Research Article

Real Representation Approach to Quaternion Matrix Equation Involving $\phi$-Hermicity

Xin Liu,1 Huajun Huang,2 and Zhuo-Heng He3

1Faculty of Information Technology, Macau University of Science and Technology, Avenida Wai Long, Taipa, Macau 999078, China
2Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849-5310, USA
3Department of Mathematics, Shanghai University, Shanghai 200444, China

Correspondence should be addressed to Zhuo-Heng He; hzh19871126@126.com

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1. Introduction

The definitions of $\phi$-Hermitian and $\phi$-skew-Hermitian quaternion matrices were first introduced by Rodman (Definition 3.6.1 in [1]). As a special case of $\phi$-(skew)-Hermitian matrix, the $\eta$-(anti)-Hermitian matrix first arises in widely linear modelling and has important applications over independent component analysis [2] and convergence analysis in statistical signal processing [3]. Recently, it was widely investigated (e.g., [4–17]). For instance, Horn and Zhang [15] gave a singular value decomposition for $\eta$-Hermitian matrix. Yuan et al. [16, 17] derived least squares $\eta$-(anti)-Hermitian solution of some classic quaternion matrix equations, while there have been a few papers to consider the $\phi$-Hermitian and $\phi$-skew-Hermitian solutions of some quaternion matrix equations. For example, the $\phi$-Hermitian solutions of the mixed pairs of Sylvester matrix equations,

\[
\begin{align*}
A_1 X - Y B_1 &= C_1, \\
A_2 Z - Y B_2 &= C_2, \\
Z &= Z_\phi,
\end{align*}
\]

were given by using Moore–Penrose inverses [18]. The $\phi$-skew-Hermitian solution of the quaternion matrix equations,

\[
\begin{align*}
B X B_\phi + C Y C_\phi &= A, \\
E Z E_\phi + D Y D_\phi &= B,
\end{align*}
\]

were discussed through matrix decomposition [19]. Very recently, some practical necessary and sufficient conditions
for the existence of a solution of the system of quaternion matrix equations
\[
\begin{cases}
A_1X_1 + (A_1X_1)_\phi + C_1Y_1(C_1)_\phi + F_1W(F_1)_\phi = E_1, \\
A_2X_2 + (A_2X_2)_\phi + C_2Y_2(C_2)_\phi + F_2W(F_2)_\phi = E_2, \\
Y_1 = (Y_1)_\phi, Y_2 = (Y_2)_\phi, W = W_\phi,
\end{cases}
\]
(3)
were given by [20] in terms of ranks and Moore–Penrose inverses. As we know that real (complex) representation is a usual method to address the problems of quaternion matrix theory, it enables us to convert problems over quaternions into the problems over real (complex) number field. Up to now, there are many existing real (complex) representations. But, as far as we know, none of those can well preserve the structure of \(\phi\)-(skew)-Hermitian matrix. Thus, we define a new real representation of a quaternion matrix, which can map a \(\phi\)-Hermitian matrix or \(\phi\)-skew-Hermitian matrix into a skew-symmetric or symmetric matrix. Motivated by the above works and its important applications, we will present a new real representation of a quaternion matrix to discuss the \(\phi\)-Hermitian or \(\phi\)-skew-Hermitian solution of the quaternion matrix equation \(AX = B\).

The paper is organized as follows. In Section 2, we give a complete characterization of the nonstandard involutions of \(\mathbb{H}\) and their conjugacy properties. In Section 3, we present a new real representation of quaternion matrix, which could well preserve the structure of \(\phi\)-(skew)-Hermitian matrix. In Section 4, we discuss the existence of the \(\phi\)-(skew)-Hermitian solution of the quaternion matrix equation \(AX = B\) and derive the solutions when it is solvable.

2. A Complete Characterization of the Nonstandard Involutions of \(\mathbb{H}\) and Their Conjugacy Properties

In this section, we give a complete characterization of the nonstandard involutions of \(\mathbb{H}\) and their conjugacy properties.

