Research Article

A Multidimensional Fixed-Point Theorem and Applications to Riemann-Liouville Fractional Differential Equations

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1. Introduction

Fixed-point theory has experienced quick improvement over the most recent quite a few years. The development has been firmly advanced by the vast number of utilizations in the existence theory of functional, fractional, differential, partial differential, and integral equations. Two fundamental theorems concerning fixed points are those of Schauder and of Banach. The Schauder theorem states that if $E$ is a closed bounded convex subset of a Banach space $X$ and $T : E \rightarrow E$ is a continuous map such that $T(E)$ is compact, then $T$ has a fixed point. In Banach’s theorem, every contraction operator on a complete metric space has a unique fixed point. In 1964, Krasnoselskii [1] gave a very important fixed-point theorem which is a blend of the nonlinear contraction principle and Schauder’s fixed-point theorem. He gave intriguing applications to differential equations by finding the existence of solutions under some hybrid conditions. In 2013, Dhage [2] proposed an important Krasnoselskii-type fixed-point theorem and used it to study the existence of the solution to the system of nonlinear fractional differential equations. In the same year, Dhage and Jadhav [3] studied the existence of solution for hybrid differential equations. Also, Lu et al. [4] developed the theory of fractional hybrid differential equations including R-L FDEs and gave an existence theorem for fractional hybrid differential equations. In 2016, Bashiri et al. [5] proposed a new version of Krasnoselskii-type fixed-point theorem based on coupled fixed-point approach and applied their results to study the existence of solution to a system of two R-L FDEs. Recently, Yang et al. [6] used the concept of fractional derivative due to Caputo to prove Krasnoselskii coupled fixed-point theorem under some certain conditions and applied their results to study the existence of the solution to a nonlinear coupled system for fractional differential equations.

In this paper, we give a new version of $N$-tupled fixed-point theorem which is utilized to prove the existence of the $N$-tupled fixed point of sum of more than two operators. Also, we apply our $N$-tupled fixed-point theorem to prove the existence of solution to the system of R-L FDEs.

Our paper is organized as follows. Section 2 contains some basic concepts concerning fixed-point theory. Notably, a fractional derivative criterion in normed spaces is stated, which is based on the concept of Riemann-Liouville derivative and Caputo fractional derivative. In Section 3, our main $N$-tupled fixed-point results are proved. Section 4 is devoted to study the existence of a solution for nonlinear R-L FDEs.
2. Preliminaries

Let \( (X, \| \cdot \|) \) be a normed space and \( F : X \times X \rightarrow X \) be a mapping. An element \((x, y) \in X \times X\) is called coupled fixed point of \( F \) in \( X \times X \) if \( F(x, y) = x \) and \( F(y, x) = y \) [7].

**Definition 1.** A mapping \( \phi : X \rightarrow X \) is said to be \( D \)-function if it is upper semicontinuous and monotone nondecreasing such that \( \phi(0) = 0 \).

**Definition 2** (see [8]). Let \( Q : \prod_{i=1}^{N} X \rightarrow X \). The point \((x_1, x_2, \ldots, x_N) \in \prod_{i=1}^{N} X \) is called the \( N \)-tupled fixed point of \( \prod_{i=1}^{N} X \) if and only if

\[
\begin{align*}
  x_1 &= Q(x_1, x_2, \ldots, x_N), \\
  x_2 &= Q(x_2, x_3, \ldots, x_N, x_1), \\
  &\vdots \\
  x_N &= Q(x_N, x_1, \ldots, x_{N-1}).
\end{align*}
\]

For more information about coupled, tripled, and \( N \)-tupled fixed points, see [9–17].

Now, we recall some definitions about the fractional derivative. The fractional derivative is defined via the fractional integral operator. The fractional integral operator of order \( \alpha \in (0, \infty) \) of a function \( g(t) \) is given as

\[
I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \, ds.
\]

Also, the Riemann-Liouville derivative of order \( \alpha > 0 \) is defined as

\[
D^\alpha g(t) = \begin{cases} 
\frac{d^m}{dt^m} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} g(s) \, ds, & \text{if } m-1 < \alpha < m, \\
\frac{d^m}{dt^m} g(t), & \text{if } \alpha = m,
\end{cases}
\]

where \( m \) is a positive integer and \( m-1 < \alpha < m \).

