Research Article

Stationary Distribution and Extinction of a Stochastic SIQR Model with Saturated Incidence Rate

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In this paper, we consider a stochastic SIQR epidemic model with saturated incidence rate. By constructing a proper Lyapunov function, we obtain the existence and uniqueness of positive solution for this SIQR model. Furthermore, we study the dynamical properties of this stochastic SIQR model; that is, (i) we establish the sufficient condition for the existence of ergodic stationary distribution of the model; (ii) we obtain the extinction of the disease under some conditions. At last, numerical simulations are introduced to illustrate our theoretical results.

1. Introduction

Due to the importance of disease transmission and control, epidemic models are attracting more and more attention from scholars. Since the pioneer work of Kermack and McKendrick (see [1]), where they described a directly transmitted viral or bacterial agent in a closed population and divided the total population into three subparts as susceptibles (S), infectives (I), and recovers (R), mathematical models have proven their adequacy to describe and analyze the propagation and control of infectious diseases. From then on, various epidemic models have been developed in recent decades, such as SIR models, SIS models, and SEIR models; see [2–6].

In modern society, due to the close interpersonal communication, the harm of disease transmission is more and more significant, and various measures have been taken to control the spread of disease. Among them, isolation measure was widely used by public health institutions and governments to control the spread of disease. In reality, it has been proved that the isolation measure is an effective method to suppress disease such as leprosy, plague, cholera, smallpox, tuberculosis, measles, rabies, foot and mouth, and ebola, which has ravaged West Africa in recent years [7]. Therefore, from the point of view of mathematic, the quarantine subpopulation (Q) was introduced into the epidemic models, and SIQ, SIQS, SIQR models have been proposed in the literature [8–10]. Afterwards Hethcote et al. [11] analyzed six SIQS and

SIQR models with bilinear, standard, or quarantine-adjusted incidence and found that a Hopf bifurcation might occur in the SIQR model with quarantine-adjusted incidence rate. Recently, Nirwani et al. [12] considered a SIQR epidemic model with saturated incidence rate as follows:

$$\begin{aligned}\frac{dS(t)}{dt} &= \Lambda - \frac{\beta S(t) I(t)}{1 + \alpha I(t)} - \mu S(t), \\ \frac{dI(t)}{dt} &= \frac{\beta S(t) I(t)}{1 + \alpha I(t)} - (\delta + \gamma + \mu + \mu_1) I(t), \\ \frac{dQ(t)}{dt} &= \delta I(t) - (\kappa + \mu + \mu_2) Q(t), \\ \frac{dR(t)}{dt} &= \gamma I(t) + \kappa Q(t) - \mu R(t),\end{aligned}\tag{1}$$

where they found that the equilibrium points are locally asymptotically stable under certain conditions.

Recently, some researchers have considered the impact of environmental noise on the dynamics of epidemic models [13–16]. In reality, the environmental noise may involve the variations of factors such as climate, habitats, health habits, medical quality, and so on, which may affect the natural birth rate, death rate, and so on [17]. Especially, for human diseases, the nature of epidemic growth and spread is inherently stochastic, as the contacts of person-to-person

are unpredictable [18]. Therefore, stochastic models may be a more accurate method to describe the spread of infectious diseases in humans (see, e.g., [19–26]). There are various approaches to include random perturbation in epidemic models, and among them, parameter perturbation method is a frequently adopted way. In this paper, we will adopt the method of Cai et al. [27]; that is, we replace $-\mu dt$ in (1) with $-\tilde{\mu}dt = -\mu dt + \sigma_i dB_i(t)$ ($i = 1, 2, 3, 4$). Therefore, we propose the following stochastic SIQR epidemic model with saturated incidence:

$$\begin{aligned} dS(t) &= \left(\Lambda - \frac{\beta S(t) I(t)}{1 + \alpha I(t)} - \mu S(t) \right) dt + \sigma_1 S dB_1(t), \\ dI(t) &= \left(\frac{\beta S(t) I(t)}{1 + \alpha I(t)} - (\delta + \gamma + \mu + \mu_1) I(t) \right) dt \\ &\quad + \sigma_2 I dB_2(t), \\ dQ(t) &= (\delta I(t) - (\kappa + \mu + \mu_2) Q(t)) dt + \sigma_3 Q dB_3(t), \\ dR(t) &= (\gamma I(t) + \kappa Q(t) - \mu R(t)) dt + \sigma_4 R dB_4(t), \end{aligned} \quad (2)$$

where all parameters are positive. For the biological meaning of each coefficient, we briefly mention below: Λ is the recruitment rate of the population, β is the average number of adequate contacts of a person per unit time, α measures the appropriate preventions taken by infectious for epidemic control, and μ is the natural death rate of the population, while μ_1, μ_2 are the extra disease-related death rate in compartments I and Q , respectively. γ, κ denote the recover rates from group I, Q to R , respectively, and δ denotes the removal rate from I . Additionally, $B_1(t), B_2(t), B_3(t)$, and $B_4(t)$ are standard one-dimensional independent Brownian motions, and $\sigma_i^2 > 0$ are the intensities of the white noises, $i = 1, 2, 3, 4$.

