Hindawi
Mathematical Problems in Engineering
Volume 2019, Article ID 2680657, 6 pages
https://doi.org/10.1155/2019/2680657

Research Article

Convergence Rates in Homogenization of the Mixed Boundary Value Problems

Jie Zhao and Juan Wang

College of Science, Zhongyuan University of Technology, Zhengzhou 450007, China

Correspondence should be addressed to Jie Zhao; kaifengajie@163.com

Received 10 December 2018; Accepted 11 February 2019; Published 5 March 2019

Academic Editor: Weimin Han

We will study the convergence rates of solutions for homogenization of the mixed boundary value problems. By utilizing the smoothing operator as well as duality argument, we deal with the mixed boundary conditions in a uniform fashion. As a consequence, we establish the sharp rate of convergence in H^1 and L^2, with no smoothness assumption on the coefficients.

1. Introduction

Convergence rates estimates of solutions are one of the main questions in homogenization theory. There are many papers about convergence of solutions for elliptic homogenization problems. Assume that all of functions are smooth enough, the $O(\varepsilon)$ error estimate in L^∞ was presented by Bensoussan, Lions, and Papanicolaou [1]. In 1987, Avellaneda and Lin [2] proved L^p convergence by the method of maximum principle. At the same year, they [3] also obtained L^∞ error estimate when f is less regular than Bensoussan, Lions, and Papanicolaou's. Recently, there were many activities in the theory of homogenization with error estimates. In 2012, Kenig, Lin, and Shen [4] obtained convergence of solutions in L^2 and $H^{1/2}$ in Lipschitz domains with Dirichlet or Neumann boundary conditions. In 2014, they [5] have also studied the asymptotic behavior of the Green and Neumann functions obtaining some error estimates of solutions. In 2015, the first author [6] obtained the pointwise as well as $W^{1,p}$ convergence results, which is based on Fourier analysis. In 2016, Shen [7] proved the H^1 convergence rates with Dirichlet or Neumann conditions.

The problems of changing type of boundary conditions in homogenization have been studied extensively in various settings in the past years. In [8] the H^1 weak convergence of solutions was obtained in homogenization problems of multi-level-junction type. In 2011, Cardone [9] considered the homogenization with mixed boundary value problems in a thin periodically perforated plate and obtained the logarithmic rate of convergence of solutions. In the monograph [10], the H^1 convergence rate was shown by the method of potentials for the solutions to the Dirichler-Fourier mixed boundary value problem in the perforated domain. The L^2 convergence rate of Steklov-type problems was studied in [11]. In 2017, Shen [12] also obtained the L^2 convergence rate with Dirichlet-Neumann mixed boundary value problem.

In homogenization problems for the Poisson equations in a domain with oscillating boundary, the H^1 convergence rate have been studied in [13–15], while work [16] deals with the multilevel oscillation of the boundary with different conflicting boundary conditions. See also [17–19] for more results on the asymptotic behavior of eigenvalues for the boundary value problems in domains with oscillating boundaries or interfaces.

In this paper, we shall establish the sharp rates of convergence in H^1 and L^2 for oscillating operators with the mixed Dirichlet-Robin boundary condition. In particular, the L^2 estimate was proved by Griso in [20, 21] for Dirichlet or Neumann boundary conditions, using the periodic unfolding method. Our results, on one hand, extend the classical Laplace operator to oscillating operator; on the other hand they extend the classical boundary value problems to a broader mixed boundary conditions settings in homogenization. Meanwhile, our approach is utilizing the smoothing operator which is much more simple and direct than periodic unfolding method.

More precisely, let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$. Suppose that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two disjoint closed sets of $\partial\Omega$. Let $u_\varepsilon \in H^1(\Omega)$ be a weak solution to the following problem:

$$\begin{aligned} L_\varepsilon u_\varepsilon &= -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) \\ &= -\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j}\right) = f \quad \text{in } \Omega, \\ u_\varepsilon &= g \quad \text{on } \Gamma_1, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + k u_\varepsilon &= h \quad \text{on } \Gamma_2, \end{aligned} \tag{1}$$

where $k \geq 0$ is a number. Here $\partial u_\varepsilon/\partial\nu_\varepsilon = n_i a_{ij}(\partial u_\varepsilon/\partial x_j)$ denotes the conormal derivative with L_ε and $n(x)$ is the outward unit normal to $\partial\Omega$ at the point x.