The Caputo fractional derivative of order \( \alpha > 0 \) with \( m-1 < \alpha < m \) is denoted by \( D^\alpha_\beta \) and is given as

\[
D^\alpha_\beta g(t) = \begin{cases} 
\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} \frac{d^m}{ds^m} g(s) \, ds, & \text{if } m-1 < \alpha < m, \\
\frac{d^m}{dt^m} g(t), & \text{if } \alpha = m.
\end{cases}
\]

For more details about the fractional calculus, we refer the reader to [18–23].

The following two lemmas are useful in what follows.

**Lemma 3** (see [21]). If \( 0 < \alpha < 1 \) and \( g(t) \in L^1([0, b], X) \), then

\[
(A1) \quad D^\alpha_\beta g(t) = g(t),
\]

\[
(A2) \quad I^\alpha D^\alpha_\beta g(t) = g(t) - ([D^{\alpha-1}_\beta g(t)]_{t=0}/\Gamma(\alpha))t^{\alpha-1}, \quad \text{for all } t \in [0, 1].
\]

**Lemma 4** (see [24]). Let \( X \) be a complete normed space and \( C \) be a nonempty closed and bounded convex subset of \( X \). Consider \( P : X \rightarrow X \) and \( Q : C \rightarrow X \) to be two mappings, and satisfying the following conditions

1. \( P \) is a contraction with constant \( p \in (0, 1) \)
2. \( Q \) is continuous and compact
3. the equation \( Pa + Qy = a \) implies that \( a \in C \), for all \( y \in C \),

then the operator equation \( Px + Qx = x \) has a solution in \( C \).

3. Main Fixed-Point Results

In this section, we give the main \( N \)-tupled fixed-point theorem which will play an important role in the proposed applications.

The following theorem was given in [5] and will be used with Lemma 4 as a tool to prove the main results.

**Theorem 5.** Let \( X \) be a complete normed space and \( C \) be a nonempty closed and bounded convex subset of \( X \). Consider \( P : X \rightarrow X \) and \( Q : C \rightarrow X \) to be two mappings and the following are satisfied:

1. \( P \) is a contraction
2. \( Q \) is continuous and compact
3. the equation \( Pa + Qy = a \) implies that \( a \in C \), for all \( y \in C \),

Then, the equation \( F(x, y) = Px + Qy \) has a coupled fixed point in \( C \times C \).

Consider \( X^N = \prod_{i=1}^{N} X \) and define the following two operations (sum and scalar multiplication in \( X^N \)) as follows:

\[
W + V = (w_1 + v_1, w_2 + v_2, \ldots, w_N + v_N),
\]

\[
\lambda V = (\lambda v_1, \lambda v_2, \ldots, \lambda v_N).
\]

Therefore, the norm in \( X^N \) can be defined as

\[
\|W\|_{X^N} = \|\langle w_1, w_2, \ldots, w_N \rangle\| = \|w_1\| + \|w_2\| + \ldots + \|w_N\|,
\]

where \( \| \cdot \| \) is the norm defined on \( X \). It is easy to prove that \( (X^N, \| \cdot \|) \) is a Banach space.

We are now ready to prove our main \( N \)-tupled fixed-point theorem for this section.

**Theorem 6.** Let \( X \) be a complete normed space and \( C \) be a nonempty closed and bounded convex subset of \( X \). Consider
Mathematical Problems in Engineering

\[ T_1 : X \rightarrow X \text{ and } T_2, T_3, \ldots, T_N : C \rightarrow X \text{ are mappings and} \]
the following are satisfied:

1. For every \( x_1, x_2 \in X \), there exist D-function \( \varphi_{T_1} \) which satisfies \( \| T_1 x_1 - T_1 x_2 \| \leq \lambda \varphi_{T_1} (\| x_1 - x_2 \|) \), where \( \lambda \in (0, 1) \) and \( \varphi_{T_1}(r) < r \), for all \( r > 0 \);

2. \( T_2, T_3, \ldots, T_N \) are continuous and compact;

3. The equation \( a = T_1 a + \sum_{i=2}^{N} T_i y_i \) implies that \( a \in C \), for all \( y_2, \ldots, y_N \in C \).