This paper is organized as follows. In Section 2, we give some preliminaries and notations which will be used in the following analysis. In Section 3, we prove that there is a unique global positive solution of system (2). In Section 4, we show the main result that there is an ergodic stationary distribution of system (2) under certain conditions. In Section 5, we establish sufficient conditions for extinction of the disease. Some numerical simulations are given to illustrate the analytical result in Section 6. Finally, we provide a brief summary of the main results and put forward some research prospects.

2. Preliminaries and Some Notations

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, which satisfies the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets) and $B_i(t)$ ($i = 1, 2, 3, 4$) are defined on this complete probability space. Denote $\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$ and $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$.

For the d -dimensional stochastic differential equation

$$dX(t) = f(X(t), t) dt + g(X(t), t) dB(t) \quad (3)$$

for $t \geq 0$,

with initial value $X(0) = X_0 \in \mathbb{R}^d$, while $B(t)$ is a d -dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, we define the differential operator L of (3) as follows:

$$\begin{aligned} L &= \frac{\partial}{\partial t} + \sum_{i=1}^d f_i(X, t) \frac{\partial}{\partial X_i} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d [g^T(X, t) g(X, t)]_{ij} \frac{\partial^2}{\partial X_i \partial X_j}. \end{aligned} \quad (4)$$

For $V(X, t) \in C^{2,1}(\mathbb{R}^d \times [0, \infty); \mathbb{R}_+)$, which represents the family of nonnegative functions defined on $\mathbb{R}^d \times [0, \infty)$ such that they are continuously twice differentiable in X and once in t , by acting L on this function V , we have

$$\begin{aligned} LV(X, t) &= V_t(X, t) + V_X(X, t) f(X, t) \\ &\quad + \frac{1}{2} \text{trace} [g^T(X, t) V_{XX}(X, t) g(X, t)]. \end{aligned} \quad (5)$$

By the Itô's formula, if $X(t) \in \mathbb{R}^d$, then

$$\begin{aligned} dV(X(t), t) &= LV(X(t), t) dt \\ &\quad + V_X(X(t), t) g(X(t), t) dB(t). \end{aligned} \quad (6)$$

At last, we give some theory about the stationary distribution. Firstly, we recall the definitions of homogeneous Markov process and stationary distribution.

Definition 1 (see [28]). The transition probability function $P(s, t, x, A)$ is said to be time-homogeneous (and the corresponding Markov process is called time-homogeneous) if the function $P(s, t, x, A)$ is independent of s , where $0 \leq s \leq t, x \in \mathbb{R}^d$, and $A \in \mathcal{B}$ and $\mathcal{B}$ denotes the σ -algebra of Borel sets in $\mathbb{R}^d$.

Definition 2 (see [29]). Denote $\mathbb{P}_\gamma$ the corresponding probability distribution of an initial distribution γ , which describes the initial state of (3) at $t = 0$. Suppose that the distribution of $X(t)$ with initial distribution γ converges in some sense to a distribution $\pi = \pi_\gamma$, i.e.,

$$\lim_{t \rightarrow \infty} \mathbb{P}_\gamma \{X(t) \in F\} = \pi(F) \quad (7)$$

for all measurable F , then we say that (3) has a stationary distribution $\pi(\cdot)$.

Let $X(t)$ be a homogeneous Markov process in E_d and satisfy the following stochastic differential equation:

$$dX(t) = b(X) dt + \sum_{r=1}^k g_r(X) dB_r(t). \quad (8)$$

The diffusion matrix is denoted as $A(x) = (a_{ij}(x))$, where $a_{ij}(x) = \sum_{r=1}^k g_r^i(x) g_r^j(x)$.

Lemma 3 (see [28]). *The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded domain $D \subset E_d$ with regular boundary Γ and*

A_1 : in the open domain D and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero;

A_2 : if $x \in E_d \setminus D$, the mean time τ which a path issuing from x reaches the set D is finite, and $\sup_{x \in K} E_x \tau < \infty$ for every compact subset $K \in E_d$.

Then

$$\mathbb{P}_x \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(X(t)) dt = \int_{E_d} f(x) \pi(dx) \right\} = 1 \quad (9)$$

for all $x \in E_d$, where $f(\cdot)$ is a function integrable with respect to the measure π .

Remark 4. As pointed in [24], to validate A_1 , we need to prove

$\tilde{A}_1$: there is a positive number M such that $\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq M |\xi|^2$, $x \in D$, $\xi \in \mathbb{R}^d$; and to validate A_2 , we need to prove

$\tilde{A}_2$: there exists a nonnegative C^2 -function V such that LV is negative for any $E_d \setminus D$. Thus, we will verify the conditions $\tilde{A}_1$ and $\tilde{A}_2$ to obtain the existence of stationary distribution for Markov process $X(t)$.

3. Existence and Uniqueness of the Positive Solution

In this section, we will give the existence and uniqueness of positive solutions for system (2), which is the basis of studying the long-time behavior of the model.

