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Research Article

A Generalized Cubic Exponential B-Spline Scheme with Shape Control

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In this paper, a generalized cubic exponential B-spline scheme is presented, which can generate different kinds of curves, including the conics. Such a scheme is obtained by generalizing the cubic exponential B-spline scheme based on an iteration from the generation of exponential polynomials and a suitable function with two parameters μ and ν. By changing the values of μ and ν, the sensitivity of the shape of the subdivision curve to the initial control value ν^0 can be changed and different kinds of curves can then be obtained by adjusting the value of ν^0. For this new scheme, we show that, with any admissible choice of μ and ν, it owns the same smoothness order and support as the cubic exponential B-spline scheme. Besides, based on a different iteration and another suitable function, we construct a similar nonstationary scheme to generate more curves with different shapes and show the role of iterations and suitably chosen functions in the construction and analysis of such schemes. Several examples are given to illustrate the performance of our new schemes.

1. Introduction

Subdivision schemes are efficient tools to generate smooth curves/surfaces from a set of initial control points and they play an important role in computer graphics, wavelets, and other fields like biomedical imaging [1]. According to whether the refinement rules depend on the recursion level, subdivision schemes can be divided into stationary and nonstationary ones. Stationary schemes can generate algebraic polynomials while nonstationary ones can generate richer function spaces, such as the exponential polynomial spaces. As a result, nonstationary schemes can generate curves like hyperbolas or surfaces like spheres and other ones with different shapes, which can not be done using stationary schemes.

Since the nonstationary schemes can generate richer function spaces and more kinds of curves, there have been continuous works on the construction and analysis of nonstationary schemes. The exponential B-spline schemes [2] are such typical examples, which can generate exponential polynomials. Besides, Conti & Romani [3] gave conditions on the symbols of nonstationary schemes to reproduce exponential polynomials. Siddiqi et al. [4] presented ternary nonstationary schemes generating hyperbolas and parabolas. Zheng & Zhang [5] applied the push-back operation in the nonstationary case and constructed a combined nonstationary scheme generating different exponential polynomials. Asghar & Mustafa [6] constructed p-ary nonstationary schemes which are new versions of the Lane-Riesenfeld algorithms. For other nonstationary schemes generating exponential polynomials, see [7–9] and the references therein. In fact, there are other nonstationary schemes generating curves with different kinds of shapes. Beccari et al. [10] proposed a ternary 4-point nonstationary interpolatory scheme, whose nonstationarity can be seen as the result of an iteration, and this scheme can generate curves with considerable variations of shapes. Similarly, Tan et al. [11] proposed a 3-point nonstationary approximating subdivision, which can be seen as constructed based on a different iteration and can also generate a wide variety of curves.

Due to nonstationary schemes's ability in curve design, as illustrated above, in this paper, we aim to propose a

new nonstationary subdivision scheme, which can generate curves with considerable variations of shapes and exponential polynomials as well. The inspiration comes from the works in [10, 11]. In fact, we start from the C^2-convergent cubic exponential B-spline scheme [12], which generates the conics. To our purpose, we see the cubic exponential B-spline scheme as obtained based on an iteration coming from the generation of exponential polynomials and treat the coefficients of the subdivision rules as functions of this iteration. Then, together with a function with two parameters μ and ν, we can obtain a generalized cubic exponential B-spline scheme, which is just the one we want. For this new scheme, by changing the values of μ and ν, we can change the sensitivity of the shape of the obtained curve to the initial control value ν^0, and different kinds of curves, including the conics, can then be obtained by adjusting ν^0. We point out that compared with the cubic exponential B-spline scheme, this newly obtained one can generate curves with more kinds of shapes and, compared with the schemes in [10, 11], this new one enjoys the advantages like shorter support and generation of exponential polynomials. For such a new scheme, we show that, with any admissible choice of μ and ν, it keeps the same smoothness order and the support as the cubic exponential B-spline scheme. Besides, based on a different iteration and another suitable function, we also present a similar nonstationary scheme to generate curves with more kinds of shapes. This also shows the role of iterations and suitably chosen functions in the construction and analysis of such schemes. The performance of our schemes is illustrated by several numerical examples.

The rest of this paper is organized as follows. In Section 2, we review some known definitions and results about subdivision schemes and iterations. Section 3 is devoted to the construction and analysis of the generalized cubic exponential B-spline scheme. In Section 4, we present several examples and compare the new scheme with several existing nonstationary ones. In Section 5, we move a further step and construct a similar scheme based on a different iteration and a different suitable function. Section 6 concludes this paper.

2. Preliminary

In this section, we recall some basic definitions and known results about subdivision schemes and iterations to form the basis of the rest of this paper. Let $l_0(\mathbb{Z})$ denote the linear space of real sequences with finite support. For a sequence $\boldsymbol{\lambda} \in l_0(\mathbb{Z})$, its support is the finite set $\{i \in \mathbb{Z}, \lambda_i \neq 0\}$. Given an initial data sequence $\mathbf{q}^0 = \{q_i^0, i \in \mathbb{Z}\} \in l_0(\mathbb{Z})$, we consider the nonstationary subdivision scheme

$$\left(\mathbf{q}^{k+1}\right)_i := \left(S_{\mathbf{a}^k}\mathbf{q}^k\right)_i = \sum_{j\in\mathbb{Z}} a_{i-2j}^k q_j^k, \quad i \in \mathbb{Z}, \tag{1}$$

where $S_{\mathbf{a}^k}$ is the k-level subdivision operator and the sequence $\mathbf{a}^k = \{a_i^k, i \in \mathbb{Z}\}$ is the k-level mask with finite support. We denote this scheme by $\{S_{\mathbf{a}^k}\}_{k\geq 0}$ and the so-called k-level symbol of the scheme $\{S_{\mathbf{a}^k}\}_{k\geq 0}$ is the Laurent polynomial $a^k(z) = \sum_{i\in\mathbb{Z}} a_i^k z^i$.