Let \(\mathbb{R}\) and \(\mathbb{H}^{\text{mox}}\) stand, respectively, for the real number field and the set of all \(m \times n\) matrices over the real quaternion algebra
\[
\mathbb{H} = \{ a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, \quad a_0, a_1, a_2, a_3 \in \mathbb{R} \}.
\]
(4)
The symbols \(I_n, 0, A^T\) stand for the \(n \times n\) identity matrix, the zero matrix with appropriate size, the transpose of \(A\), respectively. The Moore–Penrose inverse \(A^\dagger\) of a real matrix \(A\) is defined to be the unique matrix \(A^\dagger\), such that

(i) \(AA^\dagger A = A\),
(ii) \(A^\dagger A A^\dagger = A^\dagger\),
(iii) \((AA^\dagger)^T = AA^\dagger\),
(iv) \((A^\dagger A)^T = A^\dagger A\).

Furthermore, \(L_A\) and \(R_A\) stand for the projectors \(L_A = I - A^\dagger A\) and \(R_A = I - AA^\dagger\) induced by \(A\), respectively. For any \(A = A_1 + A_2i + A_3j + A_4k \in \mathbb{H}^{m\times n}\), \(A^* = A_1^T - A_2^T i - A_3^T j - A_4^T k\) is defined as the usual conjugate transpose of \(A\).

In [1], Rodman defined the involution over \(\mathbb{H}\) as follows.

**Definition 1 (involution) [1].** A map \(\phi : \mathbb{H} \rightarrow \mathbb{H}\) is called an antiautomorphism if \(\phi(xy) = \phi(y)\phi(x)\) for all \(x, y \in \mathbb{H}\), and \(\phi(x + y) = \phi(x) + \phi(y)\) for all \(x, y \in \mathbb{H}\). An antiautomorphism \(\phi\) is called an involution if \(\phi(\phi(x)) = x\) for every \(x \in \mathbb{H}\).

The matrix representations of involutions are given in the following lemma.

**Lemma 1** [1]. Let \(\phi\) be an antiautomorphism of \(\mathbb{H}\). Assume that \(\phi\) does not map \(\mathbb{H}\) into zero. Then, \(\phi\) is one-to-one and onto \(\mathbb{H}\); thus, \(\phi\) is in fact an antiautomorphism. Moreover, \(\phi\) is real linear and can be represented as a \(4 \times 4\) real matrix with respect to the basis \(\{1, i, j, k\}\). Then, \(\phi\) is an involution if and only if
\[
\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},
\]
(5)
where either \(T = -I_3\) or \(T\) is a \(3 \times 3\) real orthogonal symmetric matrix with eigenvalues of \(1, 1, -1\).

By Lemma 1, involutions can be classified into two classes: the standard involution and the nonstandard involution, as defined below.

**Definition 2 (standard involution) [1].** An involution \(\phi\) is standard if \(\phi = \begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}\).

**Definition 3 (nonstandard involution) [1].** An involution \(\phi\) is nonstandard if
\[
\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},
\]
(6)
where \(T\) is a \(3 \times 3\) real orthogonal symmetric matrix with eigenvalues of \(1, 1, -1\).

In this paper, we only consider the nonstandard involution. The following theorem shows that each nonstandard involution is in the form of \(\phi(a) = a^{m^*} = -\eta a^* \eta\), where \(\eta = u_1 i + u_2 j + u_3 k \in \mathbb{H}\) is unit and pure imaginary, and \(a^*\) stands for the conjugate transpose of the quaternion \(a\).

**Theorem 1.** A map \(\phi : \mathbb{H} \rightarrow \mathbb{H}\) is a nonstandard involution if and only if it is of the form \(\phi(a) = a^{m^*} = -\eta a^* \eta\), where \(\eta = u_1 i + u_2 j + u_3 k \in \mathbb{H}\) is unit and pure imaginary, that is, \(u_1, u_2, u_3 \in \mathbb{R}\) and \(u_1^2 + u_2^2 + u_3^2 = 1\). Moreover, let \(u = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^T\), then the matrix representation of such a nonstandard involution is \(\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}\), where \(T = I_3 - 2uu^T\) is a real Householder matrix.

