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Research Article

Acceptability Indexes for Portfolio Vectors

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In this paper, we introduce two new classes of acceptability indexes, named quasi-concave acceptability indexes and coherent acceptability indexes, for portfolio vectors. We establish the one-to-one correspondence between quasi-concave (coherent, resp.) acceptability indexes and convex (coherent, resp.) risk measures for portfolio vectors. We derive the representation results for coherent and convex risk measures. Finally, based on these results, we derive the representation results for quasi-concave acceptability indexes and coherent acceptability indexes for portfolio vectors. These new acceptability indexes can be considered as a kind of multivariate extension of univariate coherent and quasi-concave acceptability indexes introduced by Cherny and Madan (2009) and Rosazza Gianin and Sgarra (2013), respectively.

1. Introduction

Recently, several authors have focused their attention on acceptability indexes. Coherent acceptability indexes have been defined by Cherny and Madan [1] as performance measures of terminal cash flows seen as random variables, by proposing some basic axioms to be satisfied by every sound financial acceptability indexes. Further, Rosazza Gianin and Sgarra [2] introduced the broader class, named quasi-concave acceptability indexes, by dropping one of coherency axioms. More recently, Bielecki et al. [3] and Biagini and Bion-Nadal [4] extended the definitions of coherent and quasi-concave acceptability indexes to a dynamic setting, respectively. For more works about acceptability indexes, see Bielecki et al. [5] and the references therein.

In all the above-mentioned works, a financial position is described by a random variable or a one-dimensional stochastic process. However, in order to evaluate the degree of quality of a multivariate portfolio, among which there may exist some possible dependence between the components, it is natural to consider a random vector rather than random variables composed by all components. This observation motivates us to study acceptability indexes for portfolio vectors.

In the present paper, we will study multivariate coherent and quasi-concave acceptability indexes. Meanwhile, in order to investigate the correspondence between multivariate acceptability indexes and multivariate risk measures, we will make some adaptation to one of the axioms of multivariate risk measures introduced by Burgert and Rüschendorf [6] and will establish the one-to-one correspondence between quasi-concave (resp. coherent) acceptability indexes and convex (resp. coherent) risk measures for portfolio vectors. Representation results for (the adapted) convex and coherent risk measures will be given. Finally, we will provide representation results for these new acceptability indexes.

It should be mentioned that there also exist many papers about multidimensional risk measures. For scalar multivariate risk measures, see Burgert and Rüschendorf [6], Rüschendorf [7], Ekeland and Schachermayer [8], Ekeland et al. [9], Rüschendorf [10], Wei and Hu [11], Chen et al. [12], and the references therein. For set-valued multivariate risk measures, see Jouini et al. [13], Hamel [14], Hamel and Heyde [15], Hamel et al. [16], Hamel et al. [17], Labuschagne and Offwood-Le Roux [18], Farkas et al. [19], Molchanov and Cascos [20], Chen and Hu [21], and the references therein.

The rest of the paper is organized as follows. In Section 2, we briefly introduce some preliminaries. The main results

are stated in Section 3, and their proofs are postponed to Appendix.

2. Preliminaries

In this section, we will briefly introduce the preliminaries. Let $(\Omega, \mathcal{F})$ be a fixed measurable space and $(\Omega, \mathcal{F}, P)$ a fixed probability space. We denote by $\mathcal{G} = L^{\infty}(\Omega, \mathcal{F})$ the space of bounded $\mathcal{F}$-measurable random variables. Denote $\mathcal{H} = L^{p}(\Omega, \mathcal{F}, P)$ for $1 \le p \le +\infty$, where, when $p = +\infty$, $\mathcal{H}$ denotes the space of essentially bounded $\mathcal{F}$-measurable random variables and, when $1 \le p < +\infty$, $\mathcal{H}$ denotes the space of random variables with finite p-order moment. For $X, Y \in \mathcal{H}$, we will identify X with Y if $P(X = Y) = 1$. The space $\mathcal{G}$ (or $\mathcal{H}$) represents financial risk positions. Positive values of $X \in \mathcal{G}$ or $\mathcal{H}$ correspond to gains, while negative values correspond to losses. For $X \in \mathcal{G}$, define $\|X\|_{\infty} := \sup_{\omega\in\Omega}|X(\omega)|$, and then $(\mathcal{G}, \|\cdot\|_{\infty})$ is a Banach space. For $H \in \mathcal{H}$, define $\|H\|_{p} := \operatorname{esssup}|H|$, if $p = +\infty$; $[E(|H|^{p})]^{1/p}$, if $1 \le p < +\infty$, where $E(|H|^{p})$ means the integral of $|H|^{p}$ with respect to the probability P, and then $(\mathcal{H}, \|\cdot\|_{p})$ is a Banach space.

A map $\mu : \mathcal{F} \longrightarrow \mathbf{R}$ is called a finitely additive set function if for any finite collection $A_1, \ldots, A_n \in \mathcal{F}$ of mutually disjoint sets, $\mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n}\mu(A_i)$, and if $\mu(\varnothing) = 0$. The total variation of a finitely additive set function μ is defined as $\|\mu\|_{\mathrm{var}} := \sup\{\sum_{i=1}^{n}|\mu(A_i)| : A_1, \ldots, A_n$ disjoint sets in $\mathcal{F}$, $n \in \mathbb{N}\}$. By $ba(\Omega, \mathcal{F})$, we denote the space of all finitely additive measures μ with $\|\mu\|_{\mathrm{var}} < +\infty$. $E_{\mu}(X)$ denotes the integral of $X \in \mathcal{G}$ with respect to $\mu \in ba(\Omega, \mathcal{F})$. We denote by $\mathcal{M}_{s,f}(\Omega, \mathcal{F}) := \{\mu : \mathcal{F} \longrightarrow [0,1] : \mu \in ba(\Omega, \mathcal{F}), \mu(\Omega) \le 1\}$, by $\mathcal{M}_{1,f}(\Omega, \mathcal{F}) := \left\{\mu \in \mathcal{M}_{s,f}(\Omega, \mathcal{F}) : \mu(\Omega) = 1\right\}$, and by $\mathcal{M}_{s}(\Omega, \mathcal{F}) := \left\{\mu \in \mathcal{M}_{s,f}(\Omega, \mathcal{F}) : \mu \text{ is } \sigma - \text{additive}\right\}$. We also denote by $ba(\Omega, \mathcal{F}, P)$ ($\mathcal{M}_{s,f}(\Omega, \mathcal{F}, P)$, $\mathcal{M}_{1,f}(\Omega, \mathcal{F}, P)$, $\mathcal{M}_{s}(\Omega, \mathcal{F}, P)$, resp.) the subclass of $ba(\Omega, \mathcal{F})$ ($\mathcal{M}_{s,f}(\Omega, \mathcal{F})$, $\mathcal{M}_{1,f}(\Omega, \mathcal{F})$, $\mathcal{M}_{s}(\Omega, \mathcal{F})$, resp.), which are absolutely continuous with respect to P.