Throughout this paper, the summation convention is used. We assume that the matrix $A(y) = (a_{ij}(y))$ with $1 \leq i, j \leq n$ is real symmetric and satisfies the ellipticity condition, i.e.,

$$\begin{aligned} &a_{ij}(y) = a_{ji}(y), \\ &\lambda|\xi|^2 \leq a_{ij}(y)\xi_i\xi_j \leq \frac{1}{\lambda}|\xi|^2, \\ &\qquad \text{for } y \in \mathbb{R}^n \text{ and } \xi = (\xi_i) \in \mathbb{R}^n, \end{aligned} \tag{2}$$

where $\lambda > 0$, and the periodicity condition

$$A(y+z) = A(y) \quad \text{for } y \in \mathbb{R}^n \text{ and } h \in \mathbb{Z}^n. \tag{3}$$

We impose the smoothness condition

$$\begin{aligned} &A(y) \text{ is bounded measurable}, \; f \in L^2(\Omega), \; g \\ &\in H^1(\partial\Omega), \; h \in H^{-1/2}(\partial\Omega). \end{aligned} \tag{4}$$

Without loss of generality, we also assume the compatibility condition

$$\int_{\partial\Omega} u_\varepsilon(x)\, d\sigma(x) = \int_\Omega f(x)\, dx + \int_{\Gamma_2} h(x)\, d\sigma(x) = 0. \tag{5}$$

Associated with (1) is the homogenized problem

$$\begin{aligned} L_0 u_0 &= -q_{ij}\frac{\partial^2 u_0}{\partial x_i \partial x_j} = f \quad \text{in } \Omega, \\ u_0 &= g \quad \text{on } \Gamma_1, \\ \frac{\partial u_0}{\partial \nu_0} + k u_0 &= h \quad \text{on } \Gamma_2, \end{aligned} \tag{6}$$

where the constant matrix q_{ij} is known as the homogenized matrix of $a_{ij}(y)$ and $\partial u_0/\partial\nu_0 = n_i q_{ij}(\partial u_0/\partial x_j)$.

Recall that u_ε is called the weak solution of (1), if for any $\phi \in H^1(\Omega)$, function $u_\varepsilon - g \in H^1(\Omega)$ holds

$$\begin{aligned} &\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla\phi\, dx + k\int_{\partial\Omega} u_\varepsilon \phi\, d\sigma(x) \\ &\quad = \int_\Omega f\phi\, dx + \langle h, \phi\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}. \end{aligned} \tag{7}$$

The existence and uniqueness of the weak solution to the mixed boundary value problem (1) follow from Lax-Milgram theorem. It is well known that the solution u_ε converges to u_0 weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, as $\varepsilon \longrightarrow 0$.

For the Dirichlet or Neumann boundary value problems, the regularity estimates of solutions in quantitative homogenization have been studied extensively. By the compactness method, interior and boundary Hölder's estimates, $W^{1,p}$ estimates, and Lipschitz estimates, the regularity of solutions for second-order elliptic systems or equations was established by Avellaneda, Kenig, Lin, Shen, and Suslina in a serious of papers [2, 3, 22–26]. For the case of homogenization with mixed boundary value problems, the uniform interior estimates and boundary Hölder's estimates have already been established in [27], and the sharp boundary regularity estimates have obtained in [28]. See also [29–32] for more related results on uniform regularity estimates.

The novelty of this paper lies in the fact that it deals with the mixed Dirichlet-Robin boundary condition which is a more general settings, for instance, in the case of Dirichlet problem when $\Gamma_2 = \emptyset$, in the case of Robin problem when $\Gamma_1 = \emptyset$, and for the Neumann problem when $\Gamma_1 = \emptyset$ and $k = 0$. As far as the author knows, very few convergence rates results are known for (1) of such mixed boundary value problems.