**Proof.** First, for each \(\eta = u_1 i + u_2 j + u_3 k \in \mathbb{H}\) with \(u_1, u_2, u_3 \in \mathbb{R}\), \(u_3^2 + u_2^2 + u_1^2 = 1\), we show that \(\phi(a) = -\eta a^* \eta\)
defines a nonstandard involution. By direct calculation, we have
\[ \phi(1) = 1, \quad \phi(1) = 1, \quad \phi(i) = (1 - 2u^2_1)\bar{i} - 2u_1u_j - 2u_1u_j, \]
\[ \phi(j) = -2u_1u_i + (1 - 2u^2_j)j - 2u_2u_k, \]
\[ \phi(k) = -2u_1u_i - 2u_2u_j + (1 - 2u^2_k)k. \]

Hence, the matrix representation of \( \phi \) with respect to \( \{1, i, j, k\} \) is
\[ T = \begin{pmatrix} 1 - 2u^2_1 & -2u_1u_1 & -2u_1u_1 \\ -2u_1u_2 & 1 - 2u^2_2 & -2u_2u_2 \\ -2u_1u_3 & -2u_2u_3 & 1 - 2u^2_3 \end{pmatrix} = \begin{pmatrix} I_3 - 2uu^T, \\ u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \]

Then, \( T \) is a real Householder matrix, which has eigenvalues of 1, 1, -1 and is symmetric. Hence, \( \phi \) is a nonstandard involution by Lemma 1.

Second, Lemma 1 shows that every nonstandard involution \( \phi \) has the matrix representation \( \phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & T \end{pmatrix} \), where \( T \) is an orthogonal matrix with eigenvalues of 1, 1, -1. Let \( e_1 = (1, 0, 0)^T \). Then, there exists a 3 x 3 real orthogonal matrix \( U \) such that
\[ T = U \delta \phi \delta U^T = U(I_3 - 2ee^T)U^T = I_3 - 2(Ue_1)(Ue_1)^T, \]
where \( Ue_1 \) is the first column of \( U \) and is a real unit vector. Denote \( u = Ue_1 = (u_1, u_2, u_3)^T \) and \( \eta = u_1i + u_2j + u_3k \). Then, \( \phi \) is of the form \( \phi(a) = -\eta a^* \eta \), as desired.

We give two examples to understand the nonstandard involution.

**Example 1** [4]. Let \( a = a_0 + a_1i + a_2j + a_3k \in \mathbb{H} \) and \( \eta \in \{i, j, k\} \). Then, the map \( a \rightarrow a^\eta \) is a nonstandard involution, where \( a^\eta = -\eta a^* \eta \). Upon computation, we have
\[ a^i = a_0 - a_1i + a_2j + a_3k, \quad \text{if} \ \eta = i, \]
\[ a^j = a_0 + a_1i - a_2j + a_3k, \quad \text{if} \ \eta = j, \]
\[ a^k = a_0 + a_1i + a_2j - a_3k, \quad \text{if} \ \eta = k. \]

**Example 2** [18]. Let \( a = a_0 + a_1i + a_2j + a_3k \in \mathbb{H} \) and \( \eta \in \{\sqrt{2}/2 \,(i + j), \ (\sqrt{2}/2 \,(i + k), \ (\sqrt{2}/2 \,(j + k)\} \). Then, the map \( a \rightarrow a^\eta \) is a nonstandard involution, where \( a^\eta = -\eta a^* \eta \). Upon computation, we have
\[ a^{(\sqrt{2}/2 \,(i + j)} = a_0 - a_1i - a_2j + a_3k, \quad \text{if} \ \eta = \sqrt{2}/2 \,(i + j), \]
\[ a^{(\sqrt{2}/2 \,(i + k)} = a_0 - a_1i + a_2j - a_3k, \quad \text{if} \ \eta = \sqrt{2}/2 \,(i + k), \]
\[ a^{(\sqrt{2}/2 \,(j + k)} = a_0 + a_1i - a_2j - a_3k, \quad \text{if} \ \eta = \sqrt{2}/2 \,(j + k). \]

The product of two nonstandard involutions is in general not an involution. However, every product of the form \( \delta \phi \delta^{-1} = \delta \phi \delta \), where \( \delta \) and \( \phi \) are nonstandard involutions, is again a nonstandard involution. Moreover, Theorem 1 implies that all nonstandard involutions are conjugate via nonstandard involutions, as shown below.