By attaching q_i^k, $i \in \mathbb{Z}$ to the parameter values $i/2^k$, $k \geq 0$, we say the subdivision scheme $\{S_{\mathbf{a}^k}\}_{k\geq 0}$ is C^0-convergent if, for the initial data sequence $\mathbf{q}^0 \in l_0(\mathbb{Z})$, there exists a function $f_{\mathbf{q}^0} \in C^0(\mathbb{R})$ satisfying

$$\lim_{k\longrightarrow\infty} \left\| f_{\mathbf{q}^0}\left(\frac{\cdot}{2^k}\right) - S_{\mathbf{a}^k}\mathbf{q}^k \right\|_\infty = 0, \tag{2}$$

where $f_{\mathbf{q}^0}(\cdot/2^k)$ denotes the sequence $\{f_{\mathbf{q}^0}(i/2^k)\}_{i\in\mathbb{Z}}$. If $f_{\mathbf{q}^0} \in C^l(\mathbb{R})$, we say the subdivision $\{S_{\mathbf{a}^k}\}_{k\geq 0}$ is C^l-convergent.

In order to investigate the convergence and smoothness of nonstationary subdivision schemes, let us recall some definitions and results as follows.

Definition 1 (see [13]). A nonstationary subdivision scheme $\{S_{\mathbf{u}^k}\}_{k\geq 0}$ with the k-level mask $\mathbf{u}^k$ is said to be asymptotically similar to the stationary subdivision scheme $S_{\mathbf{u}}$ with the mask $\mathbf{u}$, if the k-level mask $\mathbf{u}^k$ and the mask $\mathbf{u}$ have the same support D (i.e. $a_i^k = a_i = 0$ for $i \notin D$) and satisfy

$$\lim_{k\longrightarrow\infty} u_{\boldsymbol{i}}^k = u_{\boldsymbol{i}}, \quad i \in D. \tag{3}$$

Definition 2 (see [14]). let D^n be the nth order differentiation operator. A nonstationary subdivision scheme $\{S_{\mathbf{u}^k}\}_{k\geq 0}$ with the k-level symbol $u^k(z)$ is said to satisfy the approximate sum rules of order $r+1$ if

$$\begin{aligned} \mu_k &= \left|u^k(1) - 2\right|, \\ \delta_k &= \max_{\eta\leq r} \left|2^{-k\eta} D^\eta u^k(-1)\right|, \end{aligned} \tag{4}$$

satisfy

$$\begin{aligned} \sum_k \mu^k &< \infty, \\ \sum_k 2^{kr}\delta_k &< \infty. \end{aligned} \tag{5}$$

Theorem 3 (see [14]). *Assume that the nonstationary subdivision scheme $\{S_{\mathbf{u}^k}\}_{k\geq 0}$ satisfies approximate sum rules of order $r+1$ and is asymptotically similar to a convergent stationary subdivision scheme $S_{\mathbf{u}}$ who is C^r-convergent. Then the nonstationary scheme $\{S_{\mathbf{u}^k}\}_{k\geq 0}$ is C^r-convergent.*

Now we recall some knowledge about the generation of exponential polynomials.

Definition 4 (see [3]). Let $T \in \mathbb{Z}_+$ and $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_T)$ with $\gamma_T \neq 0$ a finite set of real or imaginary numbers. The space of exponential polynomials $V_{T,\boldsymbol{\gamma}}$ is

$$V_{T,\boldsymbol{\gamma}} := \left\{ f : \mathbb{R} \longrightarrow \mathbb{C},\ f \in C^T(\mathbb{R}) : \sum_{j=0}^{T} \gamma_j D^j f = 0 \right\}. \tag{6}$$

The exponential polynomial space $V_{T,\boldsymbol{\gamma}}$ can be characterized as in the following lemma.

Lemma 5 (see [3]). *Let $\gamma(z) = \sum_{j=0}^{T} \gamma_j z^j$ and denote by $\{\theta_l, \tau_l\}_{l=1,\dots,N}$ the set of zeros with multiplicity satisfying*

$$D^r \gamma(\theta_l) = 0, \quad r = 0, \dots, \tau_l - 1, \ l = 1, \dots, N. \tag{7}$$

Then

$$T = \sum_{l=1}^{N} \tau_l,$$
$$V_{T,\boldsymbol{\gamma}} := span\left\{x^r e^{\theta_l x}, \ r = 0, \dots, \tau_l - 1, \ l = 1, \dots, N\right\}. \tag{8}$$

Definition 6 (see [3]). We say the subdivision scheme $\{S_{\mathbf{a}^k}\}_{k\geq 0}$ is $V_{T,\boldsymbol{\gamma}}$-generating if it is convergent and for $f \in V_{T,\boldsymbol{\gamma}}$, there exists an initial sequence f^0 uniformly sampled from $\tilde{f} \in V_{T,\boldsymbol{\gamma}}$, such that

$$\lim_{k\longrightarrow\infty} S_{\mathbf{a}^{m+k}} S_{\mathbf{a}^{m+k-1}} \dots S_{\mathbf{a}^m} f^0 = f, \quad m \geq 0. \tag{9}$$

Now let us now recall some known definitions and results about fixed point iterations.