The following are the main results of this paper.

Theorem 1. *Suppose that $u_\varepsilon \in H^1(\Omega)$ and $u_0 \in H^2(\Omega)$ are the weak solutions of the mixed boundary value problems (1) and (6), respectively. Then, under the assumptions (2)–(5), there exists a constant C such that*

$$\left\|u_\varepsilon - u_0 - \varepsilon\chi T_\varepsilon(\nabla u_0)\right\|_{H^1(\Omega)} \leq C\varepsilon\left\|u_0\right\|_{H^2(\Omega)}, \tag{8}$$

where T_ε is the smoothing operator and χ is the solution of the cell problem.

Theorem 2. *Under the conditions as Theorem 1, then there exists a constant C such that*

$$\left\|u_\varepsilon - u_0\right\|_{L^2(\Omega)} \leq C\varepsilon\left\|u_0\right\|_{H^2(\Omega)}. \tag{9}$$

The rest of the paper is organized as follows. Section 2 contains some basic formulas and useful propositions which play important roles to get convergence rates. In Section 3, we show that the solution u_ε of partial differential equation with mixed boundary value problems $H^1(\Omega)$ and $L^2(\Omega)$ convergence to the solutions of the corresponding homogenized problems is based on using of smoothing operator. Finally, we summarize our results and discuss possible further development in Section 4.

2. Preliminaries

We begin by specifying our notations.

Let $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ denote an open ball with center x and radius r and $\widetilde{\Omega}_\varepsilon = \{x \in \mathbb{R}^n : dist(x, \partial\Omega) \le \varepsilon\}$. Since Ω is Lipschitz, then there exists a bounded extension operator $E : H^2(\Omega) \longrightarrow H^2(\mathbb{R}^n)$, such that $\widetilde{u}_0 = E(u_0)$ is an extension of u_0 and $\|\widetilde{u}_0\|_{H^2(\mathbb{R}^n)} \le C\|u_0\|_{H^2(\Omega)}$. We set $\varphi \in C_0^\infty(\mathbb{R}^n)$ to be a smooth function and $\|\varphi\|_{H^1(\mathbb{R}^n)} \le C\|\varphi\|_{H^1(\Omega)}$. We will also use C to denote positive constant which may vary in different formulas.

Associated with operator L_ε in (1), the homogenized operator is

$$L_0 = -q_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \quad in\ \Omega, \tag{10}$$

where L_0 is a constant coefficient operator which is also called homogenized operator. The constant matrix $Q = (q_{ij})$ is given by

$$q_{ij} = \int_Y \left[a_{ij}(y) + a_{ik}(y)\frac{\partial \chi^j(y)}{\partial y_k}\right] dy, \tag{11}$$

where $Y = [0,1)^n \simeq \mathbb{R}^n/\mathbb{Z}^n$. Function $\chi(y) = (\chi^j(y))$ is a solution of the following cell problem:

$$\begin{gathered} -\frac{\partial}{\partial y_i}\left[a_{ik}(y)\frac{\partial \chi^j(y)}{\partial y_k} + a_{ij}(y)\right] = 0, \quad in\ Y, \\ \chi^j(y+h) = \chi^j(y), \\ for\ y \in \mathbb{R}^n, h \in \mathbb{Y}^n, \\ \int_Y \chi^j(y)\,dy = 0, \end{gathered} \tag{12}$$

Fix $\psi \in C_0^\infty(B_1(0))$ such that $\psi \ge 0$ and $\int_{\mathbb{R}^n} \psi dx = 1$. Define operator T_ε on L^2 as

$$T_\varepsilon(u)(x) = u * \psi_\varepsilon = \int_{\mathbb{R}^n} u(x-y)\psi_\varepsilon(y)\,dy, \tag{13}$$

where $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$. We also call it the smoothing operator.