Then, \( F(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} T_i x_i \) has an N-tupled fixed point in \( \prod_{i=1}^{N} C \).

**Proof.** Define \( C^N = \prod_{i=1}^{N} C \). It is easy to prove \( C^N \neq \emptyset \) is closed convex and bounded subset of \( X^N \). Consider \( P_1, P_2, \ldots, P_N : C^N \rightarrow C^N \) such that, for \( u = (x_1, x_2, \ldots, x_N) \in C^N \), then

\[
P_1 u = (T_1 x_1, T_1 x_2, \ldots, T_1 x_N),
\]

\[
P_2 u = (T_2 x_2, T_2 x_3, \ldots, T_2 x_N, T_2 x_1),
\]

\[
\vdots
\]

\[
P_N u = (T_N x_N, T_N x_1, T_N x_2, \ldots, T_N x_{N-1}).
\]

Therefore, if we prove that the operator equation \( u = \sum_{i=1}^{N} P_i u \) has a solution \( u = (x_1, x_2, \ldots, x_N) \in C^N \), then we obtain that

\[
(x_1, x_2, \ldots, x_N) = (T_1 x_1, T_1 x_2, \ldots, T_1 x_N) + (T_2 x_2, T_2 x_3, \ldots, T_2 x_N, T_2 x_1) + \cdots + (T_N x_N, T_N x_1, T_N x_2, \ldots, T_N x_{N-1}).
\]

Then, we have that

\[
x_1 = T_1 x_1 + T_2 x_2 + \cdots + T_N x_N,
\]

\[
x_2 = T_1 x_2 + T_2 x_3 + \cdots + T_N x_1,
\]

\[
\vdots
\]

\[
x_N = T_1 x_N + T_2 x_1 + \cdots + T_N x_{N-1},
\]

which implies that \( (x_1, x_2, \ldots, x_N) \) is an \( N \)-tupled fixed point of \( F \) in \( X^N \). Now, we prove the theorem by three steps.

**Step 1.** Prove \( P_1 \) is a contraction.

Let \( W \neq V \in C^N \), we have that

\[
\| P_1 W - P_1 V \|_{X^N} = \| (T_1 w_1, T_1 w_2, \ldots, T_1 w_N) - (T_1 v_1, T_1 v_2, \ldots, T_1 v_N) \|_{X^N} \leq \| T_1 w_1 - T_1 v_1 \| + \cdots + \| T_1 w_N - T_1 v_N \| \leq \lambda (\| w_1 - v_1 \| + \cdots + \| w_N - v_N \|),
\]

for some \( \lambda \in (0, 1) \). Then, \( P_1 \) is a contraction.

**Step 2.** \( P_2, P_3, \ldots, P_N \) are continuous and compact.

Let \( (U_n) = (x_{n1}, x_{n2}, \ldots, x_{nn}) \) be a sequence in \( C^N \) converging to \( U = (x_1, x_2, \ldots, x_N) \in C^N \) as \( n \to \infty \). Since \( T_2 \) is continuous, we get that

\[
\lim_{n \to \infty} P_1 U_n = (\lim_{n \to \infty} T_2 x_{n1}, \lim_{n \to \infty} T_2 x_{n2}, \ldots, \lim_{n \to \infty} T_2 x_{nn}) \]

\[
= (T_2 x_2, T_2 x_3, \ldots, T_2 x_N) = P_2 (x_1, x_2, \ldots, x_N) = P_2 U.
\]

This shows that \( P_2 \) is continuous. Similarly, one can obtain that \( P_3, \ldots, P_N \) are continuous. Since \( T_2 \) is bounded, then, for all \( x \in X \), there exist \( M_2 > 0 \) such that \( \| T_2 x \| \leq M_2 \).