Let $N \ge 1$ be a fixed positive integer, which represents the number of assets. For $1 \le i \le N$, let $(\Omega_i, \mathcal{F}_i)$ be a fixed measurable space and $(\Omega_i, \mathcal{F}_i, P_i)$ a fixed probability space. We denote by $\mathcal{G}_i := L^{\infty}(\Omega_i, \mathcal{F}_i)$ and $\mathcal{H}_i := L^{p_i}(\Omega_i, \mathcal{F}_i, P_i)$, $1 \le p_i \le +\infty$, $1 \le i \le N$. Let $\mathcal{X}^N$ be the product space $\mathcal{X}_1 \times \cdots \times \mathcal{X}_N$, where $\mathcal{X}_i$ is either $\mathcal{G}_i$ or $\mathcal{H}_i$ for $1 \le i \le N$. Define a norm in the space $\mathcal{X}^N$ by $\|\mathbf{M}\| := \|X_1\| + \cdots + \|X_N\|$ for $\mathbf{M} = (X_1, \ldots, X_N) \in \mathcal{X}^N$, where $\|X_i\|$ equals $\|X_i\|_{\infty}$ if $\mathcal{X}_i$ is $\mathcal{G}_i$, and $\|X_i\|_{p_i}$ if $\mathcal{X}_i$ is $\mathcal{H}_i$ for $1 \le i \le N$. Then, $(\mathcal{X}^N, \|\cdot\|)$ is a Banach space.

Next, we introduce a notation $\mathcal{M}_{s,f}(\mathcal{X}_i)$ for $1 \le i \le N$. When $\mathcal{X}_i$ is $\mathcal{G}_i$, $\mathcal{M}_{s,f}(\mathcal{X}_i)$ stands for $\mathcal{M}_{s,f}(\Omega_i, \mathrm{F}_i)$. When $\mathcal{X}_i$ is $\mathcal{H}_i$,

(1) If $p_i = +\infty$, then $\mathcal{M}_{s,f}(\mathcal{X}_i)$ means the set $\mathcal{M}_{s,f}(\Omega_i, \mathcal{F}_i, P_i)$

(2) If $p_i = 1$, then $\mathcal{M}_{s,f}(\mathcal{X}_i)$ stands for the set $\{Q \in \mathcal{M}_{s}(\Omega_i, \mathcal{F}_i, P_i) : (dQ/dP_i) \in L^{\infty}(\Omega_i, \mathcal{F}_i, P_i)\}$

(3) If $1 < p_i < +\infty$, then $\mathcal{M}_{s,f}(\mathcal{X}_i)$ means the set $\{Q \in \mathcal{M}_{s}(\Omega_i, \mathcal{F}_i, P_i) : (dQ/dP_i) \in L^{q_i}(\Omega_i, \mathcal{F}_i, P_i)\}$, where q_i is the conjugate index of p_i

For $\mathbf{M}_1 = (X_1, \ldots, X_N)$, $\mathbf{M}_2 = (Y_1, \ldots, Y_N) \in \mathcal{X}^N$, $\mathbf{M}_1 + \mathbf{M}_2$ means $(X_1 + Y_1, \ldots, X_N + Y_N)$, and $\mathbf{M}_1 \le \mathbf{M}_2$ means $X_1 \le Y_1, \ldots, X_N \le Y_N$. $a\mathbf{M}$ stands for $(aX_1, \ldots, aX_N)$ for $\mathbf{M} = (X_1, \ldots, X_N) \in \mathcal{X}^N$ and $a \in \mathbf{R}$. We set $0 := (0, \ldots, 0)$, $1 := (1, \ldots, 1)$, and for any fixed i between 1 and N, $\mathbf{e_i} := (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 occupies the ith position. For $1 \le i \le N$, $X\mathbf{e_i}$ means $(0, \ldots, 0, X, 0, \ldots, 0)$, where X occupies the ith position for any random variable X.

On a general level, a multivariate acceptability index (or a performance measure for portfolio vectors) is any map $\alpha : \mathcal{X}^N \longrightarrow [0, +\infty]$. Given a portfolio vector $\mathbf{M} \in \mathcal{X}^N$, $\alpha(\mathbf{M})$ measures the degree of quality of $\mathbf{M}$.

Definition 1. A quasi-concave acceptability index is a map $\alpha : \mathcal{X}^N \longrightarrow [0, +\infty]$ satisfying the following two axioms:

(A1) Quasi-concavity: for any real $c \in [0,1]$ and any $\mathbf{M}_1, \mathbf{M}_2 \in \mathcal{X}^N$, then

$$\alpha(c\mathbf{M}_1 + (1-c)\mathbf{M}_2) \ge \min\{\alpha(\mathbf{M}_1), \alpha(\mathbf{M}_2)\}. \tag{1}$$

(A2) Monotonicity: $\mathbf{M}_1 \ge \mathbf{M}_2$ implies $\alpha(\mathbf{M}_1) \ge \alpha(\mathbf{M}_2)$ for any $\mathbf{M}_1, \mathbf{M}_2 \in \mathcal{X}^N$.

Furthermore, a quasi-concave acceptability index α is called a coherent acceptability index if it also satisfies:

(A3) Scaling invariance: $\alpha(c\mathbf{M}) = \alpha(\mathbf{M})$ for any $\mathbf{M} \in \mathcal{X}^N$ and $c > 0$.