Proposition 3. *If $u_0 \in H^2(\mathbb{R}^n)$, then*

$$\|\nabla u_0 - T_\varepsilon(\nabla u_0)\|_{L^2(\mathbb{R}^n)} \le C\varepsilon \left\|\nabla^2 u_0\right\|_{L^2(\mathbb{R}^n)} \tag{14}$$

and

$$\left\|T_\varepsilon\left(\nabla^2 u_0\right)\right\|_{L^2(\mathbb{R}^n)} \le C\left\|\nabla^2 u_0\right\|_{L^2(\mathbb{R}^n)} \tag{15}$$

Proof. These estimates have proved by Parseval's Theorem and Hölder's inequality, which may be found in [7]. □

Proposition 4. *Let $F_{ij}(y) \in L^2(Y)$ be a periodic function, $Y = [0,1)^n$. Suppose that $\int_Y F_{ij}(y)dy = 0$ and $(\partial/\partial y_i)(F_{ij}(y)) = 0$. Then there exists $\Phi_{kij} \in H^1(Y)$ such that $F_{ij} = \partial\Phi_{kij}/\partial y_k$ and $\Phi_{kij} = -\Phi_{ikj}$.*

Proof. This proposition had been proved by Kenig, Lin, and Shen [4]. □

Remark 5. Let

$$F_{ij}(y) = q_{ij} - a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j(y)}{\partial y_k}. \tag{16}$$

Note that periodic function $F_{ij}(y)$ satisfies $\int_Y F_{ij}(y)dy = 0$ and $(\partial/\partial y_i)(F_{ij}) = 0$. It follows from Proposition 4 that there exists a function $\Phi_{kij}(y)$, such that $\Phi_{kij} = -\Phi_{ikj}$ and $F_{ij} = \partial\Phi_{kij}/\partial y_k$.

Remark 6. Under the assumption $A(y)$ is bounded measurable in $\mathbb{R}^n$, it is known that $\chi(y) \in C^{0,\alpha}(\mathbb{R}^n)$. This implies that $\Phi(y) \in C^{0,\alpha}(\mathbb{R}^n)$. In particular, $\|\chi\|_{L^\infty(\mathbb{R}^n)} + \|\Phi\|_{L^\infty(\mathbb{R}^n)} + \le C$.

Proposition 7. *If $u_0 \in H^2(\mathbb{R}^n)$, then*

$$\|T_\varepsilon(\nabla u_0)\|_{L^2(\widetilde{\Omega}_\varepsilon)} \le C\varepsilon^{1/2}\|u_0\|_{H^2(\mathbb{R}^n)}. \tag{17}$$

Proof. By Fubini's Theorem,

$$\begin{aligned} &\int_{\widetilde{\Omega}_\varepsilon} |T_\varepsilon(\nabla u_0)|^2\,dx \\ &\quad \le C\int_{\widetilde{\Omega}_\varepsilon}\int_{B_1(0)} |\nabla u_0(x-\varepsilon y)|^2\,dy\,dx \\ &\quad \le C\int_{B_1(0)} dy \int_{\widetilde{\Omega}_{2\varepsilon}} |\nabla u_0(x)|^2\,dx \le C\varepsilon\|u_0\|^2_{H^2(\mathbb{R}^n)}, \end{aligned} \tag{18}$$

where we have used the well-known estimate for the last inequality. See [33] or [12] for the proof. □

3. Proofs Theorems

The goal of this section is to establish H^1 and L^2 convergence rates of solutions.

Let

$$M_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi T_\varepsilon(\nabla\widetilde{u}_0). \tag{19}$$

In order to prove Theorem 1, it suffices to show that $\|M_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}$.

By the represented formula of M_ε, then M_ε satisfies the following boundary value problem:

$$\begin{gathered} L_\varepsilon M_\varepsilon = L_0 u_0 - L_\varepsilon u_0 - L_\varepsilon(\varepsilon\chi T_\varepsilon(\nabla\widetilde{u}_0)) \\ in\ \Omega, \\ M_\varepsilon = -\varepsilon\chi T_\varepsilon(\nabla\widetilde{u}_0) \quad on\ \Gamma_1, \\ \frac{\partial M_\varepsilon}{\partial\nu_\varepsilon} + kM_\varepsilon = \frac{\partial u_0}{\partial\nu_0} - \frac{\partial u_0}{\partial\nu_\varepsilon} - \frac{\partial(\varepsilon\chi T_\varepsilon(\nabla\widetilde{u}_0))}{\partial\nu_\varepsilon} \\ - k\varepsilon\chi T_\varepsilon(\nabla\widetilde{u}_0) \quad on\ \Gamma_2, \end{gathered} \tag{20}$$

where we have used (1) and (6) satisfied by u_ε and u_0, respectively.