Therefore, we have that

\[
\| P_2 (x_1, x_2, \ldots, x_N) \|_{X^N} = \| (T_2 x_2, T_2 x_3, \ldots, T_2 x_N, T_2 x_1) \|_{X^N} \leq \| T_2 x_2 \| + \cdots + \| T_2 x_N \| \leq M_2 N.
\]

Thus, \( P_2 \) is uniformly bounded in \( C^N \). By doing the same steps, we get that \( P_3, \ldots, P_N \) are also uniformly bounded in \( C^N \).

Since \( T_2 \) is compact in \( C \), then we get that \( T_2(C) \) is equicontinuous and uniformly bounded in \( X \). Now, we prove that \( P_2(C^N) \) is equicontinuous in \( X^N \). Since \( T_2(C) \) is equicontinuous in \( C \), for any \( z \in C \) and \( t_2 > t_1 \), we have that

\[
\| (T_2 z)(t_2) - (T_2 z)(t_1) \| \to 0, \quad \text{as } t_2 - t_1 \to 0.
\]

Hence, for any \( Z = P_2 u \in P_2(C^N) \) and \( t_2 - t_1 \to 0 \), we obtain

\[
\| Z(t_2) - Z(t_1) \|_{X^N} = \| (P_2 u)(t_2) - (P_2 u)(t_1) \|_{X^N} = \| (T_2 y_2)(t_2) - (T_2 y_2)(t_1) \|_{X^N} \]

\[
+ \| (T_2 y_3)(t_2) - (T_2 y_3)(t_1) \|_{X^N} + \cdots + \| (T_2 y_N)(t_2) - (T_2 y_N)(t_1) \|_{X^N} \]

\[
\to 0.
\]

Thus, \( P_2(C^N) \) is equicontinuous in \( X^N \). Therefore, \( P_2(C^N) \) is compact in \( X^N \). Hence, \( P_2 : D^N \to X^N \) is compact. By doing the same steps, we get \( P_3, P_4, \ldots, P_n : D^N \to X^N \) are compact.

**Step 3.** The operator equation \( u_0 = P_1 u_0 + P_2 u + \cdots + P_n u \) implies that \( u_0 \in C^N \), for all \( u \in C^N \).
Let \( u_0 = (a, a, \ldots, a) \in X^N \) and \( u = (x_1, x_2, \ldots, x_n) \in D.N \) Consider the operator equation \( u_0 = P_1 u_0 + P_2 u + \ldots + P_N u \). Hence, we get that

\[
P_1 u_0 + P_2 u + \ldots + P_N u = (T_1 a, T_1 a, \ldots, T_1 a) + (T_2 x_2, T_2 x_3, \ldots, T_2 x_1) + \ldots + (T_N x_N, T_N x_1, \ldots, T_N x_{N-1})
\]

(15)

We get that, for all \( x_2, x_3, \ldots, x_N \in D \),

\[
a = T_1 a + T_2 x_2 + \ldots + T_N x_N.
\]

(16)

It implies that \( a \in C \). Therefore, we have that \( (a, \ldots, a) \in C^N \).

Define \( B : C^N \to X^N \) as \( BU = P_1 U + P_2 U + \ldots + P_N U \), for all \( U \in C^N \). Thus, \( B \) is continuous and compact. If \( u_0 = P_1 u_0 + BV \) holds, then \( u_0 \in C^N \), for all \( V \in C^N \).