Theorem 5. *For any initial value $X_0 = (S_0, I_0, Q_0, R_0) \in \mathbb{R}_+^4$, there is a unique positive solution $X(t) = (S(t), I(t), Q(t), R(t))$ of system (2) on $t \geq 0$ and the solution will maintain in $\mathbb{R}_+^4$ with probability one.*

Proof. The proof is motivated by the work Mao et al. [30]. Since the coefficients of system (2) satisfy the local Lipschitz condition, then for any initial value $(S_0, I_0, Q_0, R_0) \in \mathbb{R}_+^4$, there exists a unique local solution $(S(t), I(t), Q(t), R(t))$ on $[0, \tau_e)$, where τ_e is the explosion time [31]. To verify that this solution is global, we only need to prove that $\tau_e = \infty$ a.s. For this purpose, let $n_0 > 0$ be large enough such that every component of X_0 lay within the interval $[1/n_0, n_0]$. For each integer $n > n_0$, define the following stopping time:

$$\begin{aligned} \tau_n &= \inf \left\{ t \in [0, \tau_e) : \min \{(S(t), I(t), Q(t), R(t))\} \right. \\ &\quad \left. \leq \frac{1}{n} \text{ or } \max \{(S(t), I(t), Q(t), R(t))\} \geq n \right\}. \end{aligned} \quad (10)$$

For empty set Φ , we define $\inf \Phi = \infty$.

It is obvious that τ_n is increasing with $n \rightarrow \infty$. Let $\tau_\infty = \limsup_{n \rightarrow \infty} \tau_n$, then $\tau_\infty \leq \tau_e$ a.s. In the following, we only need to prove that $\tau_\infty = \infty$ a.s. If this assertion is not valid, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon$. Hence, there exists an integer $n_1 \geq n_0$ such that

$$\mathbb{P}\{\tau_n \leq T\} \geq \epsilon, \quad n \geq n_1. \quad (11)$$

Define a C^2 -function $V: \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$ as follows:

$$\begin{aligned} V(S, I, Q, R) &= (S - 1 - \ln S) + (I - 1 - \ln I) \\ &\quad + (Q - 1 - \ln Q) + (R - 1 - \ln R). \end{aligned} \quad (12)$$

Utilizing the Itô's formula, we get that

$$\begin{aligned} dV(S, I, Q, R) &= LV(S, I, Q, R) dt + \sigma_1(S - 1) dB_1(t) \\ &\quad + \sigma_2(I - 1) dB_2(t) \\ &\quad + \sigma_3(Q - 1) dB_3(t) \\ &\quad + \sigma_4(R - 1) dB_4(t), \end{aligned} \quad (13)$$

where

$$\begin{aligned} LV &= \left(1 - \frac{1}{S}\right) \left(\Lambda - \frac{\beta SI}{1 + \alpha I} - \mu S\right) \\ &\quad + \left(1 - \frac{1}{I}\right) \left(\frac{\beta SI}{1 + \alpha I} - (\delta + \gamma + \mu + \mu_1) I\right) \\ &\quad + \left(1 - \frac{1}{Q}\right) \times (\delta I - (\kappa + \mu + \mu_2) Q) \\ &\quad + \left(1 - \frac{1}{R}\right) (\gamma I + \kappa Q - \mu R) \\ &\quad + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\ &= \Lambda + 4\mu + \delta + \gamma + \kappa + \mu_1 + \mu_2 \\ &\quad - \mu(S + I + Q + R) - \mu_1 I - \mu_2 Q + \frac{\beta I}{1 + \alpha I} \\ &\quad - \frac{\beta S}{1 + \alpha I} - \frac{\Lambda}{S} - \frac{\delta I}{Q} - \frac{\gamma I}{R} - \frac{\kappa Q}{R} \\ &\quad + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\ &< \Lambda + 4\mu + \left(\delta + \gamma + \kappa + \frac{\beta}{\alpha} + \mu_1 + \mu_2\right) \\ &\quad + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} =: G, \end{aligned} \quad (14)$$

where G is a positive constant which is independent of S, I, Q, R , and t . Therefore,

$$\begin{aligned} dV(S, I, Q, R) &\leq G dt + \sigma_1(S - 1) dB_1(t) \\ &\quad + \sigma_2(I - 1) dB_2(t) \\ &\quad + \sigma_3(Q - 1) dB_3(t) \\ &\quad + \sigma_4(R - 1) dB_4(t). \end{aligned} \quad (15)$$

Integrating both sides of (15) from 0 to $\tau_n \wedge T$, then

$$\begin{aligned}
& V(S(\tau_n \wedge T), I(\tau_n \wedge T), Q(\tau_n \wedge T), R(\tau_n \wedge T)) \\
& - V(S_0, I_0, Q_0, R_0) \leq G(\tau_n \wedge T) \\
& + \int_0^{\tau_n \wedge T} [\sigma_1(S-1)dB_1(t) + \sigma_2(I-1)dB_2(t) \\
& + \sigma_3(Q-1)dB_3(t) + \sigma_4(R-1)dB_4(t)]
\end{aligned} \quad (16)$$