In view of the fact that

$$\int_{\Omega} A_{\varepsilon}\nabla u_{\varepsilon}\cdot\nabla\varphi dx + k\int_{\partial\Omega} u_{\varepsilon}\varphi d\sigma(x) = \int_{\Omega} Q\nabla u_0\cdot\nabla\varphi dx + k\int_{\partial\Omega} u_0\varphi d\sigma(x), \tag{21}$$

we obtain

$$\begin{aligned}&\int_{\Omega} A_{\varepsilon}\nabla M_{\varepsilon}\cdot\nabla\varphi dx + k\int_{\partial\Omega} M_{\varepsilon}\varphi d\sigma(x)\\ &= \int_{\Omega}\left[Q\nabla u_0 - A_{\varepsilon}\nabla u_0 - \varepsilon A_{\varepsilon}\nabla\left(\chi T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\right)\right]\cdot\nabla\varphi dx\\ &\quad - k\int_{\partial\Omega}\varepsilon A_{\varepsilon}\chi T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\varphi d\sigma(x).\end{aligned} \tag{22}$$

It is easy to calculate that

$$\begin{aligned}&Q\nabla u_0 - A_{\varepsilon}\nabla u_0 - \varepsilon A_{\varepsilon}\nabla\left(\chi T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\right)\\ &= \left[Q\left(\nabla u_0 - T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\right) - A_{\varepsilon}\left(\nabla u_0 - T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\right)\right.\\ &\quad\left. - \varepsilon A_{\varepsilon}\chi T_{\varepsilon}\left(\nabla^2\tilde{u}_0\right)\right] + \left[Q - A_{\varepsilon} - A_{\varepsilon}\nabla\chi\right]T_{\varepsilon}\left(\nabla\tilde{u}_0\right).\end{aligned} \tag{23}$$

Then, it follows from the bilinear form that

$$\begin{aligned}&\int_{\Omega} A_{\varepsilon}\nabla M_{\varepsilon}\cdot\nabla\varphi dx + k\int_{\partial\Omega} M_{\varepsilon}\varphi d\sigma(x)\\ &= \int_{\Omega}\left[Q\left(\nabla u_0 - T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\right) - A_{\varepsilon}\left(\nabla u_0 - T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\right)\right.\\ &\quad\left. - \varepsilon A_{\varepsilon}\chi T_{\varepsilon}\left(\nabla^2\tilde{u}_0\right)\right]\cdot\nabla\varphi dx + \int_{\Omega}\left[Q - A_{\varepsilon} - A_{\varepsilon}\nabla\chi\right]\\ &\quad\cdot T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\cdot\nabla\varphi dx - k\int_{\partial\Omega}\varepsilon A_{\varepsilon}\chi T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\varphi d\sigma(x)\\ &\doteq I_1 + I_2 + I_3.\end{aligned} \tag{24}$$

To estimate I_1, we note that, by Proposition 3,

$$\left|I_1\right| \le C\varepsilon\left\|u_0\right\|_{H^2(\Omega)}\left\|\nabla\varphi\right\|_{L^2(\Omega)}. \tag{25}$$