Definition 7 (see [15]). We say $x^* \in \mathbb{R}$ is a fixed point for a given function $\varphi(\cdot)$ if

$$\varphi(x^*) = x^*. \tag{10}$$

The following result gives sufficient conditions for the existence and uniqueness of a fixed point and how to approximate it.

Theorem 8 (see [15]). *Let $\varphi(\cdot) \in C[a,b]$ be such that $\varphi(x) \in [a,b]$ for all $x \in [a,b]$. Suppose, in addition, that $D^1\varphi$ exists on (a,b) and that there exists a constant $0 < L < 1$ such that*

$$\left|D^1\varphi(x)\right| \leq L, \quad x \in [a,b]. \tag{11}$$

Then, there exists a unique fixed point x^* for $\varphi(\cdot)$ on $[a,b]$. Besides, for any $x^0 \in [a,b]$, the sequence defined by

$$x^n = \varphi\left(x^{n-1}\right), \quad n \geq 1, \tag{12}$$

converges to the unique fixed point x^* in $[a,b]$.

3. The Generalized Cubic Exponential B-Spline Scheme

In this section, we construct the generalized cubic exponential B-spline scheme and then investigate its convergence and smoothness.

3.1. Construction of the Generalized Cubic Exponential B-Spline Scheme. Before we construct the generalized cubic exponential B-spline scheme, we briefly review the cubic exponential B-spline scheme [12], which can be written down as

$$\begin{aligned} p_{2i}^{k+1} &= \frac{1}{4\left(1+\nu^{k+1}\right)}\left(p_{i-1}^k + p_{i+1}^k\right) \\ &\quad + \left(1 - \frac{1}{2\left(1+\nu^{k+1}\right)}\right) p_i^k, \\ p_{2i+1}^{k+1} &= \frac{1}{2}\left(p_i^k + p_{i+1}^k\right), \end{aligned} \tag{13}$$

where

$$\begin{aligned} t_k &= \frac{t}{2^k}, \\ \nu^k &= \frac{1}{2}\left(e^{t_k} + e^{-t_k}\right), \end{aligned} \tag{14}$$

with t being a nonnegative or pure imaginary constant. From the definition of ν^k in (14), we see that ν^{k+1} and ν^k satisfies the iteration

$$\nu^{k+1} = \sqrt{\frac{1+\nu^k}{2}}, \tag{15}$$

with $\nu^0 \in (-1, \infty)$.

As is known, the cubic exponential B-spline scheme (13) is C^2-convergent and can generate the function space $E := span\{1, x, e^{\pm tx}\}$ in the sense of Definition 6 [5]. Thus, it can generate conic sections.

In fact, from the cubic exponential B-spline scheme (13), we can see that it is just the iteration in (15) that forces the subdivision rules of this scheme to depend on the recursive level k. This means that the nonstationarity of this scheme results from the iteration in (15). Besides, we can see the coefficients in the corresponding subdivision rules as functions of this iteration. That is to say, there exists a function $\overline{h}(x) = 1/(1+x)$ such that the cubic exponential B-spline scheme (13) can be rewritten as

$$\begin{aligned} p_{2i}^{k+1} &= \frac{\overline{h}\left(\nu^{k+1}\right)}{4}\left(p_{i-1}^k + p_{i+1}^k\right) + \left(1 - \frac{\overline{h}\left(\nu^{k+1}\right)}{2}\right) p_i^k, \\ p_{2i+1}^{k+1} &= \frac{1}{2}\left(p_i^k + p_{i+1}^k\right). \end{aligned} \tag{16}$$

In this way, if we replace the function $\overline{h}(\cdot)$ or the iteration (15) in (16) by a different one, we can obtain a different nonstationary subdivision scheme.

Now based on this observation, let us construct the new generalized cubic exponential B-spline scheme. In fact, to obtain our new scheme, we replace the function $\overline{h}(\cdot)$ by a more generalized one as follows:

$$h(x) = \frac{1}{x^\mu + x^\nu}, \quad \mu, \nu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \ x \in \Omega_{\mu,\nu}, \tag{17}$$

with $\Omega_{\mu,\nu} := \{x : x^{\mu} + x^{\nu} \neq 0,\ \mu, \nu \in \mathbb{N}_0\}$. Then, from (16), we can obtain a new scheme, which can be written as

$$p_{2i}^{k+1} = \frac{h\left(v^{k+1}\right)}{4}\left(p_{i-1}^{k} + p_{i+1}^{k}\right) + \left(1 - \frac{h\left(v^{k+1}\right)}{2}\right) p_i^k,$$
$$p_{2i+1}^{k+1} = \frac{1}{2}\left(p_i^k + p_{i+1}^k\right). \tag{18}$$

Note that when one of the parameters μ and ν is 0 and the other is 1, $h(\cdot)$ becomes $\overline{h}(\cdot)$ and the scheme (18) reduces to the cubic exponential B-spline scheme (13). We point out that the scheme (18) is just the desired generalized cubic exponential B-spline scheme and we denote it by $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$. From the definition of v^k in (14), we see that $\lim_{k\longrightarrow\infty} v^k = 1$. As a result, it can be seen that $\lim_{k\longrightarrow\infty} h(v^{k+1}) = h(1) = 1/2$ and the limit stationary scheme of $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ is thus the cubic B-spline scheme.

In fact, we only modified the existing elements in the masks of the cubic exponential B-spline scheme. Thus, the support of the new scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ is the same as the cubic exponential B-spline scheme, i.e., $[-2, 2]$.