Further, taking the expectation for the above inequality and due to the property of Brownian motion, we obtain

$$\begin{aligned}
& \mathbb{E}V(S(\tau_n \wedge T), I(\tau_n \wedge T), Q(\tau_n \wedge T), R(\tau_n \wedge T)) \\
& \leq V(S_0, I_0, Q_0, R_0) + GT.
\end{aligned} \quad (17)$$

Let $\Omega_n = \{\tau_n \leq T\}$ for $n \geq n_1$. By (11) we have $\mathbb{P}(\Omega_n) \geq \epsilon$. From the definition of the stopping time, we can deduce that, for every $\omega \in \Omega_n$, $S(\tau_n, \omega)$ or $I(\tau_n, \omega)$ or $Q(\tau_n, \omega)$ or $R(\tau_n, \omega)$ equals either n or $1/n$. Hence, it follows from (17) that

$$\begin{aligned}
& V(S_0, I_0, Q_0, R_0) + GT \geq \mathbb{E}[1_{\Omega_n}(\omega)V(S(\tau_n, \omega), \\
& I(\tau_n, \omega), Q(\tau_n, \omega), R(\tau_n, \omega))] = \mathbb{P}(\Omega_n) \mathbb{E}V(S(\tau_n, \\
& \omega), I(\tau_n, \omega), Q(\tau_n, \omega), R(\tau_n, \omega)) \geq \epsilon \left[(n-1 - \ln n) \right. \\
& \left. \wedge \left(\frac{1}{n} - 1 + \ln n \right) \right],
\end{aligned} \quad (18)$$

where 1_{Ω_n} represents the indicator function of Ω_n . Letting $n \rightarrow \infty$ in above inequality yields a contradiction that

$$\infty > V(S_0, I_0, Q_0, R_0) + GT = \infty. \quad (19)$$

Therefore, $\tau_\infty = \infty$ a.s. and this completes the proof. $\square$

4. Stationary Distribution and Ergodicity

In the field of infectious diseases, the persistence and extinction of diseases are an important research issue. In this section, we will show that system (2) has an ergodic stationary distribution under certain condition.

Theorem 6. Assume that

$$R_0^S := \frac{\beta\Lambda}{(\mu + \sigma_1^2/2)(\gamma + \delta + \mu + \mu_1 + \sigma_2^2/2)} > 1, \quad (20)$$

then system (2) has a unique stationary distribution $\pi(\cdot)$ and it has the ergodic property.

Proof. By Theorem 5, we have obtained that, for any initial value $(S_0, I_0, Q_0, R_0) \in \mathbb{R}_+^4$, there is a unique global solution $(S, I, Q, R) \in \mathbb{R}_+^4$.

The diffusion matrix of system (2) is given by

$$A = \begin{pmatrix} \sigma_1^2 S^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 I^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 Q^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 R^2 \end{pmatrix}. \quad (21)$$

Choose $M = \min_{(S, I, Q, R) \in \overline{D}_\sigma \subset E_d} \{\sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 Q^2, \sigma_4^2 R^2\}$, where $E_d = \mathbb{R}_+^4$ and D_σ is a bounded domain in E_d , which does not contain the origin. For $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$, we have that

$$\begin{aligned}
& \sum_{i,j=1}^4 a_{ij}(S, I, Q, R) \xi_i \xi_j = \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 I^2 \xi_2^2 + \sigma_3^2 Q^2 \xi_3^2 \\
& + \sigma_4^2 R^2 \xi_4^2 \geq M |\xi|^2,
\end{aligned} \quad (22)$$

$(S, I, Q, R) \in \overline{D}_\sigma$.

Then the condition $\widetilde{A}_1$ holds.

Denote

$$\begin{aligned}
& \eta = \left(\mu + \frac{\sigma_1^2}{2} \right)^{1/2} \\
& \cdot \left[\left(\frac{\beta\Lambda}{\gamma + \delta + \mu + \mu_1 + \sigma_2^2/2} \right)^{1/2} - \left(\mu + \frac{\sigma_1^2}{2} \right)^{1/2} \right].
\end{aligned} \quad (23)$$

From the assumption $R_0^S > 1$, we know that η is positive. Construct a function $\widehat{V} : \mathbb{R}_+^4 \rightarrow \mathbb{R}$ in the following form:

$$\begin{aligned}
& \widehat{V}(S, I, Q, R) = M(-\ln S - c \ln I) \\
& + \frac{1}{\rho + 1} (S + I + Q + R)^{\rho+1} - \ln S \\
& - \ln Q - \ln R + (S + I + Q + R) \\
& := MV_1 + V_2 + V_3 + V_4 + V_5 + V_6,
\end{aligned} \quad (24)$$

where

$$\begin{aligned}
& c = \frac{\mu + \sigma_1^2/2}{\gamma + \delta + \mu + \mu_1 + \sigma_2^2/2}, \\
& 0 < \rho < \frac{2\mu}{\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2}
\end{aligned} \quad (25)$$

and M is positive constant which satisfying the following condition:

$$-2M\eta + \Lambda + B + 3\mu + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \leq -2, \quad (26)$$

where B is determined later. It is evident that

$$\liminf_{k \rightarrow \infty, (S, I, Q, R) \in \mathbb{R}_+^4 \setminus U_k} \widehat{V}(S, I, Q, R) = +\infty, \quad (27)$$

where $U_k = (1/k, k) \times (1/k, k) \times (1/k, k) \times (1/k, k)$. In addition, $\widehat{V}(S, I, Q, R)$ is a continuous function. Hence $\widehat{V}(S, I, Q, R)$ has a minimum point $(\bar{S}_0, \bar{I}_0, \bar{Q}_0, \bar{R}_0)$ in the interior of $\mathbb{R}_+^4$. Then we define a nonnegative C^2 -function $V : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$ as follows:

$$V(S, I, Q, R) = \widehat{V}(S, I, Q, R) - \widehat{V}(\bar{S}_0, \bar{I}_0, \bar{Q}_0, \bar{R}_0). \quad (28)$$

Making use of Itô's formula, we have

$$LV_1 = -\left(\frac{\Lambda}{S} + \frac{c\beta S}{1+\alpha I}\right) + \frac{\beta I}{1+\alpha I} + \mu + \frac{\sigma_1^2}{2} + c\left[(\gamma + \delta + \mu + \mu_1) + \frac{\sigma_2^2}{2}\right]. \quad (29)$$

Utilizing the inequality $a + b \geq 2\sqrt{ab}$, $a, b > 0$, yields

$$\begin{aligned} LV_1 &\leq -2\left(\frac{c\beta\Lambda}{1+\alpha I}\right)^{1/2} + \frac{\beta I}{1+\alpha I} + \mu + \frac{\sigma_1^2}{2} + c\left(\gamma + \delta + \mu + \mu_1 + \frac{\sigma_2^2}{2}\right) \\ &= -2\left[\frac{\beta\Lambda(\mu + \sigma_1^2/2)}{(\gamma + \delta + \mu + \mu_1 + \sigma_2^2/2)(1+\alpha I)}\right]^{1/2} \\ &\quad + 2\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{\beta I}{1+\alpha I} = -2\left(\mu + \frac{\sigma_1^2}{2}\right)^{1/2} \\ &\quad \cdot \left\{\left[\frac{\beta\Lambda}{(\gamma + \delta + \mu + \mu_1 + \sigma_2^2/2)(1+\alpha I)}\right]^{1/2}\right. \\ &\quad \left.- \left(\mu + \frac{\sigma_1^2}{2}\right)^{1/2}\right\} + \frac{\beta I}{1+\alpha I}. \end{aligned} \quad (30)$$

Meanwhile,

$$\begin{aligned} LV_2 &= (S + I + Q + R)^\rho \\ &\quad \cdot [\Lambda - \mu S - (\mu + \mu_1)I - (\mu + \mu_2)Q - \mu R] + \frac{1}{2} \\ &\quad \cdot \rho(S + I + Q + R)^{\rho-1} \\ &\quad \times (\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_3^2 Q^2 + \sigma_4^2 R^2) \\ &\leq (S + I + Q + R)^\rho [\Lambda - \mu(S + I + Q + R)] + \frac{1}{2} \\ &\quad \cdot \rho(S + I + Q + R)^{\rho+1} (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \\ &= \Lambda(S + I + Q + R)^\rho \\ &\quad - \left[\mu - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\right](S + I + Q + R)^{\rho+1} \\ &\leq B - \frac{1}{2}\left[\mu - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\right] \\ &\quad \cdot (S + I + Q + R)^{\rho+1} \leq B \\ &\quad - \frac{1}{2}\left[\mu - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\right] \\ &\quad \cdot (S^{\rho+1} + I^{\rho+1} + Q^{\rho+1} + R^{\rho+1}), \end{aligned} \quad (31)$$

where

$$\begin{aligned} B &= \sup_{(S,I,Q,R) \in \mathbb{R}_+^4} \left\{ \Lambda(S + I + Q + R)^\rho \right. \\ &\quad \left. - \frac{1}{2}\left[\mu - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\right] \right. \\ &\quad \left. \cdot (S + I + Q + R)^{\rho+1} \right\} < \infty. \end{aligned} \quad (32)$$

We can also obtain that

$$LV_3 = -\frac{\Lambda}{S} + \frac{\beta I}{1+\alpha I} + \mu + \frac{\sigma_1^2}{2}, \quad (33)$$

$$LV_4 = -\frac{\delta I}{Q} + (\kappa + \mu + \mu_2) + \frac{\sigma_3^2}{2}, \quad (34)$$

$$LV_5 = -\frac{\gamma I}{R} - \frac{\kappa Q}{R} + \mu + \frac{\sigma_4^2}{2} \quad (35)$$

and

$$LV_6 = \Lambda - \mu(S + I + Q + R) - \mu_1 I - \mu_2 Q \leq \Lambda. \quad (36)$$