Next, we shall estimate I_2. Let $F_{ij} = q_{ij} - a_{ij}(y) - a_{ik}(y)(\partial\chi^j(y)/\partial y_k)$. Note that F_{ij} is periodic and satisfies the conditions of Proposition 4. Then, in view of Remark 5, there exists a periodic function $\Phi_{kij} \in H^1(Y)$, such that $\Phi_{kij} = -\Phi_{ikj}$, and $q_{ij} - a_{ij}(y) - a_{ik}(y)(\partial\chi^j(y)/\partial y_k) = \partial\Phi_{kij}/\partial y_k$. Thus, by the divergence theorem, it gives

$$\begin{aligned}I_2 &= \int_{\Omega}\frac{\partial}{\partial x_k}\left(\varepsilon\Phi_{kij}\right)T_{\varepsilon}\left(\frac{\partial\tilde{u}_0}{\partial x_j}\right)\cdot\frac{\partial\varphi}{\partial x_i}dx\\ &= -\int_{\Omega}\varepsilon\Phi_{kij}T_{\varepsilon}\left(\frac{\partial^2\tilde{u}_0}{\partial x_k\partial x_j}\right)\frac{\partial\varphi}{\partial x_i}dx\\ &\quad - \int_{\Omega}\varepsilon\Phi_{kij}T_{\varepsilon}\left(\frac{\partial\tilde{u}_0}{\partial x_j}\right)\frac{\partial^2\varphi}{\partial x_k\partial x_i}dx\\ &= -\int_{\Omega}\varepsilon\Phi_{kij}T_{\varepsilon}\left(\frac{\partial^2\tilde{u}_0}{\partial x_k\partial x_j}\right)\frac{\partial\varphi}{\partial x_i}dx.\end{aligned} \tag{26}$$

Note that the second term vanishes in view of the antisymmetry of Φ_{kij}.

As a result, using Proposition 3 and Remark 6, we get that

$$\left|I_2\right| \le C\varepsilon\left\|u_0\right\|_{H^2(\Omega)}\left\|\nabla\varphi\right\|_{L^2(\Omega)}. \tag{27}$$

It remains to estimate I_3. It follows from Proposition 7 that

$$\begin{aligned}\left|I_3\right| &\le k\int_{\partial\Omega}\varepsilon\left|A_{\varepsilon}\chi T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\varphi\right| d\sigma(x)\\ &\le C\varepsilon\int_{\widetilde{\Omega}_{\varepsilon}}\left|T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\right|\cdot\left|\varphi\right| dx\\ &\le C\varepsilon\left\|T_{\varepsilon}\left(\nabla\tilde{u}_0\right)\right\|_{L^2(\widetilde{\Omega}_{\varepsilon})}\left\|\varphi\right\|_{L^2(\Omega_{\varepsilon})}\\ &\le C\varepsilon^{3/2}\left\|u_0\right\|_{H^2(\Omega)}\left\|\varphi\right\|_{L^2(\Omega_{\varepsilon})}.\end{aligned} \tag{28}$$

This, together with (25) and (27), gives that

$$\begin{aligned}&\left|\int_{\Omega} A_{\varepsilon}\nabla M_{\varepsilon}\cdot\nabla\varphi dx + k\int_{\partial\Omega} M_{\varepsilon}\varphi d\sigma(x)\right|\\ &\quad \le C\varepsilon\left\|u_0\right\|_{H^2(\Omega)}\left\|\nabla\varphi\right\|_{L^2(\Omega)}.\end{aligned} \tag{29}$$

By the coercive condition of bilinear form and duality argument, we get the desired result, which completes the proof of Theorem 1.

It follows from Theorem 1 and Proposition 3, by Minkowski's inequality, that

$$\begin{aligned}\left\|u_{\varepsilon} - u_0\right\|_{L^2(\Omega)} &\le C\varepsilon\left\|u_0\right\|_{H^2(\Omega)} + \left\|\varepsilon\chi T_{\varepsilon}\left(\nabla u_0\right)\right\|_{L^2(\Omega)}\\ &\le C\varepsilon\left\|u_0\right\|_{H^2(\Omega)}.\end{aligned} \tag{30}$$

This completes the proof of Theorem 2.

4. Conclusions and Perspectives

In this paper, we research the convergence rates of solutions for homogenization of the mixed Dirichlet-Robin boundary value problems. Our approach is utilizing the smoothing operator, which is much more simple and direct to deal with boundary discrepancies. As a consequence, we obtain the H^1 and L^2 convergence rates results, which extend the classical boundary value problems to a broader mixed boundary condition settings.