The above axioms have natural financial interpretation. (A1), quasi-concavity, means that a diversified portfolio performs at higher level than its components. (A2), monotonicity, states that if $\mathbf{M}_1$ dominates $\mathbf{M}_2$, then $\mathbf{M}_1$ is acceptable at least at the same level as $\mathbf{M}_2$ is. (A3), scaling invariance, means that cash flows with the same direction of trade have the same level of acceptance. For more explanations about these axioms, see Cherny and Madan [1], Rosazza Gianin and Sgarra [2], Biagini and Bion-Nadal [4], Bielecki, Cialenco, and Zhang [3], and Bielecki et al. [5].

As mentioned in Cherny and Madan [1] and Rosazza Gianin and Sgarra [2], there is a strong relationship between univariate acceptability indexes and univariate risk measures. For multivariate acceptability indexes, it is natural to link them with multivariate risk measures. Theorem 1 shows that this is indeed the case. The delicate issue however is what family of multivariate risk measures should be used. It turns out that to produce a quasi-concave or coherent acceptability index for portfolio vectors, one needs to make an adaptation to the definition of coherent and convex risk measures for portfolio vectors introduced by Burgert and Rüschendorf [6]; see also Rüschendorf [10] and Wei and Hu [11]. Definition 2 is such an adaptation.

Definition 2. A map $\rho : \mathcal{X}^N \longrightarrow \mathbf{R}$ is called a convex risk measure, if it satisfies the following three axioms:

(R1) Monotonicity: $\mathbf{M}_1 \ge \mathbf{M}_2$ implies $\rho(\mathbf{M}_1) \le \rho(\mathbf{M}_2)$ for any $\mathbf{M}_1, \mathbf{M}_2 \in \mathcal{X}^N$

(R2) Translation invariance: $\rho(\mathbf{M}+m1)=\rho(\mathbf{M})-m$ for any $\mathbf{M}\in\mathcal{X}^N$ and $m\in\mathbf{R}$

(R3) Convexity: $\rho(c\mathbf{M}_1+(1-c)\mathbf{M}_2)\le c\rho(\mathbf{M}_1)+(1-c)\rho(\mathbf{M}_2)$ for any $c\in[0,1]$ and $\mathbf{M}_1,\mathbf{M}_2\in\mathcal{X}^N$

Furthermore, a convex risk measure ρ is called a coherent risk measure if it also satisfies

(R4) Positive homogeneity: $\rho(c\mathbf{M})=c\rho(\mathbf{M})$ for any $\mathbf{M}\in\mathcal{X}^N$ and $c>0$

Definition 3. We call a map $f:\mathcal{X}^N\longrightarrow\mathbf{R}$ has the Fatou property, if $\{\mathbf{M}_n;n\ge1\}$ is a bounded sequence in $\mathcal{X}^N$ which converges to $\mathbf{M}\in\mathcal{X}^N$, then

$$f(\mathbf{M})\le\liminf_{n\longrightarrow\infty}f(\mathbf{M}_n). \tag{2}$$

Definition 4. A family of risk measures $(\rho_x)_{x\in(0,+\infty)}$ is called increasing if $\rho_x(\mathbf{M})\ge\rho_y(\mathbf{M})$ for all $x\ge y>0$ and $\mathbf{M}\in\mathcal{X}^N$.

3. Main Results

In this section, we will state the representation results for quasi-concave and coherent acceptability indexes for portfolio vectors defined in Definition 1. We need first to give some propositions and theorems, whose proofs will be given in the Appendix.

The following proposition shows that a quasi-concave (coherent, resp.) acceptability index can induce an increasing family of convex (coherent, resp.) risk measures, and vice versa, an increasing family of convex (coherent, resp.) risk measures can also induce a quasi-concave (coherent, resp.) acceptability index.

Proposition 1

(1) Assume that $\alpha:\mathcal{X}^N\longrightarrow[0,+\infty]$ is a quasi-concave (coherent, resp.) acceptability index. Then, the set of functions ρ_x, $x\in(0,+\infty)$, defined by

$$\rho_x(\mathbf{M}):=\inf\{m\in\mathbf{R}:\alpha(\mathbf{M}+m1)\ge x\},\quad \mathbf{M}\in\mathcal{X}^N, \tag{3}$$

is an increasing family of convex (coherent, resp.) risk measures.

(2) Assume that $(\rho_x)_{x\in(0,+\infty)}$ is an increasing family of convex (coherent, resp.) risk measures. Then, the function α defined by

$$\alpha(\mathbf{M}):=\sup\{x\in(0,+\infty):\rho_x(\mathbf{M})\le0\},\quad \mathbf{M}\in\mathcal{X}^N, \tag{4}$$

is a quasi-concave (coherent, resp.) acceptability index.

Remark 1. If α is a quasi-concave (coherent, resp.) acceptability index with the Fatou property, then $(\rho_x)_{x\in(0,+\infty)}$ defined by (3) also has the Fatou property. And vise versa, if $(\rho_x)_{x\in(0,+\infty)}$ is an increasing family of convex (coherent, resp.) risk measures with the Fatou property, then α defined by (4) also has the Fatou property.

The following theorem shows that a quasi-concave (coherent, resp.) acceptability index can be represented by a family of convex (coherent, resp.) risk measures, and vise versa, convex (coherent, resp.) risk measures can be represented by a quasi-concave (coherent, resp.) acceptability index.

Theorem 1

(1) If α is a quasi-concave (coherent, resp.) acceptability index, then there exists an increasing family of convex (coherent, resp.) risk measures $(\rho_x)_{x\in(0,+\infty)}$, such that

$$\alpha(\mathbf{M})=\sup\{x\in(0,+\infty):\rho_x(\mathbf{M})\le0\}. \tag{5}$$

(2) If $(\rho_x)_{x\in(0,+\infty)}$ is an increasing family of convex (coherent, resp.) risk measures, then there exists a quasi-concave (coherent, resp.) acceptability index α, such that

$$\rho_x(\mathbf{M})=\inf\{m\in\mathbf{R}:\alpha(\mathbf{M}+m1)\ge x\}, \tag{6}$$

here we assume that $\inf\emptyset=\infty$ *and* $\sup\emptyset=0$.

Next, we will state the representation results for coherent and convex risk measures defined in Definition 2.