**Theorem 2.** Every nonstandard involution \( \phi \) of \( \mathbb{H} \) is conjugate via a nonstandard involution to \( \phi_1 \) where \( \phi_1(a) = a^\eta = -ia^*i \). In other words, there exists a nonstandard involution \( \delta \) such that
\[ \phi = \delta \phi_1 \delta^{-1} = \delta \phi_1 \delta. \]

**Proof.** The matrix representation of \( \phi_1 \) is \( \phi_1 = \text{diag}(1, -1, 1, 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I_3 - 2ee^T \end{pmatrix} \). By Theorem 1, suppose the nonstandard involution \( \phi \) has the matrix representation \( \phi = \begin{pmatrix} 1 & 0 \\ 0 & I_3 - 2uu^T \end{pmatrix} \), where \( u = (u_1, u_2, u_3)^T \) is a real unit vector. We hope to find a real Householder matrix \( V = I_3 - 2uv^T \), where \( v = (v_1, v_2, v_3)^T \) is a real unit vector, such that
\[ I_3 - 2uu^T = V(I_3 - 2ee^T)V^T = I_3 - 2(Ve_1)(Ve_1)^T. \]

In other words, we need \( (I_3 - 2vv^T)e_1 = Ve_1 = u \), explicitly,
\[ 1 - 2v_1^2 = u_1, \quad (14a) \]
\[ -2v_2v_1 = u_2, \quad (14b) \]
\[ -2v_3v_1 = u_3. \quad (14c) \]

If \( u_1 = 1 \), then \( u_2 = u_3 = 0 \), and (14a) gives \( v_1 = 0 \); we simply choose \( v_2, v_3 \in \mathbb{R} \) such that \( v_1^2 + v_2^2 = 1 \). Otherwise \( u_1 < 1 \), we may choose \( v_1 = \sqrt{(1 - u_1)}/2 \neq 0 \) to satisfy (14a) and then solve \( v_2 \) and \( v_3 \) by (14b) and (14c). The equations (14a)–(14c) imply that
\[ 1 = u_1^2 + u_2^2 + u_3^2 = (1 - 2v_1^2)^2 + (-2v_2v_1)^2 \]
\[ + (-2v_3v_1)^2 = 1 + 4v_1^2(1 - v_2^2 + v_3^2) = 1 + 4v_1^2(1 - v_2^2 + v_3^2). \]
In both cases above, we will have \( v_1^2 + v_2^2 + v_3^2 = 1 \). Now let \( \delta \) be the nonstandard involution with the matrix representation \( \delta = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} \). Then, (12) holds.

For a given nonstandard involution \( \phi \) over \( \mathbb{H} \) and \( A \in \mathbb{H}^{m \times n} \), Rodman [1] denotes by \( A_{\phi} \) the \( n \times m \) matrix obtained by applying \( \phi \) entrywise to the transposed matrix \( A^T \). Then, by Theorem 1, the general form of \( A_{\phi} \) follows.

**Lemma 2.** Let \( \phi \) be a nonstandard involution. Then, there exists a unit pure imaginary quaternion \( \eta \) such that \( A_{\phi} \) is in the form of

\[
A_{\phi} = -\eta A^* \eta. \tag{16}
\]

We make an example in the following.

**Example 3.** If \( A_{\phi} = -jA^*j \) for \( A \in \mathbb{H}^{m \times n} \), then

\[
\begin{pmatrix}
i & i + k \\ 2 - i & 2 + i
\end{pmatrix}_\phi = \begin{pmatrix} i & 2 - i \\ i + k & 2 + i \\ -j - k & i - 2j \\ i - j + k & -3j + k
\end{pmatrix}.
\]

The following are the definitions of \( \phi- \) (skew)-Hermitian matrix. They are the special cases of \( \phi- \)(skew)-Hermitian.

**Definition 4** (\( \phi- \)Hermitian or \( \phi- \)skew-Hermitian) [1]. \( A \in \mathbb{H}^{m \times n} \) is said to be \( \phi- \)Hermitian or \( \phi- \)skew-Hermitian if \( A = A_{\phi} \) or \( A = -A_{\phi} \), where \( \phi \) is a nonstandard involution.

From Lemma 2 and Definition 4, there follows an equivalent description of Definition 4: let \( \phi \) be a nonstandard involution, that is, \( \phi(a) = -\eta \bar{a} \eta \), where \( \eta \in \mathbb{H} \) is unit and pure imaginary. Then, \( A \in \mathbb{H}^{m \times n} \) is said to be \( \phi- \)Hermitian or \( \phi- \)skew-Hermitian if \( A = -\eta A^* \eta \) or \( A = \eta A^* \eta \), It will play important role in solving our \( \phi- \)Hermicity problem.

**Example 4.** For \( \eta \in \{i, j, k\} \), \( A \in \mathbb{H}^{m \times n} \) is said to be \( \eta- \)Hermitian or \( \eta- \)anti-Hermitian if \( A = A^* \) or \( A = -A^* \), where \( A^* = -\eta A^* \eta \). Obviously, \( \eta- \)Hermitian (\( \eta- \)anti-Hermitian) matrix is a special case of \( \phi- \)Hermitian (\( \phi- \)skew-Hermitian) matrix.