Hence by (30), (31), (33), (34), (35), and (36), we can deduce that

$$\begin{aligned} LV &\leq -2M\left(\mu + \frac{\sigma_1^2}{2}\right)^{1/2} \\ &\quad \cdot \left\{\left[\frac{\beta\Lambda}{(\gamma + \delta + \mu + \mu_1 + \sigma_2^2/2)(1+\alpha I)}\right]^{1/2}\right. \\ &\quad \left.- \left(\mu + \frac{\sigma_1^2}{2}\right)^{1/2}\right\} + \frac{(M+1)\beta I}{1+\alpha I} - \frac{1}{2}\left[\mu \right. \\ &\quad \left.- \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\right](S^{\rho+1} + I^{\rho+1} + Q^{\rho+1} \\ &\quad + R^{\rho+1}) - \frac{\Lambda}{S} - \frac{\delta I}{Q} - \frac{\gamma I}{R} - \frac{\kappa Q}{R} + \Lambda + B + 3\mu + \kappa \\ &\quad + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2}. \end{aligned} \quad (37)$$

Next, we will construct a compact subset D such that the condition A_2 in Lemma 3 holds. Thus, we define the following bounded subset:

$$\begin{aligned} D &= \left\{ \epsilon_1 \leq S \leq \frac{1}{\epsilon_1}, \epsilon_1 \leq I \leq \frac{1}{\epsilon_1}, \epsilon_2 \leq Q \leq \frac{1}{\epsilon_2}, \epsilon_2 \leq R \right. \\ &\quad \left. \leq \frac{1}{\epsilon_2} \right\}, \end{aligned} \quad (38)$$

where $\epsilon_1, \epsilon_2 > 0$ are small enough and satisfy the following conditions:

$$-2M\tilde{\eta} + (M+1)\beta\epsilon_1 + \Lambda + B + 3\mu + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \leq -1, \quad (39)$$

$$-\frac{\Lambda}{\epsilon_1} + E \leq -1, \quad (40)$$

$$-\frac{\delta}{\epsilon_1} + E \leq -1, \quad (41)$$

$$-\frac{\gamma}{\epsilon_1} + E \leq -1, \quad (42)$$

$$-\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] \frac{1}{\epsilon_1^{\rho+1}} + E \leq -1, \quad (43)$$

$$-\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] \frac{1}{\epsilon_2^{\rho+1}} + E \leq -1, \quad (44)$$

where

$$\tilde{\eta} := \left(\mu + \frac{\sigma_1^2}{2} \right)^{1/2} \cdot \left\{ \left[\frac{\beta\Lambda}{(\gamma + \delta + \mu + \mu_1 + \sigma_2^2/2)(1 + \alpha\epsilon_1)} \right]^{1/2} - \left(\mu + \frac{\sigma_1^2}{2} \right)^{1/2} \right\}. \quad (45)$$

Note that, for sufficiently small ϵ_1 , (39) is valid due to inequality (26).

It can be easily seen that $\mathbb{R}_+^4 \setminus D$ is divided into eight domains, which can be denoted as

$$\begin{aligned} D_1 &= \{(S, I, Q, R) \in \mathbb{R}_+^4, 0 < S < \epsilon_1\}, \\ D_2 &= \{(S, I, Q, R) \in \mathbb{R}_+^4, 0 < I < \epsilon_1\}, \\ D_3 &= \{(S, I, Q, R) \in \mathbb{R}_+^4, S \geq \epsilon_1, I \geq \epsilon_1, 0 < Q < \epsilon_2\}, \\ D_4 &= \{(S, I, Q, R) \in \mathbb{R}_+^4, I \geq \epsilon_1, 0 < R < \epsilon_2\}, \\ D_5 &= \left\{ (S, I, Q, R) \in \mathbb{R}_+^4, S > \frac{1}{\epsilon_1} \right\}, \\ D_6 &= \left\{ (S, I, Q, R) \in \mathbb{R}_+^4, I > \frac{1}{\epsilon_1} \right\}, \\ D_7 &= \left\{ (S, I, Q, R) \in \mathbb{R}_+^4, Q > \frac{1}{\epsilon_2} \right\}, \\ D_8 &= \left\{ (S, I, Q, R) \in \mathbb{R}_+^4, R > \frac{1}{\epsilon_2} \right\}. \end{aligned} \quad (46)$$

Consequently, we consider the condition A_2 from the following eight cases.