Theorem 2 (representation result for convex risk measures). *A function $\rho:\mathcal{X}^N\longrightarrow\mathbf{R}$ is a convex risk measure if and only if there exists a function $F:\prod_{i=1}^N\mathcal{M}_{s,f}(\mathcal{X}_i)\longrightarrow\mathbf{R}\cup\{+\infty\}$ with*

$$\inf_{\substack{(Q_1,\dots,Q_N)\in\mathcal{M}_{s,f}(\mathcal{X}_1)\times\cdots\times\mathcal{M}_{s,f}(\mathcal{X}_N)\\ \sum_{i=1}^N Q_i(1)=1}}F(Q_1,\dots,Q_N)\in\mathbf{R}, \tag{7}$$

such that

$$\rho(\mathbf{M})=\sup_{\substack{(Q_1,\dots,Q_N)\in\mathcal{M}_{s,f}(\mathcal{X}_i)\times\cdots\times\mathcal{M}_{s,f}(\mathcal{X}_N)\\ \sum_{i=1}^N Q_i(1)=1}}\left\{E_{Q_1}(-X_1)+\cdots+E_{Q_N}(-X_N)-F(Q_1,\dots,Q_N)\right\}, \tag{8}$$

for any $\mathbf{M} = (X_1, \ldots, X_N) \in \mathcal{X}^N$, *where* F *can be chosen as*

$$F_{\min}(Q_1, \ldots, Q_N) := \sup_{(X_1,\ldots,X_N)\in\mathcal{A}_\rho} \left\{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N)\right\}, \tag{9}$$

where

$$\mathcal{A}_\rho := \left\{\mathbf{M} \in \mathcal{X}^N : \rho(\mathbf{M}) \le 0\right\}. \tag{10}$$

Remark 2. A function $\rho : \mathcal{X}^N \longrightarrow \mathbf{R}$ is a convex risk measure with the Fatou property if and only if there exists a function $F : \prod_{i=1}^N \mathcal{M}_s(\mathcal{X}_i) \longrightarrow \mathbf{R} \cup \{+\infty\}$ with

$$\inf_{\substack{(Q_1,\ldots,Q_N)\in\mathcal{M}_{s,f}(\mathcal{X}_1)\times\cdots\times\mathcal{M}_{s,f}(\mathcal{X}_N)\\ \sum_{i=1}^N Q_i(1)=1}} F(Q_1, \ldots, Q_N) \in \mathbf{R}, \tag{11}$$

such that

$$\rho(\mathbf{M}) = \sup_{\substack{(Q_1,\ldots,Q_N)\in\mathcal{M}_{s,f}(\mathcal{X}_i)\times\cdots\times\mathcal{M}_{s,f}(\mathcal{X}_N)\\ \sum_{i=1}^N Q_i(1)=1}} \left\{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N) - F(Q_1, \ldots, Q_N)\right\}, \tag{12}$$

for any $\mathbf{M} = (X_1, \ldots, X_N) \in \mathcal{X}^N$, where F can be chosen as

$$F_{\min}(Q_1, \ldots, Q_N) := \sup_{(X_1,\ldots,X_N)\in\mathcal{A}_\rho} \left\{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N)\right\}, \tag{13}$$

where

$$\mathcal{A}_\rho := \left\{\mathbf{M} \in \mathcal{X}^N : \rho(\mathbf{M}) \le 0\right\}. \tag{14}$$

Corollary 1 (representation result for coherent risk measures). *A function* $\rho : \mathcal{X}^N \longrightarrow \mathbf{R}$ *is a coherent risk measure if and only if there exists a subset*

$$\mathcal{D} \subset \left\{Q := (Q_1, \ldots, Q_N) \in \prod_{i=1}^N \mathcal{M}_{s,f}(\mathcal{X}_i) : \sum_{i=1}^N Q_i(1) = 1\right\}, \tag{15}$$

such that

$$\rho(\mathbf{M}) = \sup_{(Q_1,\ldots,Q_N)\in\mathcal{D}} \left\{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N)\right\}, \tag{16}$$

for any $\mathbf{M} = (X_1, \ldots, X_N) \in \mathcal{X}^N$.

Remark 3. A function $\rho : \mathcal{X}^N \longrightarrow \mathbf{R}$ is a coherent risk measure with the Fatou property if and only if there exists a subset

$$\mathcal{D} \subset \left\{Q := (Q_1, \ldots, Q_N) \in \prod_{i=1}^N \mathcal{M}_s(\mathcal{X}_i) : \sum_{i=1}^N Q_i(1) = 1\right\}, \tag{17}$$

such that

$$\rho(\mathbf{M}) = \sup_{(Q_1,\ldots,Q_N)\in\mathcal{D}} \left\{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N)\right\}, \tag{18}$$

for any $\mathbf{M} = (X_1, \ldots, X_N) \in \mathcal{X}^N$.

Now, we are in a position to state the main results of this paper, whose proofs will be provided in the next section.

Theorem 3 (representation result for quasi-concave acceptability indexes). *A map* $\alpha : \mathcal{X}^N \longrightarrow [0, +\infty]$ *is a quasi-concave acceptability index if and only if there exists an decreasing family* $(F_x)_{x\in(0,+\infty)}$ *of functionals such that*

$$\alpha(\mathbf{M}) = \sup\left\{x \in (0, +\infty) : \inf_{\substack{(Q_1,\ldots,Q_N)\in\mathcal{M}_{s,f}(\mathcal{X}_1)\times\cdots\times\mathcal{M}_{s,f}(\mathcal{X}_N)\\ \sum_{i=1}^N Q_i(1)=1}} \left\{E_{Q_1}(X_1) + \cdots + E_{Q_N}(X_N) + F_x(Q_1, \ldots, Q_N)\right\} \ge 0\right\}, \tag{19}$$

for any $\mathbf{M} \in \mathcal{X}^N$, *where* F *can be chosen as*

$$F_x(Q_1, \ldots, Q_N) := \sup_{\mathbf{M}\in\mathcal{X}^N, \alpha(\mathbf{M})\ge x} -\left\{E_{Q_1}(X_1) + \cdots + E_{Q_N}(X_N)\right\}, \tag{20}$$

and here we assume that $\inf\varnothing = \infty$ *and* $\sup\varnothing = 0$.