Indeed, it is expected that one could obtain the $W^{1,p}$ convergence rates, for any $1 \le p < \infty$. To the best of our knowledge, such estimates for the mixed boundary value problems in homogenization have not been reported so far in the literature. Hence, how to utilize the smoothing operator and avoid difficult of the terms from boundary discrepancies for such problems are an interesting problem. This is one further possible direction to be developed.

Generally, many other types of equations with the mixed boundary conditions settings could be considered by this method. One may naturally try to extend the classical second-order equations to $2m$-order higher-order equations or nonlinear elliptic equations. It is expected that the method of this work could contribute to a better solving of the mixed boundary value problem in homogenization.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work has been supported by National Natural Science Foundation of China (No. 11626239), China Scholarship Council (No. 201708410483), and the Education Department of Henan Province (No. 18A110037). A part of this work was done while the first author was visiting school of mathematics and applied statistics, University of Wollongong, Australia.

References

[1] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, 1978.

[2] M. Avellaneda and F.-H. Lin, "Homogenization of elliptic problems with L^p boundary data," *Applied Mathematics & Optimization*, vol. 15, no. 2, pp. 93–107, 1987.

[3] M. Avellaneda and F.-H. Lin, "Compactness methods in the theory of homogenization," *Communications on Pure and Applied Mathematics*, vol. 40, no. 6, pp. 803–847, 1987.

[4] C. E. Kenig, F. Lin, and Z. Shen, "Convergence rates in L^2 for elliptic homogenization problems," *Archive for Rational Mechanics and Analysis*, vol. 203, no. 3, pp. 1009–1036, 2012.

[5] C. E. Kenig, F. Lin, and Z. Shen, "Periodic homogenization of Green and Neumann functions," *Communications on Pure and Applied Mathematics*, vol. 67, no. 8, pp. 1219–1262, 2014.

[6] J. Zhao, "Homogenization of the boundary value for the Neumann problem," *Journal of Mathematical Physics*, vol. 56, no. 2, Article ID 021508, 9 pages, 2015.

[7] Z. Shen, "Boundary estimates in elliptic homogenization," *Analysis & PDE*, vol. 10, no. 3, pp. 653–694, 2017.

[8] T. A. Melnik and G. A. Chechkin, "Asymptotic analysis of boundary value problems in thick three-dimensional multilevel junctions," *Sbornik: Mathematics*, vol. 200, no. 3, pp. 49–74, 2009.

[9] G. Cardone, S. A. Nazarov, and A. L. Piatnitski, "On the rate of convergence for perforated plates with a small interior Dirichlet zone," *Zeitschrift für angewandte Mathematik und Physik ZAMP*, vol. 62, no. 3, pp. 439–468, 2011.

[10] G. A. Chechkin, *Topics on Concentration Phenomena and Problems with Multiple Scales*, vol. 2, Springer, Berlin, Germany, 2006.

[11] Y. Amirat, O. Bodart, G. A. Chechkin, and A. L. Piatnitski, "Asymptotics of a spectral-sieve problem," *Journal of Mathematical Analysis and Applications*, vol. 435, no. 2, pp. 1652–1671, 2016.

[12] Z. Shen and J. Zhuge, "Convergence rates in periodic homogenization of systems of elasticity," *Proceedings of the American Mathematical Society*, vol. 145, no. 3, pp. 1187–1202, 2017.

[13] Y. Amirat, O. Bodart, G. A. Chechkin, and A. L. Piatnitski, "Boundary homogenization in domains with randomly oscillating boundary," *Stochastic Processes and Their Applications*, vol. 121, no. 1, pp. 1–23, 2011.

[14] G. A. Chechkin and T. P. Chechkina, "On homogenization of problems in domains of the "infusorium" type," *Journal of Mathematical Sciences*, vol. 120, pp. 1470–1482, 2003.

[15] Y. Amirat, O. Bodart, U. De Maio, and A. Gaudiello, "Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary," *SIAM Journal on Mathematical Analysis*, vol. 35, no. 6, pp. 1598–1616, 2004.