3. New Real Representation of Quaternion Matrix

Real representation method is a powerful tool to study the quaternions. There are several existing real (complex) representations of quaternion matrix, for example, the following real representation in [21, 22].

**Definition 5.** For \( X \in \mathbb{H}^{m \times n} \), \( X = X_0 + X_1 i + X_2 j + X_3 k \), \( X_0, X_1, X_2, X_3 \in \mathbb{R}^{m \times n} \), define

\[
X^\sigma = \begin{bmatrix} X_0 & -X_1 & -X_2 & -X_3 \\ X_1 & X_0 & -X_3 & X_2 \\ X_2 & X_3 & X_0 & -X_1 \\ X_3 & -X_2 & X_1 & X_0
\end{bmatrix} \in \mathbb{R}^{4m \times 4n}. \tag{18}
\]

The above real representation of a quaternion matrix can map an Hermitian matrix or a skew-Hermitian matrix over \( \mathbb{H} \) into a symmetric matrix or skew-symmetric matrix over \( \mathbb{R} \). In this section, we aim at exploring the \( \phi- \)(skew)-Hermitian solutions of the quaternion matrix \( AX = B \). To well preserve the structure of \( \phi- \)(skew)-Hermitian matrix, in this section, we present a new representation of quaternion matrix basing on the above one. Let \( \eta = u_1 i + u_2 j + u_3 k \in \mathbb{H} \) be a unit and pure imaginary quaternion. Now, applying the product rule \( (XY)^\sigma = X^\sigma Y^\sigma \) to the matrix equality

\[
-\eta^2 I_n = I_n, \tag{19}
\]

we have

\[
F_n = (\eta I_n)^\sigma, \tag{20}
\]

where

\[
F_n = \begin{bmatrix} 0 & -u_1 I_n & -u_2 I_n & -u_3 I_n \\ u_1 I_n & 0 & -u_3 I_n & u_2 I_n \\ u_2 I_n & u_3 I_n & 0 & -u_1 I_n \\ -u_3 I_n & -u_2 I_n & u_1 I_n & 0
\end{bmatrix}. \tag{21}
\]

Note that \( F_n^T = F_n \) and \( -F_n^2 = I_{4n} \), thus \( F_n \) is an orthogonal matrix. Next, we can use \( F_n \) to define the new real representation.

**Definition 6.** For \( X = X_0 + X_1 i + X_2 j + X_3 k \in \mathbb{H}^{m \times n} \), \( X_0, X_1, X_2, X_3 \in \mathbb{R}^{m \times n} \) and \( \eta = u_1 i + u_2 j + u_3 k \in \mathbb{H} \), which is unit and pure imaginary. We define

\[
X'^n = F_n X^\sigma. \tag{22}
\]

It follows from Theorem 1 and Lemma 2 that the real representation \( \sigma^n \) can map a \( \phi- \)Hermitian matrix or \( \phi- \)skew-Hermitian matrix over \( \mathbb{H} \) into a skew-symmetric or...
symmetric matrix over $\mathbb{R}$. This will be a critical technique in tackling the problem of the paper.

Denoting

$$P_n = \begin{bmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & I_n & 0 \end{bmatrix},$$

$$L_n = \begin{bmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & -I_n & 0 \\ I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \end{bmatrix},$$

$$T_n = \begin{bmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_n \\ 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix},$$

Then, by direct calculation, we have the following lemma.

**Lemma 3.** $P_n, L_n, T_n, F_n$ are all orthogonal matrices, and

$$P_n F_n = F_n P_n,$$

$$L_n F_n = F_n L_n,$$

$$T_n F_n = F_n T_n.$$  \((23)\)

Next, we summarize the properties of real representations $X'^x, X'^y$ in the following proposition. Some properties of $\sigma$ are given by [21, 22] and the others can be verified directly.