Corollary 2 (representation result for coherent acceptability indexes). *A map* $\alpha : \mathcal{X}^N \longrightarrow [0, +\infty]$ *is a coherent acceptability index if and only if there exists a family of subsets* $(\mathcal{D}_x)_{x\in(0,+\infty)}$ *of* $\left\{Q := (Q_1, \ldots, Q_N) \in \prod_{i=1}^N \mathcal{M}_{s,f}(\mathcal{X}_i) : \sum_{i=1}^N Q_i(1) = 1\right\}$ *with* $\mathcal{D}_x \subseteq \mathcal{D}_y$ *for* $0 < x \le y$, *such that*

$$\alpha(\mathbf{M}) = \sup\left\{x \in (0,+\infty) : \inf_{(Q_1,\ldots,Q_N)\in\mathscr{D}_x}\left\{E_{Q_1}(X_1)+\cdots+E_{Q_N}(X_N)\right\}\geq 0\right\}, \tag{21}$$

and here we assume that $\inf\varnothing = \infty$ *and* $\sup\varnothing = 0$.

Remark 4

(a) A map $\alpha : \mathscr{X}^N \longrightarrow [0,+\infty]$ is a quasi-concave acceptability index with the Fatou property if and only if there exists an decreasing family $(F_x)_{x\in(0,+\infty)}$ of functionals such that

$$\alpha(\mathbf{M}) = \sup\left\{x \in (0,+\infty) : \inf_{\substack{(Q_1,\ldots,Q_N)\in\mathscr{M}_s(\mathscr{X}_1)\times\cdots\times\mathscr{M}_s(\mathscr{X}_N)\\ \Sigma_{i=1}^N Q_i(1)=1}}\left\{E_{Q_1}(X_1)+\cdots+E_{Q_N}(X_N)+F_x(Q_1,\ldots,Q_N)\right\}\geq 0\right\}, \tag{22}$$

for any $\mathbf{M} \in \mathscr{X}^N$, where F can be chosen as

$$F_x(Q_1,\ldots,Q_N) := \sup_{\mathbf{M}\in\mathscr{X}^N,\alpha(\mathbf{M})\geq x} -\left\{E_{Q_1}(X_1)+\cdots+E_{Q_N}(X_N)\right\}, \tag{23}$$

and here we assume that $\inf\varnothing = \infty$ and $\sup\varnothing = 0$.

(b) A map $\alpha : \mathscr{X}^N \longrightarrow [0,+\infty]$ is a coherent acceptability index with the Fatou property if and only if there exists a family of subsets $(\mathscr{D}_x)_{x\in(0,+\infty)}$ of $\left\{Q := (Q_1,\ldots,Q_N) \in \prod_{i=1}^N \mathscr{M}_s(\mathscr{X}_i) : \sum_{i=1}^N Q_i(1) = 1\right\}$ with $\mathscr{D}_x \subseteq \mathscr{D}_y$ for $0 < x \leq y$, such that

$$\alpha(\mathbf{M}) = \sup\left\{x \in (0,+\infty) : \inf_{(Q_1,\ldots,Q_N)\in\mathscr{D}_x}\left\{E_{Q_1}(X_1)+\cdots+E_{Q_N}(X_N)\right\}\geq 0\right\}, \tag{24}$$

and here we assume that $\inf\varnothing = \infty$ and $\sup\varnothing = 0$.

(c) Let $N = 1$ and $\mathscr{X} = L^\infty(\Omega,\mathscr{F},P)$, and then representation results (22) and (24) are reduced to the one-dimensional case which coincides with the representation results of Rosazza Gianin and Sgarra (2013, Proposition 3) and Cherny and Madan (2009, Theorem 1), respectively.

Appendix

In the appendix, we will provide all the proofs of the results stated in Section 3.

Proof of Proposition 1

(1) Assume that $\alpha : \mathscr{X}^N \longrightarrow \mathbf{R}$ is a quasi-concave acceptability index, and then we will show that $(\rho_x)_{x\in(0,+\infty)}$ defined by (3) is an increasing family of convex risk measures.

Assume that $x, y \in \mathbf{R}$ with $0 < x \leq y$, and then for any $\mathbf{M} \in \mathscr{X}^N$, $\{m \in \mathbf{R} : \alpha(\mathbf{M}+m1)\geq x\} \supset \{m \in \mathbf{R} : \alpha(\mathbf{M}+m1)\geq y\}$, which implies $\rho_x(\mathbf{M}) \leq \rho_y(\mathbf{M})$.

Next we will show that ρ_x satisfies (R1), monotonicity. Let $\mathbf{M}_1, \mathbf{M}_2 \in \mathscr{X}^N$ with $\mathbf{M}_1 \leq \mathbf{M}_2$, and then from the monotonicity of α, we have

$$\begin{aligned}\rho_x(\mathbf{M}_1) &= \inf\{m \in \mathbf{R} : \alpha(\mathbf{M}_1+m1)\geq x\}\\ &\geq \inf\{m \in \mathbf{R} : \alpha(\mathbf{M}_2+m1)\geq x\}\\ &= \rho_x(\mathbf{M}_2).\end{aligned} \tag{A.1}$$

Now we will show ρ_x satisfies (R2), translation invariance. For any $\mathbf{M} \in \mathscr{X}^N$ and $h \in \mathbf{R}$, we have that

$$\begin{aligned}\rho_x(\mathbf{M}+h1) &= \sup\{m \in \mathbf{R} : \alpha(\mathbf{M}+(h+m)1)\geq x\}\\ &= \sup\{m-h \in \mathbf{R} : \alpha(\mathbf{M}+m1)\geq x\}\\ &= \rho_x(\mathbf{M})-h.\end{aligned} \tag{A.2}$$

