_{N} \) are desired, and the proof of (8) is completed. Theorem 2 is proved.

The proof of Corollary 1 is completed.

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References

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