Case 1. If $(S, I, Q, R) \in D_1$, from (37), we can get that

$$\begin{aligned} LV &\leq -\frac{\Lambda}{S} + 2M \left(\mu + \frac{\sigma_1^2}{2} \right) + \frac{(M+1)\beta}{\alpha} + \Lambda + B \\ &\quad + 3\mu + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} = -\frac{\Lambda}{S} + E \\ &\leq -\frac{\Lambda}{\epsilon_1} + E, \end{aligned} \quad (47)$$

where

$$\begin{aligned} E &= 2M \left(\mu + \frac{\sigma_1^2}{2} \right) + \frac{(M+1)\beta}{\alpha} + \Lambda + B + 3\mu + \kappa \\ &\quad + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2}. \end{aligned} \quad (48)$$

Then it infers from (40) that

$$LV \leq -1. \quad (49)$$

Case 2. If $(S, I, Q, R) \in D_2$, we obtain that

$$\begin{aligned} LV &\leq -2M \left(\mu + \frac{\sigma_1^2}{2} \right)^{1/2} \cdot \left\{ \left[\frac{\beta\Lambda}{(\gamma + \delta + \mu + \mu_1 + \sigma_2^2/2)(1 + \alpha I)} \right]^{1/2} \right. \\ &\quad \left. - \left(\mu + \frac{\sigma_1^2}{2} \right)^{1/2} \right\} + (M+1)\beta I + \Lambda + B + 3\mu + \kappa \\ &\quad + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \leq -2M\tilde{\eta} + (M+1)\beta\epsilon_1 + \Lambda \\ &\quad + B + 3\mu + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2}. \end{aligned} \quad (50)$$

In view of (39), we have that

$$LV \leq -1. \quad (51)$$

Case 3. If $(S, I, Q, R) \in D_3$, we have

$$\begin{aligned} LV &\leq -\frac{\delta I}{Q} + 2M \left(\mu + \frac{\sigma_1^2}{2} \right) + \frac{(M+1)\beta}{\alpha} + \Lambda + B \\ &\quad + 3\mu + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \leq -\frac{\delta\epsilon_1}{\epsilon_2} + E. \end{aligned} \quad (52)$$

Choosing

$$\epsilon_2 = \epsilon_1^2, \quad (53)$$

and combining with (41), we obtain that

$$LV \leq -\frac{\delta}{\epsilon_1} + E \leq -1. \quad (54)$$

Case 4. If $(S, I, Q, R) \in D_4$, using the similar technique in Case 3 and (42), we have

$$\begin{aligned} LV &\leq -\frac{\gamma I}{R} + 2M \left(\mu + \frac{\sigma_1^2}{2} \right) + \frac{(M+1)\beta}{\alpha} + \Lambda + B \\ &\quad + 3\mu + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \leq -\frac{\gamma \epsilon_1}{\epsilon_2} + E \\ &= -\frac{\gamma}{\epsilon_1} + E \leq -1. \end{aligned} \quad (55)$$

Case 5. If $(S, I, Q, R) \in D_5$, we can derive that

$$\begin{aligned} LV &\leq -\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] S^{\rho+1} \\ &\quad + 2M \left(\mu + \frac{\sigma_1^2}{2} \right) + \frac{(M+1)\beta}{\alpha} + \Lambda + B + 3\mu \\ &\quad + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\ &\leq -\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] \frac{1}{\epsilon_1^{\rho+1}} + E. \end{aligned} \quad (56)$$

By (43), we can conclude that $LV \leq -1$ on D_5 .

Case 6. If $(S, I, Q, R) \in D_6$, using the analogous estimation in Case 5 and (43), we have

$$\begin{aligned} LV &\leq -\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] I^{\rho+1} \\ &\quad + 2M \left(\mu + \frac{\sigma_1^2}{2} \right) + \frac{(M+1)\beta}{\alpha} + \Lambda + B + 3\mu \\ &\quad + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\ &\leq -\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] \frac{1}{\epsilon_1^{\rho+1}} + E \\ &\leq -1. \end{aligned} \quad (57)$$

Case 7. If $(S, I, Q, R) \in D_7$, it infers that

$$\begin{aligned} LV &\leq -\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] Q^{\rho+1} \\ &\quad + 2M \left(\mu + \frac{\sigma_1^2}{2} \right) + \frac{(M+1)\beta}{\alpha} + \Lambda + B + 3\mu \\ &\quad + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\ &\leq -\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] \frac{1}{\epsilon_2^{\rho+1}} + E. \end{aligned} \quad (58)$$

By (44), we can conclude that $LV \leq -1$ on D_7 .

Case 8. At last, if $(S, I, Q, R) \in D_8$, by (44), we have

$$\begin{aligned} LV &\leq -\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] R^{\rho+1} \\ &\quad + 2M \left(\mu + \frac{\sigma_1^2}{2} \right) + \frac{(M+1)\beta}{\alpha} + \Lambda + B + 3\mu \\ &\quad + \kappa + \mu_2 + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\ &\leq -\frac{1}{2} \left[\mu - \frac{1}{2} \rho (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \right] \frac{1}{\epsilon_2^{\rho+1}} + E \\ &\leq -1. \end{aligned} \quad (59)$$

According to the discussion of the above eight cases, we obtain that

$$LV \leq -1 \quad \forall (S, I, Q, R) \in \mathbb{R}_+^4 \setminus D. \quad (60)$$

Therefore, condition $\tilde{A}_2$ is satisfied, and we can obtain that system (2) is ergodic and has a unique stationary distribution. The proof is complete. $\square$

Remark 7. Theorem 6 gives the sufficient condition of the existence of stationary distribution for the solution of (2). In epidemiology, this means that the stochastic model (2) oscillates around the endemic equilibrium of the corresponding deterministic model and implies the persistence of the disease a.s. under some condition.