Finally, we will show that ρ_x satisfies (R3), the convexity. For any $m_1 \in \{m \in \mathbf{R} : \alpha(\mathbf{M}_1+m1)\geq x\}$, $m_2 \in \{m \in \mathbf{R} : \alpha(\mathbf{M}_2+m1)\geq x\}$, and $\lambda \in [0,1]$, by the quasi-concavity of α, we have

$$\begin{aligned}&\alpha(\lambda\mathbf{M}_1+(1-\lambda)\mathbf{M}_2+\lambda m_1 1+(1-\lambda)m_2 1)\\&=\alpha(\lambda(\mathbf{M}_1+m_1 1)+(1-\lambda)(\mathbf{M}_2+m_2 1))\\&\geq\min\{\alpha(\mathbf{M}_1+m_1 1),\alpha(\mathbf{M}_2+m_2 1)\}\geq x,\end{aligned}\tag{A.3}$$

which implies $\lambda m_1+(1-\lambda)m_2\in\{m\in(0,+\infty):\alpha(\lambda\mathbf{M}_1+(1-\lambda)\mathbf{M}_2+m1)\geq x\}$, i.e., $\lambda m_1+(1-\lambda)m_2\geq\inf\{m\in(0,+\infty):\alpha(\lambda\mathbf{M}_1+(1-\lambda)\mathbf{M}_2+m1)\geq x\}$. Hence,

$$\lambda m_1+(1-\lambda)m_2\geq\rho_x(\lambda\mathbf{M}_1+(1-\lambda)\mathbf{M}_2).\tag{A.4}$$

By taking infimum in (A.4), first with respect to m_1 and then with respect to m_2, we have $\lambda\rho_x(\mathbf{M}_1)+(1-\lambda)\rho_x(\mathbf{M}_2)\geq\rho_x(\lambda\mathbf{M}_1+(1-\lambda)\mathbf{M}_2)$. Hence, $(\rho_x)_{x\in(0,+\infty)}$ is an increasing family of convex risk measures.

In particular, if α additionally satisfies scale invariance, then by the scaling invariant of α, we have

$$\begin{aligned}\rho_x(c\mathbf{M})&=\sup\{m\in\mathbf{R}:\alpha(c\mathbf{M}+m1)\geq x\}\\&=\sup\left\{m\in\mathbf{R}:\alpha\left(\mathbf{M}+\frac{m}{c}1\right)\geq x\right\}\\&=\sup\{cm\in\mathbf{R}:\alpha(\mathbf{M}+m1)\geq x\}\\&=c\rho_x(\mathbf{M}),\end{aligned}\tag{A.5}$$

for any $\mathbf{M}\in\mathcal{X}^N$ and $c>0$, which yields that ρ_x satisfies (R4), positive homogeneity.

(2) Assume that $(\rho_x)_{x\in(0,+\infty)}$ is an increasing family of convex risk measures. We will check that α defined by (4) is a quasi-concave acceptability index.

If $\mathbf{M}_1,\mathbf{M}_2\in\mathcal{X}^N$ are such that $\alpha(\mathbf{M}_1)\geq x_0$ and $\alpha(\mathbf{M}_2)\geq x_0$, then, by the definition (4) of α, and the monotonicity of ρ_x in x, we conclude that for any $x<x_0$,

$$\begin{aligned}\rho_x(\mathbf{M}_1)&\leq 0,\\\rho_x(\mathbf{M}_2)&\leq 0.\end{aligned}\tag{A.6}$$

By convexity of ρ_x, for any $\lambda\in[0,1]$ and $x<x_0$, we have

$$\rho_x(\lambda\mathbf{M}_1+(1-\lambda)\mathbf{M}_2)\leq\lambda\rho_x(\mathbf{M}_1)+(1-\lambda)\rho_x(\mathbf{M}_2)\leq 0.\tag{A.7}$$

Hence, $\sup\{x\in(0,+\infty):\rho_x(\lambda\mathbf{M}_1+(1-\lambda)\mathbf{M}_2)\leq 0\}\geq x_0$, and thus, by the definition (4) of α, we have $\alpha(\lambda\mathbf{M}_1+(1-\lambda)\mathbf{M}_2)\geq x_0$. This yields the quasi-concavity of α. The monotonicity is clear. Hence, α defined by (4) is a quasi-concave acceptability index.

In particular, if $(\rho_x)_{x\in(0,+\infty)}$ is an increasing family of coherent risk measures, then it is easy to check that α defined by (4) is scale invariant, which implies that it is a coherent acceptability index.

Proof of Theorem 1

(1) Assume that α is a quasi-concave (coherent, resp.) acceptability index. For every $x\in(0,+\infty)$, define ρ_x as follows:

$$\rho_x(\mathbf{M}):=\inf\{m\in\mathbf{R}:\alpha(\mathbf{M}+m1)\geq x\},\tag{A.8}$$

for any $\mathbf{M}\in\mathcal{X}^N$. By Proposition 1, $(\rho_x)_{x\in(0,+\infty)}$ is an increasing family of convex (coherent, resp.) risk measures. We will show that

$$\alpha(\mathbf{M})=\sup\{x\in(0,+\infty):\rho_x(\mathbf{M})\leq 0\},\tag{A.9}$$

for any $\mathbf{M}\in\mathcal{X}^N$. It is easy to see that, for any $\mathbf{M}\in\mathcal{X}^N$ and $x\in(0,+\infty)$, $\alpha(\mathbf{M})\geq x$ if and only if $\rho_x(\mathbf{M})\leq 0$. Hence,

$$\begin{aligned}\alpha(\mathbf{M})&=\sup\{x\in(0,+\infty):\alpha(\mathbf{M})\geq x\}\\&=\sup\{x\in(0,+\infty):\rho_x(\mathbf{M})\leq 0\}.\end{aligned}\tag{A.10}$$

(2) Assume that $(\rho_x)_{x\in(0,+\infty)}$ is an increasing family of convex (coherent, resp.) risk measures. Define the function α as follows:

$$\alpha(\mathbf{M}):=\sup\{x\in(0,+\infty):\rho_x(\mathbf{M})\leq 0\},\tag{A.11}$$

for all $\mathbf{M}\in\mathcal{X}^N$. By Proposition 1, α is a quasi-concave (coherent, resp.) acceptability index. Finally, it is easy to see that for any $\mathbf{M}\in\mathcal{X}^N$, $x\in(0,+\infty)$ and $m\in\mathbf{R}$, $\rho_x(\mathbf{M})\geq m$ if and only if $\alpha(\mathbf{M}+m1)\geq x$. In fact, if $\alpha(\mathbf{M}+m1)\geq x$, then $\sup\{y\in(0,+\infty):\rho_y(\mathbf{M}+m1)\leq 0\}\geq x$, i.e., $\sup\{y\in(0,+\infty):\rho_y(\mathbf{M})\leq m\}\geq x$, which, together with the property that $(\rho_x)_{x\in(0,+\infty)}$ is increasing with respect to x, implies that $\rho_x(\mathbf{M})\leq m$. On the other hand, if $\rho_x(\mathbf{M})\leq m$, then $\rho_x(\mathbf{M}+m1)\leq 0$. Thus, $\sup\{y\in(0,+\infty):\rho_y(\mathbf{M}+m1)\leq 0\}\geq x$, i.e., $\alpha(\mathbf{M}+m1)\geq x$.