From Lemma 3, the operator equation \( U = P_1 U + BU \) has a solution in \( C^N \). Then, if the solution is \( (x_1, x_2, \ldots, x_n) \), we get that

\[
(x_1, x_2, \ldots, x_N)
\]

\[
= (T_1 x_1, T_1 x_2, \ldots, T_1 x_N)
\]

Define \( C_{[0,b]}(\mathbb{R}) = \{ f : [0, b] \to \mathbb{R} : f \text{ is continuous} \} \). Recall that \( C_{[0,b]}(\mathbb{R}) \) is a Banach space with respect to the norm ||x||_{C_{[0,b]}} = \sup_{t \in [0, b]} |x(t)|. Also, we recall that \( L_{[0,b]}^1(\mathbb{R}) = \{ f : [0, b] \to \mathbb{R} : \int_0^b |x(t)| dt < \infty \} \); therefore, \( L_{[0,b]}^1(\mathbb{R}) \) is a Banach space with the norm ||x||_{L_{[0,b]}^1} = \int_0^b |x(t)| dt. We study the existence of a solution for system (19) under the following hypotheses:

(H1) For all \( t \in [0, b] \), the mapping \( x \mapsto x - f_1(t, x) \) is increasing in \( \mathbb{R} \).

(H2) There exist two constants \( M_1, d \) such that \( M_1 \geq L > 0 \) and

\[
|f_i(t, y(t)) - f_i(t, z(t))| \leq \frac{d |y(t) - z(t)|}{2(M_1 + |y(t) - z(t)|)}
\]

(20)

for all \( t \in [0, b] \) and \( y, z \in \mathbb{R} \).

(H3) There exists a constant, \( P_* = \max \{ |f(t, 0)| : t \in [0, b] \} \).

(H4) There are \( h_2, h_3, \ldots, h_N \in C_{[0,b]}(\mathbb{R}) \), such that

\[
f_2(t, y(t), z(t)) \leq h_2(t),
\]

\[
f_3(t, y(t), z(t)) \leq h_3(t),
\]

\[
\vdots
\]

\[
f_N(t, y(t), z(t)) \leq h_N(t),
\]

(21)

for all \( t \in [0, b] \) and \( y, z \in \mathbb{R} \).

Lu et al. [4] proved that if the hypothesis \( (H_2) \) holds, \( g \in C_{[0,b]}(\mathbb{R}), \alpha \in (0, 1) \), \( f : [0, b] \times \mathbb{R} \to \mathbb{R} \) is continuous, and \( f(0, 0) = 0 \), and the fractional differential equation

\[
D^\alpha [x(t) - f(t, x(t))] = g(t),
\]

(22)

\[
t \in [0, b], \ x(0) = 0
\]

Therefore, we have that

\[
x_1 = T_1 x_1 + T_2 x_2 + \ldots + T_N x_N = F(x_1, x_2, \ldots, x_N),
\]

\[
x_2 = T_1 x_2 + T_2 x_3 + \ldots + T_2 x_1 = F(x_2, x_3, \ldots, x_1),
\]

\[
\vdots
\]

\[
x_N = T_1 x_N + T_2 x_1 + \ldots + T_N x_{N-1} = F(x_N, x_1, \ldots, x_{N-1}).
\]

(17)

Hence, \( F \) has an \( N \)-tupled fixed point.

\[\square\]

4. Existence of the Solution of the System of R-L FDEs

In this section, we prove the existence of a mild solution of the system of \( N \)-fractional evolution equations:
has a unique solution which is the solution of the following integral equation:

\[ x(t) = f(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds, \quad t \in [0, b]. \tag{23} \]

**Theorem 7.** If the conditions (H2), (H3), and (H4) hold, the system of fractional differential equations (19) has a solution on \([0, b]\).

**Proof.** System (19) has a solution if the following system of integral equations has a solution:

\[ x_1(t) = f_1(t, x_1(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_2(s, x_2(s), I^\beta(x_2(s)))}{(t-s)^{1-\alpha}} ds + \ldots \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_N(s, x_N(s), I^\beta(x_N(s)))}{(t-s)^{1-\alpha}} ds, \quad t \in [0, b], \tag{24} \]

\[ x_2(t) = f_1(t, x_2(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_2(s, x_3(s), I^\beta(x_3(s)))}{(t-s)^{1-\alpha}} ds + \ldots \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_N(s, x_1(s), I^\beta(x_1(s)))}{(t-s)^{1-\alpha}} ds, \quad t \in [0, b], \]

\[ x_N(t) = f_1(t, x_N(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_2(s, x_{N-1}(s), I^\beta(x_{N-1}(s)))}{(t-s)^{1-\alpha}} ds + \ldots \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_N(s, x_1(s), I^\beta(x_1(s)))}{(t-s)^{1-\alpha}} ds, \quad t \in [0, b]. \]