**Proposition 1.** Let $A, B \in \mathbb{H}^{m \times n}, C \in \mathbb{H}^{n \times n}, b \in \mathbb{R}, \eta \in \{i, j, k\}$. Then

(a) $(A + B)^\eta = A^\eta + B^\eta, (bA)^\eta = bA^\eta$;

(b) $(AC)^\eta = A^\eta C^\eta, (AC)^\eta = -A^\eta C^\eta$;

(c) $P_m^T A^\eta P_n = A^\eta L_m^T A^\eta L_n = A^\eta, T_m^T A^\eta T_n = A^\eta$;

(d) $(A^\eta)^\eta = (A^\eta)^T$;

(e) $A^\eta = G_m A^\eta, A^\eta = H_m A^\eta, A^\eta = K_m A^\eta$.

As corollary of Proposition 1, we have:

**Proposition 2.** Let $A, B \in \mathbb{H}^{m \times n}, C \in \mathbb{H}^{n \times n}, b \in \mathbb{R}, \eta \in \{i, j, k\}$. Then

(a) $(A + B)^\eta = A^\eta + B^\eta, (bA)^\eta = bA^\eta$;

(b) $(AC)^\eta = A^\eta C^\eta, (AC)^\eta = -A^\eta C^\eta$;

(c) $P_m^T A^\eta P_n = A^\eta L_m^T A^\eta L_n = A^\eta, T_m^T A^\eta T_n = A^\eta$;

(d) $(A^\eta)^\eta = (A^\eta)^T$;

(e) $A^\eta = G_m A^\eta, A^\eta = H_m A^\eta, A^\eta = K_m A^\eta$.

4. $\phi$-Hermitian or $\phi$-Skew-Hermitian Solution to $AX = B$

In this section, we apply some new real representations given above to discuss the solvability conditions and $\phi$-Hermitian or $\phi$-skew-Hermitian solution of the quaternion matrix equation $AX = B$. We first begin with a useful lemma, which can be slightly modified from $C$ to $R$.

**Lemma 4** [23]. Let $A, B \in \mathbb{R}^{m \times n}$. Then, the matrix equation $AX = B$ has a solution $X = X^T \in \mathbb{R}^{m \times n}$ if and only if $R_A B = \frac{1}{2}(A^T A - B B^T)$.
Let $X$ be a $\phi$-Hermitian (or $\phi$-skew-Hermitian) solution to the quaternion matrix equation
\[ AX = B, \tag{27} \]
with $\phi \in H^{4\times 4}$. Suppose that $X$ is a $\phi$-Hermitian solution of (27), then it follows from (b) and (d) of Proposition 2 that $-A^\phi F_n X^\phi = B^\phi$ and $X^\phi = -(X^\phi)^T$. Postmultiplying the both sides with $F_n$ gives
\[ -A^\phi F_n X^\phi F_n = B^\phi F_n. \tag{33} \]

Since $F_n^T = -F_n$, then (33) can be rewritten as $A^\phi F_n X^\phi F_n = B^\phi F_n$. Hence, $F_n X^\phi F_n^T$ is a skew-symmetric solution to (28).

Conversely, if (28) has a skew-symmetric solution $Y$, i.e., $A^\phi Y = B^\phi F_n Y = -Y^T$. Then, using (ii) in (c) of Proposition 1 gives
\[ (P_m A^\phi P_n) Y = (P_m B^\phi P_n) F_n, \tag{34} \]
\[ (I_m A^\phi I_n) Y = (I_m B^\phi I_n) F_n, \tag{35} \]
\[ (T_m A^\phi T_n) Y = (T_m B^\phi T_n) F_n. \tag{36} \]

Then, by Lemma 3,
\[ A^\phi (P_n Y P_n^T) = B^\phi F_n, \tag{37} \]
\[ A^\phi (I_n Y I_n^T) = B^\phi F_n, \tag{38} \]
\[ A^\phi (T_n Y T_n^T) = B^\phi F_n, \tag{39} \]
which follows that $P_n Y P_n^T + I_n Y I_n^T + T_n Y T_n^T$ are skew-symmetric solutions of (28). Then, so is $Y := (1/4) (Y + P_n Y P_n^T + L_n Y L_n^T + T_n Y T_n^T)$. For the given skew-symmetric solution $Y$,
\[ Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix}. \tag{37} \]
By direct computation
\[
\mathbf{Y} = \begin{bmatrix}
Z_1 & -Z_2 & -Z_3 & -Z_4 \\
Z_2 & Z_1 & -Z_4 & Z_3 \\
Z_3 & Z_4 & Z_1 & -Z_2 \\
Z_4 & -Z_3 & Z_2 & Z_1
\end{bmatrix},
\]
where
\[
\begin{align*}
Z_1 &= \frac{1}{4} (Y_{11} + Y_{22} + Y_{33} + Y_{44}), \\
Z_2 &= \frac{1}{4} (Y_{21} - Y_{12} + Y_{43} - Y_{34}), \\
Z_3 &= \frac{1}{4} (Y_{31} - Y_{13} + Y_{42} - Y_{24}), \\
Z_4 &= \frac{1}{4} (Y_{41} + Y_{32} - Y_{23} - Y_{14}).
\end{align*}
\] (38)