3.2. Convergence and Smoothness. Now let us investigate the convergence and smoothness of the generalized cubic exponential B-spline scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$. In fact, we have the following result.

Theorem 9. *For $\mu, \nu \in \mathbb{N}_0$, the generalized cubic exponential B-spline scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ is C^2-convergent.*

Proof. We show that the generalized cubic exponential B-spline scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ satisfies approximate sum rules of order 3. Then by Theorem 3, we can conclude that the scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ is C^2-convergent.

To show that the scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ satisfies approximate sum rules of order 3, we denote by $a_{\mu,\nu}^k(z)$ the k-level symbol of the scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ and then $a_{\mu,\nu}^k(z)$ can be written down as

$$a_{\mu,\nu}^k(z) = \frac{h\left(v^{k+1}\right)}{4}\left(z^2 + z^{-2}\right) + \frac{1}{2}\left(z + z^{-1}\right) + 1 - \frac{h\left(v^{k+1}\right)}{2}. \tag{19}$$

It can be computed that $a_{\mu,\nu}^k(z)$ contains the factor $(1+z)^2/2$ and that

$$a_{\mu,\nu}^k(1) = 2,$$
$$a_{\mu,\nu}^k(-1) = D^1 a_{\mu,\nu}^k(-1) = 0, \tag{20}$$
$$D^2 a_{\mu,\nu}^k(-1) = 2h\left(v^{k+1}\right) - 1.$$

Therefore, from Definition 2, for μ_k and δ_k, we have

$$\mu_k = \left|a_{\mu,\nu}^k(1) - 2\right| = 0,$$
$$\delta_k = \max_{\eta\leq 2} 2^{-k\eta}\left|D^{\eta} a_{\mu,\nu}^k(-1)\right| = 2^{1-2k}\left|h\left(v^{k+1}\right) - \frac{1}{2}\right|. \tag{21}$$

Together with the definition of $h(\cdot)$, it can be seen that there exists a constant c_1 independent of k such that

$$\sum_{k=0}^{\infty} 2^{2k}\delta_k = \sum_{k=0}^{\infty} 2\left|h\left(v^{k+1}\right) - \frac{1}{2}\right| \leq 2c_1 \sum_{k=0}^{\infty}\left|v^{k+1} - 1\right|. \tag{22}$$

Therefore, to show that the scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ satisfies approximate sum rules of order 3, we only have to show $\sum_{k=0}^{\infty}|v^{k+1} - 1| < \infty$.

Let $v^{k+1} = \varphi(v^k) = \sqrt{v^k/2 + 1/2}$, with $\varphi(x) = \sqrt{x/2 + 1/2}$, $x \in (-1, \infty)$. When $v^0 > 1$, choose $M > v^0$; then for $x \in [1, M]$, we see that there exists a constant $L = 1/2\sqrt{2} < 1$ satisfying

$$1 \leq \varphi(x) \leq M,$$
$$\left|\frac{d\varphi(x)}{dx}\right| \leq L. \tag{23}$$

Therefore, from Theorem 8, there exists a unique fixed point v^* for $\varphi(\cdot)$ on $[1, M]$ and that the sequence $\{v^k\}_{k\geq 0}$ converges to v^*. In fact, the fixed point is $v^* = 1$. In this way, we have

$$\left|v^k - 1\right| = \left|\varphi\left(v^{k-1}\right) - \varphi(1)\right| \leq L\left|v^{k-1} - 1\right| \leq \cdots \leq L^{k-1}\left|v^1 - 1\right| \leq L^k\left|v^0 - 1\right|. \tag{24}$$

When $v^0 \in (-1, 1]$, notice that $v^k \in [0, 1]$ for $k \geq 1$. For $x \in [0, 1]$, we still have

$$0 \leq \varphi(x) \leq 1,$$
$$\left|\frac{d\varphi(x)}{dx}\right| \leq L. \tag{25}$$

Therefore, from Theorem 8, there exists a unique fixed point for $\varphi(\cdot)$ on $[0, 1]$, which, in fact, is just $v^* = 1$. In this case, similar to (24), we have

$$\left|v^k - 1\right| = \left|\varphi\left(v^{k-1}\right) - \varphi(1)\right| \leq L\left|v^{k-1} - 1\right| \leq \cdots \leq L^{k-1}\left|v^1 - 1\right| \leq c_2 L^k\left|v^0 - 1\right|, \tag{26}$$

where c_2 is a positive constant independent of k.

Therefore, by combining (24) and (26), we see that, for $v^0 \in (-1, \infty)$,

$$\sum_{k=0}^{\infty}\left|v^{k+1} - 1\right| \leq \sum_{k=0}^{\infty} L^{k+1}\left|v^0 - 1\right| < \infty, \tag{27}$$

and thus the scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ satisfies approximate sum rules of order 3. Then, from Theorem 3, the scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ is C^2-convergent. □

