Analytic Hierarchy Process by Least Square Method Revisit

Scott Shu-Cheng Lin

Department of Hotel Management, Lee-Ming Institute of Technology, Taiwan

Correspondence should be addressed to Scott Shu-Cheng Lin; scottsclin@mail.lit.edu.tw

Received 2 December 2018; Revised 9 February 2019; Accepted 19 March 2019; Published 3 April 2019

1. Introduction

Since AHP is widely accepted as a useful instrument for decision-making, the study of eliciting and evaluating subjective judgments becomes an interesting research topic. Saaty sets the criterion of transitivity for a consistent judgment regarding several attributes; i.e., a comparison matrix is said to be consistent if \( a_{ij} a_{ik} = a_{jk} \) for \( i, j, k = 1, 2, \ldots, n \). The eigenvector of the matrix denotes the ratio of comparison among the attributes. But a strictly consistent comparison matrix is difficult to obtain in the practice. Saaty sets the index of 0.1 for assessing the consistency ratio of a comparison matrix. Many researches were made related to this problem: Sajjad [1] studied how such hesitations can be considered in the environment of AHP and how the resulting hesitations for relative priorities of alternatives can be estimated. Several important properties of the Perron eigenvalue for a positive reciprocal matrix which can be used in the study of the AHP were examined by Aupetit and Genest [2], Vargas [3], Tummala and Wan [4], and Chen and Chu [5] have tried to verify the consistency ratio of 0.1 in assessing attribute weights.

Given a positive, reciprocal matrix \( A = [a_{ij}] \) with \( a_{ij} > 0 \) and \( a_{ij} a_{ji} = 1 \), we want to estimate the underlying ratio \( (a_i / a_j) \) from the matrix \( A \), where \( a_i \) is the weight for alternative \( i \). Several different methods for synthesizing the information contained in the matrix \( A \) have been suggested to find the best estimation, say \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \). Except the eigenvalue method, for example, Saaty [6], Saaty and Vargas [7], and Vargas [3, 8], there are the least square method, logarithmic least square method, respectively. We prove that the prediction of Saaty and Vargas is valid. Our result will provide a patch work for the theoretic foundation for Analytic Hierarchy Process.
positive, reciprocal three by three matrix, the solution for the eigenvalue method is equal to the solution for the logarithmic least square method, as Theorem 3 of Saaty and Vargas [7]. Therefore, they ascertained that \( w = u \).

As far as the solution of the least square method is concerned, they proclaimed that the solution is approximately given by \( w = (1/3, 1/3, 1/3)^T \) without giving further explanation since the LSM solution was not always unique. We will prove that, for this special problem, the LSM solution is unique and equals \( (1/3, 1/3, 1/3)^T \).

There are several papers that are worthy to discuss to show the development for analytical hierarchy process. Mustafa and Al-Bahar [15] used Analytic Hierarchy Process to assess project risks for a real problem of building the Jamuna Multipurpose Bridge in Bangladesh. Handfield et al. [16] applied Analytic Network Process for supplier assessment with environmental issues. Xiong et al. [17] discussed selection and valuation of numeric scale in Analytic Hierarchy Process.

2. Our Mathematical Formulation

Since \( v = (v_1, v_2, v_3)^T \) with \( v_1, v_2, \) and \( v_3 \) is derived by minimizing

\[
\sum_{j=1}^{3} \left( a_{ij} - \frac{v_i}{v_j} \right)^2
\]

under the constraint of \( v_1 + v_2 + v_3 = 1 \).

Combine (1) and (2); if we let \( x = (u_j/u_i) \) and \( y = (u_j/u_i) \), then \( xy = (u_j/u_i) \); hence we need to solve the following minimum problem, for \( x > 0 \) and \( y > 0 \),

\[
f(x, y) = (2 - x)^2 + \left( \frac{1}{2} - \frac{1}{y} \right)^2 + \left( \frac{1}{2} - \frac{1}{x} \right)^2 \]

\[
+ \left( 2 - \frac{1}{xy} \right)^2 + (2 - y)^2 + \left( \frac{1}{2} - xy \right)^2.
\]

Lemma 1. The minimum problem of \( f(x, y) \), for \( x > 0 \) and \( y > 0 \), has solutions.