Hence,

$$\begin{aligned}\rho_x(\mathbf{M}) &= \inf\{m \in \mathbf{R} : \rho_x(\mathbf{M}) \geq m\} \\ &= \inf\{m \in \mathbf{R} : \alpha(\mathbf{M} + m1) \geq x\}.\end{aligned} \tag{A.12}$$

The proof of Theorem 1 is completed.

Proof of Theorem 2. The sufficiency is obvious. Next, we prove the necessity. First, we show that

$$\rho(\mathbf{M}) \geq \sup_{\substack{(Q_1,\ldots,Q_N)\in\mathscr{M}_{s,f}(\mathscr{X}_1)\times\cdots\times\mathscr{M}_{s,f}(\mathscr{X}_N) \\ \sum_{i=1}^N Q_i(1)=1}} \left\{E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N) - F_{\min}(Q_1,\ldots,Q_N)\right\}, \tag{A.13}$$

for any $\mathbf{M} = (X_1,\ldots,X_N) \in \mathscr{X}^N$.

To this end, given $\mathbf{M} = (X_1,\ldots,X_N) \in \mathscr{X}^N$, note that $\tilde{M} = (\tilde{X}_1,\ldots,\tilde{X}_N) := \mathbf{M} + \rho(\mathbf{M})1 \in \mathscr{A}_\rho$. Thus, for any $(Q_1,\ldots,Q_N) \in \mathscr{M}_{s,f}(\mathscr{X}_1) \times \cdots \times \mathscr{M}_{s,f}(\mathscr{X}_N)$ with $\sum_{i=1}^N Q_i(1) = 1$, we have that

$$\begin{aligned}F_{\min}(Q_1,\ldots,Q_N) &\geq E_{Q_1}(-\tilde{X}_1) + \cdots + E_{Q_N}(-\tilde{X}_N) \\ &= E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N) \\ &\quad - \rho(\mathbf{M})\sum_{i=1}^N Q_i(1) \\ &= E_{Q_1}(-X_1) + \cdots + E_{Q_N}(-X_N) - \rho(\mathbf{M}),\end{aligned} \tag{A.14}$$

which implies (A.13).

For $\mathbf{M} = (X_1,\ldots,X_N)$ given, we will now construct $Q_{\mathbf{M}} := (Q_{1\mathbf{M}},\ldots,Q_{N\mathbf{M}}) \in \mathrm{M}_{s,f}(\mathrm{X}_1) \times \cdots \times \mathrm{M}_{s,f}(\mathrm{X}_N)$ with $\sum_{i=1}^N Q_i(1) = 1$, such that

$$\begin{aligned}\rho(\mathbf{M}) &\leq E_{Q_{1\mathbf{M}}}(-X_1) + \cdots + E_{Q_{N\mathbf{M}}}(-X_N) \\ &\quad - F_{\min}(Q_{1\mathbf{M}},\ldots,Q_{N\mathbf{M}}),\end{aligned} \tag{A.15}$$

which, together with (A.13), proves representation (8).

By translation invariance, it suffices to prove (A.15) for $\mathbf{M}_0 = (X_1^0,\ldots,X_N^0) \in \mathscr{X}^N$ with $\rho(\mathbf{M}_0) = 0$. Moreover, we may assume, without loss of generality, that $\rho(0) = 0$. Then, $\mathbf{M}_0$ is not contained in the nonempty convex set

$$\mathscr{B} := \left\{\mathbf{M} \in \mathscr{X}^N : \rho(\mathbf{M}) < 0\right\}. \tag{A.16}$$

Since $\mathscr{B}$ is a convex set, by the Hahn-Banach theorem, there exists a nonzero continuous linear functional λ (depending on $\mathbf{M}_0$) on $\mathscr{X}^N$ such that

$$\lambda(\mathbf{M}_0) \leq \inf_{\mathbf{M}\in\mathscr{B}} \lambda(\mathbf{M}) := b. \tag{A.17}$$

For $1 \leq i \leq N$, define

$$\lambda_i : \mathscr{X}_i \longrightarrow \mathbf{R}, \quad \lambda_i(X_i) := \lambda(X_i\mathbf{e_i}). \tag{A.18}$$

Then, obviously $\lambda_1,\ldots,\lambda_N$ are continuous linear functionals on $\mathscr{X}_1,\ldots,\mathscr{X}_N$, respectively, and satisfy

$$\lambda(\mathbf{M}) = \lambda(X_1\mathbf{e}_1 + \cdots + X_N\mathbf{e_N}) = \lambda_1(X_1) + \cdots + \lambda_N(X_N), \tag{A.19}$$

for any $\mathbf{M} = (X_1,\ldots,X_N) \in \mathscr{X}^N$. We further claim that λ and λ_i, $1 \leq i \leq N$, have the following two properties:

(1) $\lambda(\mathbf{M}) \geq 0$ for any $\mathbf{M} \geq 0$. Particularly, for $1 \leq i \leq N$, $\lambda_i(X_i) \geq 0$ for any $\mathscr{X}_i \ni X_i \geq 0$.

(2) $\lambda(1) > 0$.

First, we prove (1). For any $c > 0$ and any $\mathbf{M} \geq 0$, $c\mathbf{M} + 1 \in \mathscr{B}$. Hence,

$$\lambda(\mathbf{M}_0) \leq \lambda(c\mathbf{M} + 1) = c\lambda(\mathbf{M}) + \lambda(1), \quad \text{for any } c > 0, \tag{A.20}$$

which could not be true if $\lambda(\mathbf{M}) < 0$.