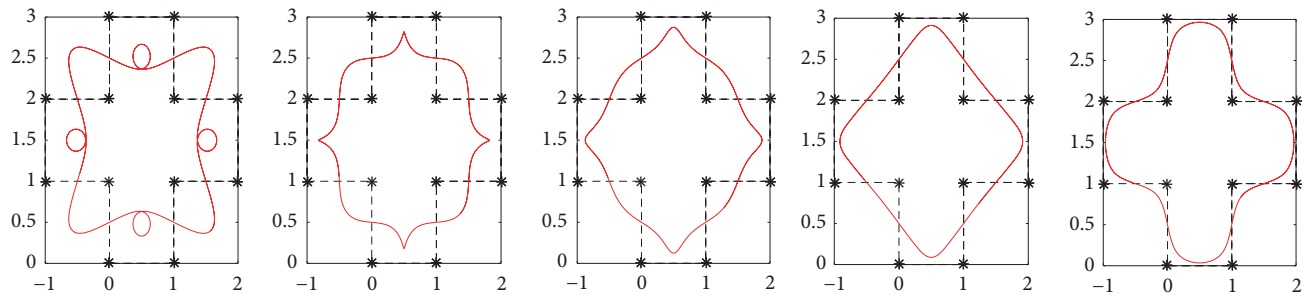


Figure 1: Curves generated by the scheme $\{S_{\mathbf{a}^k_{2,1}}\}_{k\geq 0}$ with $\nu^0 = -0.88, -0.7, -0.5, -0.2, 1.5$ (left to right).

Table 1: Comparison between different schemes.

Schemes	$\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$	$\{S_{\mathbf{b}^k}\}_{k\geq 0}$	$\{S_{\mathbf{c}^k}\}_{k\geq 0}$
support	[-2, 2]	[-2.5,2.5]	[-3, 3]
smoothness order	2	2	3
type	approximating	interpolatory	approximating
p-ary	binary	ternary	binary

Remark 10. Note that $\sum_{k=0}^{\infty} |\nu^k - 1| < \infty$ can also be derived from the fact that $|\nu^k - 1| \leq c_3 2^{-k}$ with c_3 being a constant independent of k. Here, we used the technique of fixed point iteration, which, we point out that, can also be used in the analysis of other nonstationary subdivision schemes, such as the ones in [10, 11]. This will be shown in Section 5.

4. Examples and Comparison

In this section, we present several numerical examples and compare it with some existing subdivision schemes to illustrate the performance of the scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$.

Figure 1 shows the curves generated by the scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ with $\mu = 2$, $\nu = 1$, and different values of ν^0. From Figure 1, we can see that, for $\nu^0 \in (-1, \infty)$, with a suitable choice of μ and ν, the scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ can indeed generate curves with a wide variety of shapes. Figure 2 shows how the parameters μ and ν affect the shape of the obtained curve and gives some hints on how to choose them to generate the curve we want.

Figure 2 shows the curves generated by the scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ with different values of μ, ν, and ν^0 starting from the initial control points uniformly sampled from the unit circle. From Figure 2, we see that when both μ and ν are nonzero, the shape of the obtained curve is sensitive to the change of ν^0 near $\nu^0 = -1$. But, this is not the case if only one of μ and ν is 0. In particular, when $\mu = 0$ and $\nu = 1$, the scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ reduces to the cubic exponential B-spline scheme and the curve generated with $\nu^0 = \cos(\pi/3)$ is exactly a circle. Similarly, the parabola and hyperbola can also be generated in this case. Besides, when $\mu = \nu = 0$, the scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ reduces to the cubic B-spline scheme, and the shape of the obtained curve can not be changed. From Figure 2, we can also see that when both μ and ν are nonzero, the scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ will get more sensitive to ν^0 near -1 if $\mu + \nu$ becomes bigger and thus can generate more kinds of curves with the change of ν^0 near -1. Besides, with the increasing of ν^0, the obtained curve will tend to the initial control polygon.

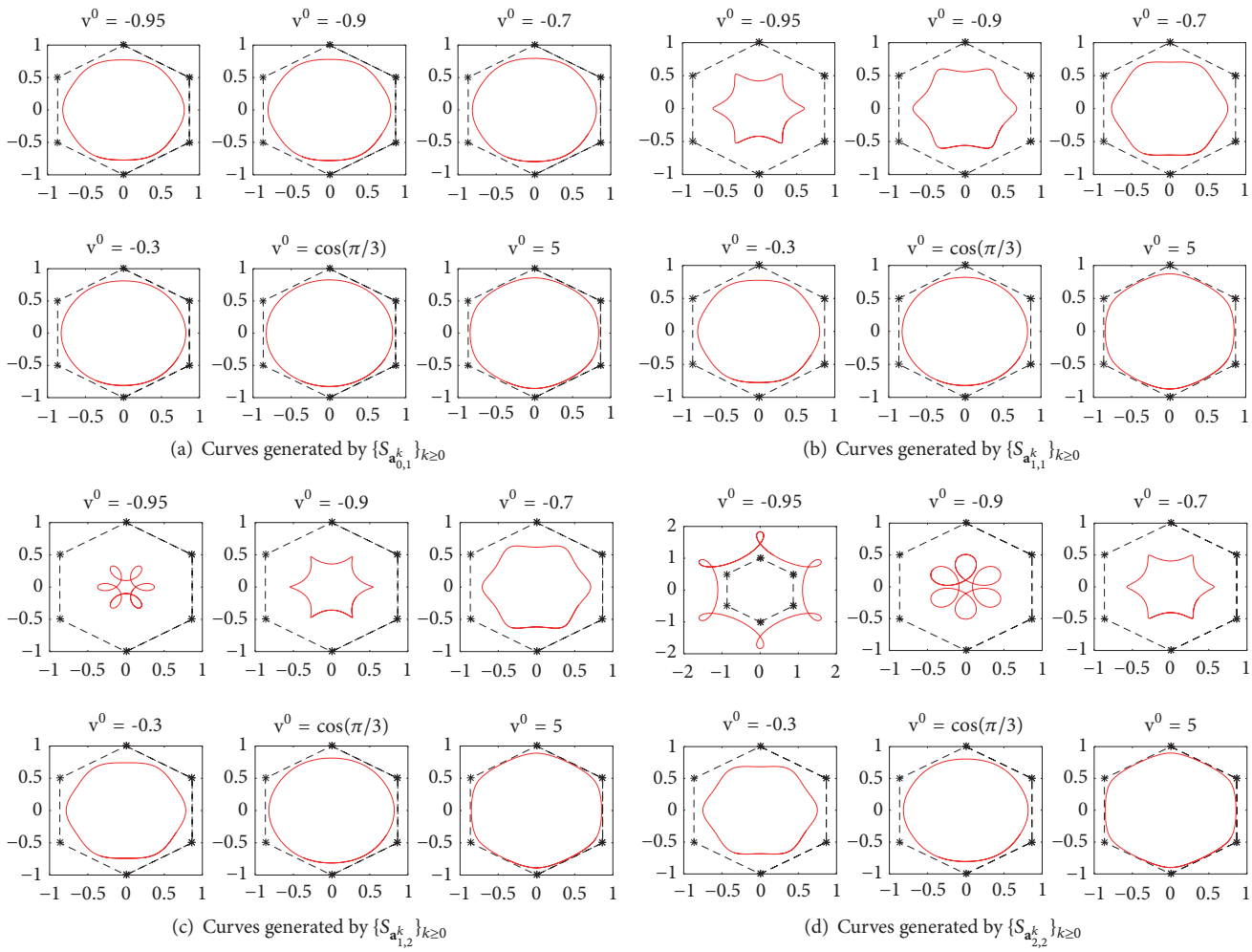
Figure 3 shows the curves generated by the cubic B-spline scheme, the cubic exponential B-spline scheme (13) and the scheme $\{S_{\mathbf{a}^k_{2,3}}\}_{k\geq 0}$ with $\nu^0 = 4$. We can see from Figure 3 that, for $\nu^0 = 4 > 1$, when μ and ν are chosen large enough, the scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ performs better than the cubic B-spline scheme and the cubic exponential B-spline scheme.