Now, construct a quaternion matrix
\[
Z = Z_1 + Z_2k + Z_3j + Z_4k
\]
\[
= \frac{1}{16} \begin{bmatrix}
I_m & U_m & jI_m & kI_m
\end{bmatrix} \begin{bmatrix}
Y + P_m Y P_n^T \\
\end{bmatrix}
\]
\[
+ L_m X_n^T + T_m Y_n^T
\]
\[
- iI_n \\
- jI_n
\]
\[
- kI_n
\] (40)

Remark 1. If we set \( \eta \in \{i,j,k\} \), then \( \phi \)-Hermitian or \( \phi \)-skew-Hermitian matrix is reduced to the well-known \( \eta \)-Hermitian or \( \eta \)-anti-Hermitian matrix. Thus, the \( \eta \)-Hermitian or \( \eta \)-anti-Hermitian solution to (27) can be discussed by simply replacing \( F_n \) by \( G_n, H_n, K_n \) and \( \sigma_\eta \) by \( \sigma_i, \sigma_j, \sigma_k \), respectively.

Example 5. Find a \( \phi \)-skew-Hermitian solution to the quaternion matrix equation \( AX = B \), where \( X_\phi = - (\sqrt{2}/2)(i + j) X^\ast (\sqrt{2}/2)(i + j) \).

\[
A = \begin{bmatrix}
2 & 3 \\
-2 & 1
\end{bmatrix} + \begin{bmatrix}
0 & -2 \\
1 & 1
\end{bmatrix} + \begin{bmatrix}
3 & 2 \\
-3 & 2
\end{bmatrix} + \begin{bmatrix}
4 & -2 \\
1 & 1
\end{bmatrix} k,
\]
\[
B = \begin{bmatrix}
-8 & -5 \\
-1 & 4
\end{bmatrix} + \begin{bmatrix}
3 & -4 \\
-4 & -3
\end{bmatrix} + \begin{bmatrix}
3 & 8 \\
-2 & -9
\end{bmatrix} + \begin{bmatrix}
-8 & 7 \\
1 & 2
\end{bmatrix} k.
\] (42)

By Theorem 3, its corresponding real matrix equation is
\[
A^\ast Y = B^\ast F_2, \quad \text{where} \ \eta = \frac{\sqrt{2}}{2}(i + j).
\] (43)

Since \( R_{A^\ast}(B^\ast F_2) = 0 \) and \( B^\ast F_2 (A^\ast)^T = A^\ast (B^\ast F_2)^T \). Thus, the real equation has a symmetric solution
\[
Y = \sqrt{2} \begin{bmatrix}
-1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\
-1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & -1
\end{bmatrix} k. \] (44)

By the formula of \( X \) in (30) in Theorem 3, the quaternion matrix equation also has a solution
\[
X = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} + \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} + \begin{bmatrix}
1 & 1 \\
-1 & 0
\end{bmatrix} k. \] (45)

which one can verify that \( X = -X_\phi \).

5. Conclusion
In this paper, we give a complete characterization of the nonstandard involutions \( \phi \) of \( \mathbb{H} \) and their conjugacy properties. Basing on the characterization of the nonstandard involutions, we present a new real representation of a quaternion matrix, which maps a \( \phi \)-Hermitian or \( \phi \)-skew-Hermitian quaternion matrix into a skew-symmetric or symmetric real matrix. By using this approach, we derive the necessary and sufficient conditions for the existence of a \( \phi \)-Hermitian solution or \( \phi \)-skew-Hermitian solution of the quaternion matrix equation (25). Furthermore, we get
solutions of (25) when it is solvable. Moreover, Example 5 is presented to illustrate our results.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Disclosure**

The first author gave a presentation of the abstract of this paper at “The 5th International Conference on Matrix Inequalities and Matrix Equations (MIME 2019)” that was held in Guilin, China, on June 7–9, 2019.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**