Next, we prove (2). Since λ is nonzero, there exists an $\mathbf{M}^* = (X_1^*,\ldots,X_N^*) \in \mathscr{X}^N$ such that $\lambda(\mathbf{M}^*) > 0$. Taking the truncation argument into account, with no loss of generality, we assume that $\|\mathbf{M}^*\| < 1$. Thus, $\|X_i^*\| < 1$, $1 \leq i \leq N$, where $\|X_i^*\|$ means that $\|X_i^*\| := \|X_i^*\|_\infty$ if $X_i^* \in \mathscr{X}_i = \mathscr{G}_i$, and $\|X_i^*\| := \|X_i^*\|_{p_i}$ if $X_i^* \in \mathscr{X}_i = \mathscr{H}_i$. Thus, $\lambda(1 - X_1^*,\ldots,1 - X_N^*) \geq 0$ by property (1). Hence, $\lambda(1) = \lambda(\mathbf{M}^* + 1 - \mathbf{M}^*) = \lambda(\mathbf{M}^*) + \lambda(1 - X_1^*,\cdots,1 - X_N^*) > 0$. Therefore, we can choose λ such that $\lambda(1) = 1$.

Consequently, $\lambda_1,\ldots,\lambda_N$ are continuous linear functionals on $\mathscr{X}_i,\ldots,\mathscr{X}_N$, respectively, and satisfy $\sum_{i=1}^N \lambda_i(1) = 1$.

By the preceding discussion and Theorem A.51 of Föllmer and Schied [22], we conclude that there exist $Q_{i\mathbf{M}_0} \in \mathscr{M}_{s,f}(\mathscr{X}_i)$, $1 \leq i \leq N$, such that

$$\begin{aligned}&\sum_{i=1}^N Q_i(1) = 1, \\ &E_{Q_{i\mathbf{M}_0}}(X_i) = \lambda_i(X_i), \quad \text{for any } (X_1,\ldots,X_N) \in \mathscr{X}^N.\end{aligned} \tag{A.21}$$

Note that $B \subset \mathrm{A}_\rho$, we know that

$$F_{\min}\left(Q_{1\mathbf{M}_0},\ldots,Q_{N\mathbf{M}_0}\right) = \sup_{(X_1,\ldots,X_N)\in\mathscr{A}_\rho}\left\{E_{Q_{1\mathbf{M}_0}}(-X_1)+\cdots+E_{Q_{N\mathbf{M}_0}}(-X_N)\right\} = \sup_{(X_1,\ldots,X_N)\in\mathscr{A}_\rho}\{\lambda_1(-X_1)+\cdots+\lambda_N(-X_N)\}$$
$$= \sup_{\mathbf{M}\in\mathscr{A}_\rho}\lambda(-\mathbf{M}) = -\inf_{\mathbf{M}\in\mathscr{A}_\rho}\lambda(\mathbf{M}) \geq -\inf_{\mathbf{M}\in\mathscr{B}}\lambda(\mathbf{M}) = -b. \tag{A.22}$$

On the other hand, $\mathbf{M}+\varepsilon 1\in\mathscr{B}$ for any $\mathbf{M}\in\mathscr{A}_\rho$ and each $\varepsilon>0$. This shows that

$$F_{\min}\left(Q_{1\mathbf{M}_0},\ldots,Q_{N\mathbf{M}_0}\right) = -b, \tag{A.23}$$

which, together with (A.17), implies that

$$E_{Q_{1\mathbf{M}_0}}\left(-X_1^0\right)+\cdots+E_{Q_{N\mathbf{M}_0}}\left(-X_N^0\right)-F_{\min}\left(Q_{1\mathbf{M}_0},\ldots,Q_{N\mathbf{M}_0}\right)$$
$$= b-\lambda(\mathbf{M}_0)\geq 0=\rho(\mathbf{M}_0). \tag{A.24}$$

Thus, $Q_{1\mathbf{M}_0},\ldots,Q_{N\mathbf{M}_0}$ are desired, and the proof of (8) is completed. Theorem 2 is proved.

Proof of Corollary 1. Assume that ρ is a coherent risk measure, and then $\mathscr{A}_\rho$ is a cone. Thus,

$$F_{\min}(Q_1,\ldots,Q_N) = \sup_{(X_1,\ldots,X_N)\in\mathscr{A}_\rho}\left\{E_{Q_1}(-X_1)+\cdots+E_{Q_N}(-X_N)\right\}$$
$$= \sup_{c(X_1,\ldots,X_N)\in\mathscr{A}_\rho} c\left\{E_{Q_1}(-X_1)+\cdots+E_{Q_N}(-X_N)\right\}$$
$$= cF_{\min}(Q_1,\ldots,Q_N), \tag{A.25}$$

for all $(Q_1,\ldots,Q_N)\in\{(Q_1,\ldots,Q_N)\in\mathscr{M}_{s,f}(\mathscr{X}_1)\times\mathscr{M}_{s,f}(\mathscr{X}_N):\sum_{i=1}^N Q_i(1)=1\}$ and $c>0$. Hence, $F_{\min}(Q_1,\ldots,Q_N)$ can take only the values 0 and $+\infty$. Set

$$\mathscr{D} := \left\{(Q_1,\ldots,Q_N)\in\mathscr{M}_{s,f}(\mathscr{X}_1)\times\mathscr{M}_{s,f}(\mathscr{X}_N):\sum_{i=1}^N Q_i(1)=1, F_{\min}(Q_1,\ldots,Q_N)=0\right\}, \tag{A.26}$$

and then, by Theorem 2, we have

$$\rho(\mathbf{M}) = \sup_{(Q_1,\ldots,Q_N)\in\mathscr{D}}\left\{E_{Q_1}(-X_1)+\cdots+E_{Q_N}(-X_N)\right\}. \tag{A.27}$$

The proof of Corollary 1 is completed.

Proof of Theorem 3. Assume that α is a quasi-concave acceptability index, and then by Theorem 1, we know that there exists an increasing family of convex risk measures $(\rho_x)_{x\in(0,+\infty)}$, such that (5) holds. By Theorem 2, for each ρ_x, $x\in(0,+\infty)$, there exists a function F_x such that (8) holds. Since $(\rho_x)_{x\in(0,+\infty)}$ is increasing, $(F_x)_{x\in(0,+\infty)}$ is decreasing, which, together with (5) and (8) yields (19). The proof of Theorem 3 is completed.

Proof of Corollary 2. By the same argument as in the proof of Theorem 3, and using Theorem 1 and Corollary 1, one can steadily show Corollary 2.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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