Recall that the nonstationary subdivision schemes in [10, 11] can also generate different kinds of curves and we denote them by $\{S_{\mathbf{b}^k}\}_{k\geq 0}$ and $\{S_{\mathbf{c}^k}\}_{k\geq 0}$, respectively. The scheme $\{S_{\mathbf{b}^k}\}_{k\geq 0}$ is a ternary interpolatory one while the schemes $\{S_{\mathbf{c}^k}\}_{k\geq 0}$ and $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ are binary approximating ones. Table 1 shows the comparison between these schemes. From Table 1, we see that the new scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ outperforms the other two in terms of support.

5. Further Discussion

Note that the scheme $\{S_{\mathbf{a}^k_{\mu,\nu}}\}_{k\geq 0}$ is obtained based on the iteration (15) and the function $h(\cdot)$. In this section, based on a different iteration and another suitable function, we move a further step and try to obtain a similar scheme, which can also generate different kinds of curves. This will show us the role of fixed point iterations and suitably chosen functions in the construction and analysis of such schemes.

Recall that the iteration (15) can be written as $\nu^{k+1} = \varphi(\nu^k)$ with $\varphi(x) = \sqrt{(1+x)/2}$ for $x \in (-1, \infty)$. Now we replace $\varphi(\cdot)$ by a different one $\psi(\cdot)$, which maps its domain I onto I. Meanwhile, we also replace the function $h(\cdot)$ by a different one $g(\cdot)$. Recall from Section 4 that when both μ and ν are nonzero, the shape of the obtained curve is sensitive to the



(a) Curves generated by $\{S_{\mathbf{a}_{0,1}^k}\}_{k\geq 0}$

(b) Curves generated by $\{S_{\mathbf{a}_{1,1}^k}\}_{k\geq 0}$

(c) Curves generated by $\{S_{\mathbf{a}_{1,2}^k}\}_{k\geq 0}$

(d) Curves generated by $\{S_{\mathbf{a}_{2,2}^k}\}_{k\geq 0}$

FIGURE 2: Curves generated by the scheme $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$ with different values of μ, ν, and $v^0 = -0.95, -0.9, -0.7, -0.3, \cos(\pi/3), 5$.

change of v^0 near $v^0 = -1$. Note that $v^0 = -1$ leads to $v^1 = 0$ and that $\lim_{x\longrightarrow 0} h(x) = \infty$. Therefore, to obtain a scheme similar to $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$, we assume that there exists a point $x^* \in I$ such that $\lim_{x\longrightarrow x^*} g(x) = \infty$.

Specifically speaking, we take $\psi(x) = \sqrt{x+2}$ with $I = (-2, \infty)$ and the function $g(x) = 2/x^2$ with $x \neq 0$. In this way, from $\{S_{\mathbf{a}_{\mu,\nu}^k}\}_{k\geq 0}$, we can derive a new scheme, which can be written as

$$p_{2i}^{k+1} = \frac{g\left(x^{k+1}\right)}{4}\left(p_{i-1}^k + p_{i+1}^k\right) + \left(1 - \frac{g\left(x^{k+1}\right)}{2}\right) p_i^k,$$
$$p_{2i+1}^{k+1} = \frac{1}{2}\left(p_i^k + p_{i+1}^k\right), \tag{28}$$

where the sequence $\{x^k\}_{k\geq 0}$ is generated by $x^{k+1} = \psi(x^k)$ with $x^0 \in (-2, \infty)$. We denote the scheme (28) by $\{S_{\mathbf{e}^k}\}_{k\geq 0}$.