Then, we can say that an element \((x_1, x_2, \ldots, x_N) \in X^N\) is an \(N\)-tupled solution of system (19) if and only if \((x_1, x_2, \ldots, x_N)\) is the solution to the system for the following operator equations:

\[ x_1(t) = (T_1x_1)(t) + (T_2x_2)(t) + \cdots + (T_Nx_N)(t), \quad t \in [0, b], \tag{26} \]

\[ x_2(t) = (T_1x_2)(t) + (T_2x_3)(t) + \cdots + (T_Nx_N)(t), \quad t \in [0, b], \tag{27} \]

\[ \vdots \]

\[ x_N(t) = (T_1x_N)(t) + (T_2x_1)(t) + \cdots + (T_Nx_{N-1})(t), \quad t \in [0, b]. \]

Define the mapping \(Q : D^N \to X\) such that \(Q(x_1, x_2, \ldots, x_N) = \sum_{i=1}^N T_i x_i\). Thus, system (19) has a solution if and only if \(Q(x_1, x_2, \ldots, x_N)\) has an \(N\)-tupled fixed point. We apply Theorem 6 to prove the result.

Now we apply three steps to prove the theorem.

**Step 1.** The operator \(T_1\) is a contraction.

Since condition H2 holds, we get that, for any \(x, y \in X\), we have

\[ \|T_1 x(t) - T_1 y(t)\| \leq \frac{d}{2} \frac{\|x(t) - y(t)\|}{\|x(t) - y(t)\|}. \tag{28} \]

Thus, by defining \(\theta(r) = dr/(M_1 + r)\), we get that \(\theta(r) < r\). Hence, by taking the sup, we get that \(\|T_1 x - T_1 y\| \leq (1/2)\|x - y\|\). Therefore, \(T_1\) is contraction.

**Step 2.** The operators \(T_2, T_3, \ldots, T_N\) are continuous and compact on \(D\).
Let $x^* \in D$ and $\{x_n\} \subset D$ such that $\lim_{n \to \infty} x_n = x^*$. Using the Dominated Convergence Theorem of Lebesgue integral, we get that

$$
\lim_{n \to \infty} T_n x_n(t) = \frac{1}{\Gamma (\alpha)} \left[ \int_0^t f_2(s, x_n(s), I^\beta (x_n(s))) \frac{ds}{(t-s)^{1-\alpha}} \right]_{s=t_1}^{s=t_2}
$$

Hence, $T_n$ is a continuous operator on $D$ for all $n \in \mathbb{N}$. Similarly, by doing the same steps, we get that

$$
\lim_{n \to \infty} T_N x_n(t) = T_N x^*(t), \quad \forall t \in [0, b].
$$

Thus, $T_1, \ldots, T_N$ are continuous operators on $D$. From (H3), for all $x \in D$, we get that

$$
T_2 x(t) = \frac{1}{\Gamma (\alpha)} \left[ \int_0^t f_2(s, x(s), I^\beta (x(s))) \frac{ds}{(t-s)^{1-\alpha}} \right]_{s=t_1}^{s=t_2}.
$$

Hence, when $t_2 - t_1 \to 0$, we have that

$$
T_2 x(t_1) - T_2 x(t_2) \to 0.
$$

Thus, $T_2(D)$ is equicontinuous on $[0, b]$. Hence, $T_2(D)$ is uniformly bounded operator on $D$ and $T_2$ is a compact operator. Similarly, by doing the same steps, we get $T_3, \ldots, T_N : D \to D$ are compact operators.