[16] G. A. Chechkin, C. D'Apice, and U. De Maio, "On the rate of convergence of solutions in domain with periodic multilevel oscillating boundary," *Mathematical Methods in the Applied Sciences*, vol. 33, no. 17, pp. 2019–2036, 2010.

[17] G. Griso, "On the spectrum of deformations of compact double-sided flat hypersurfaces," *Analysis & PDE*, vol. 6, no. 5, pp. 1051–1088, 2013.

[18] Y. Amirat, G. A. Chechkin, and R. Gadyl'shin, "Spectral boundary homogenization in domains with oscillating boundaries," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 6, pp. 4492–4499, 2010.

[19] M. Lobo, S. A. Nazarov, and E. Perez, "Eigen-oscillations of contrasting non-homogeneous elastic bodies: asymptotic and uniform estimates for eigenvalues," *IMA Journal of Applied Mathematics*, vol. 70, no. 3, pp. 419–458, 2005.

[20] G. Griso, "Error estimate and unfolding for periodic homogenization," *Asymptotic Analysis*, vol. 40, no. 3-4, pp. 269–286, 2004.

[21] G. Griso, "Interior error estimate for periodic homogenization," *Analysis and Applications*, vol. 4, no. 1, pp. 61–79, 2006.

[22] M. Avellaneda and F.-H. Lin, "Compactness methods in the theory of homogenization. II. Equations in nondivergence form," *Communications on Pure and Applied Mathematics*, vol. 42, no. 2, pp. 139–172, 1989.

[23] M. Avellaneda and F.-H. Lin, "L^p bounds on singular integrals in homogenization," *Communications on Pure and Applied Mathematics*, vol. 44, no. 8-9, pp. 897–910, 1991.

[24] C. E. Kenig, F. Lin, and Z. Shen, "Homogenization of elliptic systems with Neumann boundary conditions," *Journal of the American Mathematical Society*, vol. 26, no. 4, pp. 901–937, 2013.

[25] T. A. Suslina, "Homogenization of the Dirichlet problem for elliptic systems: L_2-operator error estimates," *Mathematika. A Journal of Pure and Applied Mathematics*, vol. 59, no. 2, pp. 463–476, 2013.

[26] T. Suslina, "Homogenization of the Neumann problem for elliptic systems with periodic coefficients," *SIAM Journal on Mathematical Analysis*, vol. 45, no. 6, pp. 3453–3493, 2013.

[27] S. Gu and Z. Shen, "Homogenization of stokes systems and uniform regularity estimates," *SIAM Journal on Mathematical Analysis*, vol. 47, no. 5, pp. 4025–4057, 2015.

[28] S. Gu and Q. Xu, "Optimal boundary estimates for Stokes systems in homogenization theory," *SIAM Journal on Mathematical Analysis*, vol. 49, no. 5, pp. 3831–3853, 2017.

[29] J. Geng, "$W^{1,p}$ estimates for elliptic problems with Neumann boundary conditions in Lipschitz domains," *Advances in Mathematics*, vol. 229, no. 4, pp. 2427–2448, 2012.

[30] J. Geng, Z. Shen, and L. Song, "Uniform $W^{1,p}$ estimates for systems of linear elasticity in a periodic medium," *Journal of Functional Analysis*, vol. 262, no. 4, pp. 1742–1758, 2012.

[31] S. N. Armstrong and Z. Shen, "Lipschitz estimates in almost-periodic homogenization," *Communications on Pure and Applied Mathematics*, vol. 69, no. 10, pp. 1882–1923, 2016.

[32] S. N. Armstrong and C. K. Smart, “Quantitative stochastic homogenization of convex integral functionals,” *Annales Scientifiques de l’Ecole Normale Superieure*, vol. 49, no. 2, pp. 423–481, 2016.

[33] M. A. Pakhnin and T. A. Suslina, “Operator error estimates for the homogenization of the elliptic Dirichlet problem in a bounded domain,” *Algebra i Analiz*, vol. 24, no. 6, pp. 139–177, 2012.