Similar to the iteration $v^{k+1} = \varphi(v^k)$, it can be shown that, for $x^0 \in (-2, \infty)$, we have $\lim_{k\longrightarrow\infty} x^k = 2 := x^*$, which, in fact, is the fixed point of $\psi(\cdot)$. Besides, notice that $g(x^*) = 1/2$. Thus, the limit stationary scheme of the scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$ is also the cubic B-spline scheme.

Now we investigate the convergence and smoothness of the newly obtained scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$. In fact, similar to Theorem 8, we have the following result.

Theorem 11. *The new scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$ is C^2-convergent.*

Proof. Denote by $e^k(z)$ the k-level symbol of the scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$. Then we have

$$e^k(z) = \frac{(1+z)^2}{4}\left(g\left(x^{k+1}\right)\left(z^2+1\right)\right.$$
$$\left. + 2\left(1 - g\left(x^{k+1}\right)\right)z\right)z^{-2}. \tag{29}$$

Similar to the proof of Theorem 8, from Definition 6, for μ_k and δ_k, we have

$$\mu_k = \left|a^k(1) - 1\right| = 0,$$
$$\delta_k = \max_{\eta\leq 2}\left|2^{-k\eta} D^\eta a^k(-1)\right| = 2^{1-2k}\left|g\left(x^{k+1}\right) - \frac{1}{2}\right|. \tag{30}$$

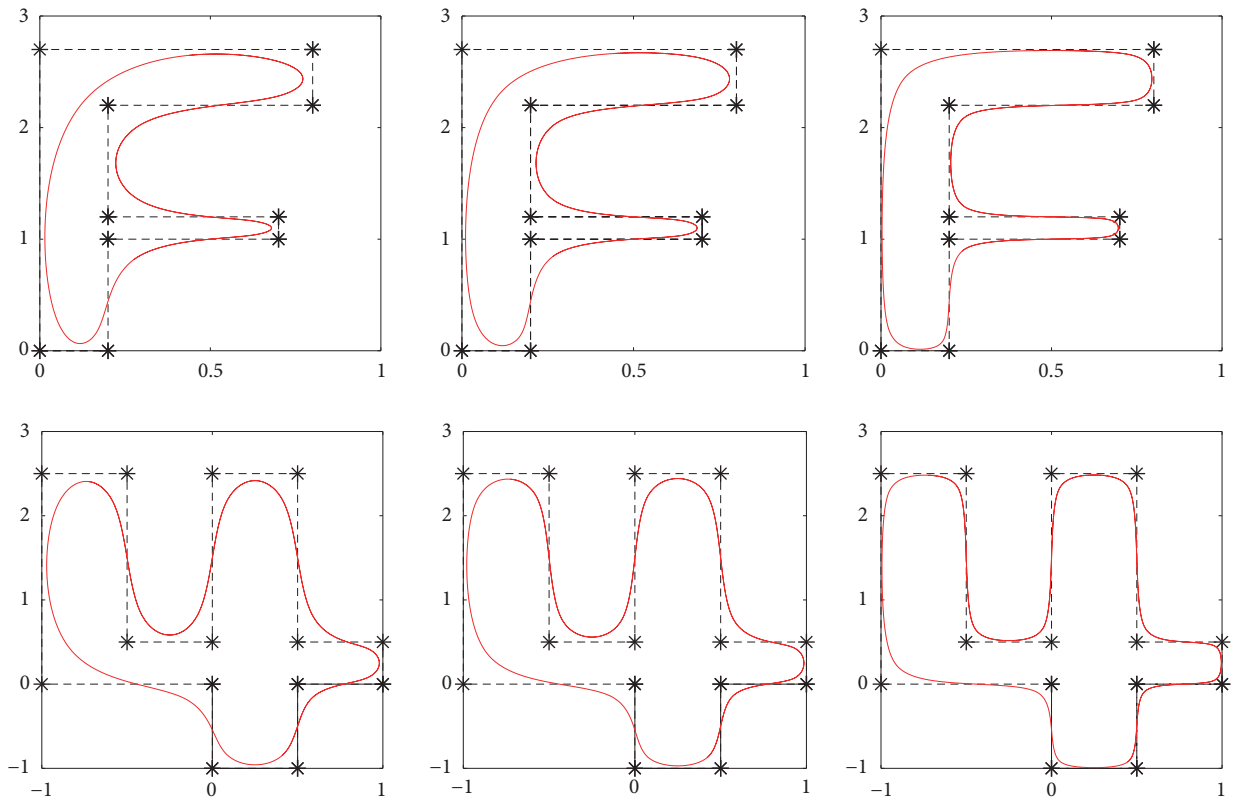


FIGURE 3: Curves generated by the cubic B-spline scheme (left column) cubic exponential B-spline (middle column), and the scheme $\{S_{\mathbf{a}_{2,3}^k}\}_{k\geq 0}$ (right column) with $\nu^0 = 4$.

Therefore, since $g(x^*) = 1/2$, we have

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{2k}\delta &= \sum_{k=0}^{\infty} 2\left|g\left(x^{k+1}\right) - \frac{1}{2}\right| \\ &= \sum_{k=0}^{\infty} 2\left|g\left(x^{k+1}\right) - g\left(x^*\right)\right| \\ &\leq \sum_{k=0}^{\infty} 2\left|D^1 g\left(\theta\right)\right|\left|x^{k+1} - x^*\right| \\ &\leq c_4 \sum_{k=0}^{\infty}\left|x^{k+1} - x^*\right|,\end{aligned} \tag{31}$$

where $\theta \in (\min\{x^{k+1}, x^*\}, \max\{x^{k+1}, x^*\})$ and c_4 is a positive constant independent of k. Now we show that $\sum_{k=0}^{\infty}|x^{k+1} - x^*| < \infty$ so that the scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$ satisfies approximate sum rules of order 3.