**Step 3.** Prove that the equation $a = T_1 a + T_2 y_2 + T_3 y_3 + \ldots + T_N y_N$ implies that $a \in D$, for all $y_2, \ldots, y_N \in D$.

Let $x \in D$ and $y_2, \ldots, y_N \in D$. Let $x = T_1 x + T_2 y_2 + T_3 y_3 + \ldots + T_N y_N$. Using condition (H4), we get that

$$
\|x(t)\| \leq \|T_1 x(t)\| + \|T_2 y_2(t)\| + \ldots + \|T_N y_N(t)\| \leq \left( \|f(t, x(t))\| + \|f(t, 0)\| \right) + \ldots + \left( \|f(t, y_N(t))\| + \|f(t, 0)\| \right)
$$

Thus, $T_2(D)$ is uniformly bounded on $[0, b]$. Hence, $T_2(D)$ is uniformly bounded operator on $D$ and $T_2$ is a compact operator. Similarly, by doing the same steps, we get $T_3, \ldots, T_N : D \to D$ are compact operators.
which implies
\[
\|x\| \leq d + F^*
+ \frac{b^\alpha}{\Gamma(\alpha + 1)} \left( \|h_2\|_{L^{\alpha+1}_1} + \ldots + \|h_N\|_{L^{\alpha+1}_1} \right).
\] (37)

Thus, \(x \in D\). By applying Theorem 6, we get that the system of fractional equations (19) has a solution in \([0,b]\).

**Example 8.** Consider the following system of R-L FDEs:

\[
D^{1/2} \left[ x_1(t) - \frac{\sin(t) x_1(t)}{2(2 + |x_1(t)|)} \right]
= tx_2(t) + 5x_2(t), \quad t \in [0,\pi],
\]

\[
D^{1/2} \left[ x_2(t) - \frac{\sin(t) x_2(t)}{2(2 + |x_2(t)|)} \right]
= tx_3(t) + 5x_3(t), \quad t \in [0,\pi],
\]

\[
D^{1/2} \left[ x_3(t) - \frac{\sin(t) x_3(t)}{2(2 + |x_3(t)|)} \right]
= tx_3(t) + 5x_3(t), \quad t \in [0,\pi],
\]

\[x_1(0) = x_2(0) = x_3(0) = 0.\]

In system (38), let \(f_1(t, x(t)) = \sin(t)|x(t)|/2(2 + |x(t)|)\), such that \(f_2(t, x(t), I^\beta x(t))) = tx(t)/(1 + |x(t)|)\) is defined as

\[
f_3(t, x(t), I^\beta x(t))) = \begin{cases}
5, & \text{if } x \leq 0, \\
5 + \frac{5x}{1 + 2x}, & \text{if } 0 < x < 2, \\
7, & \text{if } x \geq 2.
\end{cases}
\] (39)

Let \(x, y \in X\). Then, for all \(t \in [0,\pi]\), we have that

\[
|f_1(t, x(t)) - f_1(t, y(t))| \leq \frac{|x(t) - y(t)|}{2(2 + |x(t) - y(t)|)}. \] (40)

Therefore, \(f_2(t, x(t), I^\beta x(t))) \leq h_3(t)\), where \(h_3(t) = t\). Also \(f_3(t, x(t), I^\beta x(t))) \leq h_3(t)\), where \(h_3(t) = 7\). Let \(F^* = 0\), \(d = 1, M_1 = 2, b = \pi\), and \(h_1(t) = t, h_2(t) = 7\) in Theorem 7; we obtain \(K \geq 22\). Then, system (38) has a solution.

**5. Conclusions**

This paper introduced a new version of Krasnoselskii fixed-point theorem. This generalization is more general because we can study the \(N\)-tupled fixed point by it. Another advantage of the proposed \(N\)-tupled fixed point is to study the existence of systems of more than two operator equations. For that, we applied the abstract proposed fixed-point theorem to prove the existence of solution of the system of \(N\)-R-L-FDEs. We gave an example to illustrate the abstract proposed fixed-point theorem.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

The authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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