Proof. First we arbitrarily pick a point; for example, we choose (2, 2) and then \( f(2, 2) = 245/16 \). We have \( \lim_{x \to \infty} f(x, y) = \infty \), for any \( y > 0 \) and \( \lim_{y \to \infty} f(x, y) = \infty \), for any \( x > 0 \). We obtain \( \lim_{x \to \infty} f(x, y) = \infty \), for any \( y > 0 \) and \( \lim_{y \to \infty} f(x, y) = \infty \), for any \( x > 0 \). Hence we know that there exists an \( \epsilon > 0 \) and \( M > 0 \) with \( \epsilon < 2 < M \) such that \( f(x, y) > f(2, 2) \), for \( (x, y) \) in the following restricted subdomain,

\[
\{(x, y) : x < \epsilon, \text{ or } y < \epsilon, \text{ or } x > M, \text{ or } y > M\}.
\]

Hence we obtain that

\[
\min_{x,y \geq 0} f(x, y) = \min_{x,y \leq M} f(x, y)
\]

Owing to the fact that \( \{(x, y) : \epsilon \leq x, y \leq M\} \) is a compact set, the continuous function \( f(x, y) \) attains its minimum. Consequently, for the original domain,

\[
\{(x, y) : x > 0, y > 0\}, \quad \min_{x,y \leq M} f(x, y)
\]

If we take the first partial derivatives of \( f(x, y) \), then we induce that

\[
\frac{df}{dx} = \frac{1}{x^3 y^2} \left( 2x^4 y^4 + 2x^3 y^5 + x y^2 + 4 x y - x^3 y^3 \right)
\]

\[
-4x^3 y^2 - 2y^2 - 2 \right)
\]

and

\[
\frac{df}{dy} = \frac{1}{x^2 y^3} \left( 2x^4 y^4 + 2x^2 y^5 + x^3 y + 4 x y - x^3 y^3 \right)
\]

\[
-4x^3 y^2 - 2y^2 - 2 \right).
\]

If we solve the simultaneous system \( \frac{df}{dx} = 0 \) and \( \frac{df}{dy} = 0 \), then it follows

\[
2x^4 y^2 + x y^2 - 4x^3 y^2 - 2y^2
\]

\[
= 2x^2 y^4 + x^2 y - 4x^2 y^3 - 2x^2,
\]

which means

\[
(y - 2) x^2 (2y^3 + 1) = (x - 2) y^2 (2x^3 + 1).
\]

Motivated by (10), we study the following auxiliary function, \( g(t) \): for \( t > 0 \),

\[
g(t) = \frac{1}{t^2} (t^2 - 2) (2t^3 + 1).
\]

Lemma 2. \( g(t) \) is a strictly increasing function, for \( t > 0 \).

Proof. Since

\[
\frac{dg}{dt} = \frac{1}{t^3} (4t^4 - 4t^3 - t + 4),
\]

we assume an auxiliary function, say \( p(t) \), with

\[
p(t) = 4t^4 - 4t^3 - t + 4,
\]

for \( t > 0 \). We will show that \( p(t) \) is a positive function.

From \( dp(t)/dt = 16t^3 - 12t^2 - 1 \) and \( d^2 p(t)/dt^2 = 48t^2 - 24t \), we have that \( p(t) \) is concave down on \((0, 1/2)\) and concave up on \((1/2, \infty)\). Let \( dp(t)/dt = k(t) \), with

\[
k(t) = 16t^3 - 12t^2 - 1,
\]
Recall (9); we find that $\frac{\partial f}{\partial y} = 0$, under the restriction $x = y$, and then
\[
2x^8 + 2x^6 + x^3 + 4x^2 = x^6 + 4x^5 + 2x^2 + 2. \tag{17}
\]
By (17), a new auxiliary function, $h(x) = 2x^8 + 2x^6 + x^3 + 2x^5 - 2$, for $x > 0$ is examined by us.