In fact, similar to the sequence $\{\nu^k\}_{k\geq 0}$, for $L = 1/2\sqrt{2}$, we still have $|D^1\psi(x)| \leq L$ for $x \in (-2, \infty)$ and that

$$\left|x^{k+1} - x^*\right| \leq c_5 L^{k+1}\left|x^0 - x^*\right|, \tag{32}$$

where c_5 is a positive constant independent of k. Thus, $\sum_{k=0}^{\infty}|x^{k+1} - x^*| \leq \sum_{k=0}^{\infty} c_5 L^{k+1}|x^0 - x^*| < \infty$ and the scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$ satisfies approximate sum rules of order 3. Then, by Theorem 3, the scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$ is C^2-convergent, since it is asymptotic similar to the C^2-convergent cubic B-spline scheme. □

Figures 4 and 5 show some curves generated by the scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$ with different values of x^0. From Figures 4 and 5, we see that, with $x^0 \in (-2, \infty)$, the scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$ can indeed generate a wide variety of curves, including some interesting ones.

6. Conclusion

In this paper, a generalized cubic exponential B-spline scheme is presented, which can generate different kinds of curves, including the conics. The key ingredients of the construction and analysis is the iteration (15) coming from the generation of exponential polynomials and a suitable function $h(\cdot)$ with two parameters μ and ν. By adjusting the values of μ and ν, we can change the sensitivity of the shape of the obtained curve to the initial control value so as to generate various kinds of curves, including the conics. For this new scheme, we show that, with any admissible choice of μ and ν, it keeps the same smoothness order and support as the cubic exponential B-spline scheme. Some hints on how to choose μ and ν to generate the curves we want are also given. Besides, based on a different iteration and another suitable function, we also obtained a different nonstationary scheme, which can also generate a wide variety of curves with the change of the initial control value. This shows the role of fixed point

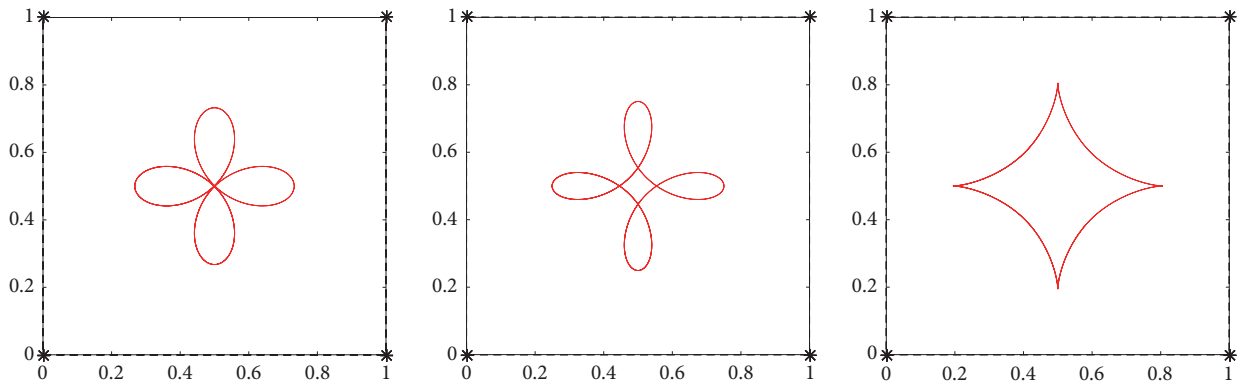


FIGURE 4: Curves generated the scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$ with $x^0 = -1.25, -1.2, -1$ (left to right: a qudarifomun, a simple 'China knot' and an astroid).

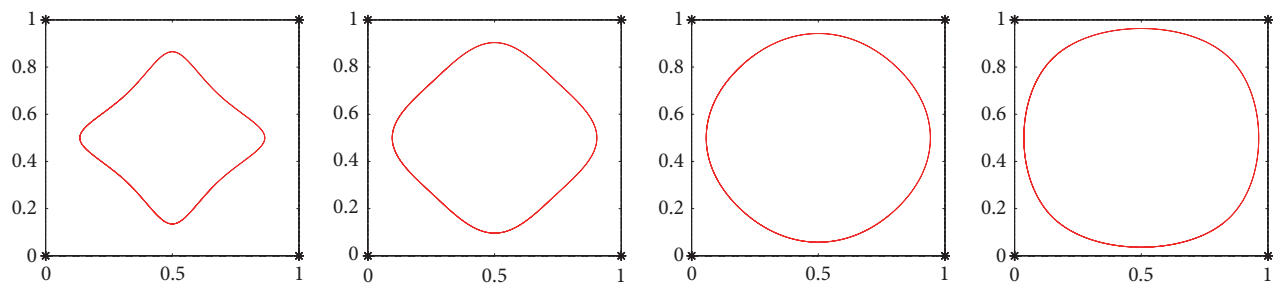


FIGURE 5: Curves generated the scheme $\{S_{\mathbf{e}^k}\}_{k\geq 0}$ with $x^0 = -0.6, -0.1, 1.1, 2.5$ (left to right).

iterations and suitably chosen functions in the construction and analysis of such schemes.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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