**Lemma 3.** $h(x)$ has a unique root at $x = 1$, for $x > 0$.

**Proof.** We factor $h(x)$ as
\[
h(x) = (x - 1) \left(2x^7 + 2x^5 + 3x^4 - x^3 + 2x + 2\right). \tag{18}
\]
If $0 < x < 1$, then $2 > x^4 + x^3$. On the other hand, if $x > 1$, then $3x^5 > x^4 + x^3$. From the above observation, we derive that $h(x) = 0$ has a unique solution at $x = 1$, for $x > 0$. $\blacksquare$

From Lemma 3 and (17), we obtain that $\frac{\partial f}{\partial x} = 0$, with $x = y$, only occurs at $x = 1$, and hence the solution for the simultaneous system $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ only happens at $(1, 1)$. Usually, the solutions for the simultaneous system $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are the candidates for the local extreme points. By recalling Lemma 1, $(1, 1)$ is the absolute minimum point; that means the solution of (2) occurs at $v_1 = v_2 = v_3$. Finally, we normalize it, so $v = (1/3, 1/3, 1/3)^T$ which coincides with the optimal solution of $w$ and $u$, for EM and LLSM, respectively.

### 3. Numerical Example

We cited a numerical example from Mustafa and Al-Bahar [15] for the comparison matrix in Table 1 to assess the risk of constructing a bridge project. The relative weights (relative importance) are computed by the eigenvalue method by Mustafa and Al-Bahar [15]. For easy comparison, we also list the relative weights by the least square method in the sixth column of Table 1.

Based on our numerical computation, the results from (a) the eigenvalue method and (b) the least square method are different.

In the following, we will show that the least square method should be superior to the eigenvalue method. Let us recall the motivation for Saaty [6] to develop Analytic Hierarchy Process that is first to consider consistent comparison matrices. For those consistent comparison matrices, say $(a_{ij})_{n \times n}$, the normalized eigenvector, say $(v_{ij})_{n \times 1}$, corresponding the maximum eigenvalue, say $\lambda_{\text{max}}$, satisfying
\[
(a_{ij})_{n \times n} (v_{ij})_{n \times 1} = \lambda_{\text{max}} (v_{ij})_{n \times 1}, \tag{19}
\]
under the restriction $\sum_{i=1}^{n} v_{ij} = 1$. 

### Table 1: Reproduction of Table 1 of Mustafa and Al-Bahar [15].

<table>
<thead>
<tr>
<th>With respect to goal</th>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>Relative importance by EM</th>
<th>Relative importance by LLSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>0.635</td>
<td>0.540</td>
</tr>
<tr>
<td>$F_2$</td>
<td>1/3</td>
<td>1</td>
<td>5</td>
<td>0.287</td>
<td>0.375</td>
</tr>
<tr>
<td>$F_3$</td>
<td>1/6</td>
<td>1/5</td>
<td>1</td>
<td>0.078</td>
<td>0.085</td>
</tr>
</tbody>
</table>
Saaty [6] mentioned that the normalized eigenvector satisfied

\[ a_{ij} = \frac{v_i}{v_j} \]  

(20)

for \( i, j = 1, 2, \ldots, n \) which is consistent with our institution. Consequently, for any comparison matrices, Saaty [6] adopted the eigenvalue method to decide relative weights for the given comparison matrix.

According to (20), researchers should directly consider the minimum problem of

\[ \sum_{i,j=1,2,\ldots,n} \left( a_{ij} - \left( \frac{v_i}{v_j} \right) \right)^2 \]  

(21)

to use its normalized solution as the relative weights, which is the least square method. Hence, we claim that the least square method should be adopted for future research to derive relative weights for a given comparison matrix.

4. Direction for Future Research

We envision that it will be an interesting research topic in next step to study a real-world problem by Analytic Hierarchy Process with three different methods: EM, LLSM, and LSM. For example, Garg et al. [20] studied the water content of Ivy tree and Bermuda grass for calculating the performance of vegetation growth and controlled manual irrigation with data analytics. By following Garg et al. [20] and examining real world data, we would be able to reveal more essential properties of EM, LLSM, and LSM pertaining to their similarity and difference.

5. Conclusion

We complete the verification of the equivalence for the solutions such that three methods, the eigenvalue, logarithmic least square, and least square methods, can induce the same solution for this inconsistent comparison matrix proposed by Saaty and Vargas. The second level of interpretation of our work can be highlighted use of data analytics such as investigation of infiltration rate for soil-biochar composites of water hyacinth as Handfield et al. [19] and Ishizaka and Labib [19].

Data Availability

The source of the discussed matrix is cited from Saaty and Vargas (1984) that had been clearly indicated in the manuscript.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research is partially supported by Ministry of Science and Technology, with Grant no. MOST 107-2410-H-241-001. The English language of the paper was revised by Jason Chou (ts883088@hotmail.com).

References