5. Extinction of the Disease

In this section, we will establish the sufficient conditions for the extinction of the disease in the system (2), which is another important issue in epidemiology. For simplicity, we define

$$\langle x(t) \rangle = \frac{1}{t} \int_0^t x(u) du. \quad (61)$$

Lemma 8. Let $(S(t), I(t), Q(t), R(t))$ be the solution of system (2) with any initial value $(S_0, I_0, Q_0, R_0) \in \mathbb{R}_+^4$. Then

$$\lim_{t \rightarrow \infty} \frac{S(t) + I(t) + Q(t) + R(t)}{t} = 0, \quad a.s. \quad (62)$$

Moreover, if $\mu > (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)/2$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t S(u) dB_1(u) &= 0, \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(u) dB_2(u) &= 0, \end{aligned} \quad (63)$$

a.s.,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q(u) dB_3(u) &= 0, \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t R(u) dB_4(u) &= 0, \end{aligned} \quad (64)$$

a.s.

Lemma 8 can be proved by using the same methods as that in Lemmas 2.1 and 2.2 of Zhao and Jiang [32], so we omit it here.

Define a constant as follows:

$$\hat{R}_0^S := \frac{\beta\Lambda/\mu - \sigma_2^2/2}{\delta + \gamma + \mu + \mu_1} = R_0 - \frac{\sigma_2^2}{2(\delta + \gamma + \mu + \mu_1)}, \quad (65)$$

where $R_0 = \beta\Lambda/\mu(\delta + \gamma + \mu + \mu_1)$ is the threshold value of the corresponding deterministic system for (2). Therefore, we obtain the following main result in this section.

Theorem 9. Assume that $\mu > (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)/2$ and $(S(t), I(t), Q(t), R(t))$ is the solution of system (2) with any initial value $(S_0, I_0, Q_0, R_0) \in \mathbb{R}_+^4$. If $\hat{R}_0^S < 1$, then

$$\lim_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq (\delta + \gamma + \mu + \mu_1) (\hat{R}_0^S - 1) \quad a.s., \quad (66)$$

and

$$\begin{aligned} \lim_{t \rightarrow \infty} \langle S(t) \rangle &= \frac{\beta\Lambda}{\mu}, \\ \lim_{t \rightarrow \infty} Q(t) &= 0, \\ \lim_{t \rightarrow \infty} R(t) &= 0 \end{aligned} \quad (67)$$

a.s.

Proof. By system (2), we have

$$\begin{aligned} d(S(t) + I(t) + Q(t) + R(t)) &= [\Lambda \\ &- \mu(S(t) + I(t) + Q(t) + R(t)) - \mu_1 I(t) \\ &- \mu_2 Q(t)] dt + \sigma_1 S(t) dB_1(t) + \sigma_2 I(t) dB_2(t) \\ &+ \sigma_3 Q(t) dB_3(t) + \sigma_4 R(t) dB_4(t). \end{aligned} \quad (68)$$

Integrating the both sides of (68) yields

$$\begin{aligned} \frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} + \frac{Q(t) - Q(0)}{t} \\ + \frac{R(t) - R(0)}{t} \\ = \Lambda - \mu \langle S(t) \rangle - (\mu + \mu_1) \langle I(t) \rangle \\ - (\mu + \mu_2) \langle Q(t) \rangle - \mu \langle R(t) \rangle \\ + \frac{\sigma_1}{t} \int_0^t S(u) dB_1(u) + \frac{\sigma_2}{t} \int_0^t I(u) dB_2(u) \\ + \frac{\sigma_3}{t} \int_0^t Q(u) dB_3(u) + \frac{\sigma_4}{t} \int_0^t R(u) dB_4(u), \end{aligned} \quad (69)$$

then

$$\begin{aligned} \langle S(t) \rangle &= \frac{\Lambda}{\mu} - \frac{\mu + \mu_1}{\mu} \langle I(t) \rangle - \frac{\mu + \mu_2}{\mu} \langle Q(t) \rangle \\ &- \langle R(t) \rangle + \varphi(t), \end{aligned} \quad (70)$$

where $\varphi(t)$ is denoted as below:

$$\begin{aligned} \varphi(t) &= -\frac{1}{\mu} \left[\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} \right. \\ &+ \frac{Q(t) - Q(0)}{t} + \frac{R(t) - R(0)}{t} \\ &- \frac{\sigma_1}{t} \int_0^t S(u) dB_1(u) - \frac{\sigma_2}{t} \int_0^t I(u) dB_2(u) \\ &- \left. \frac{\sigma_3}{t} \int_0^t Q(u) dB_3(u) - \frac{\sigma_4}{t} \int_0^t R(u) dB_4(u) \right]. \end{aligned} \quad (71)$$

Due to (62)-(64), we have

$$\lim_{t \rightarrow \infty} \varphi(t) = 0 \quad a.s. \quad (72)$$

By utilizing Itô's formula to the I component of system (2), we have

$$\begin{aligned} d(\ln I) &= \left[\frac{\beta S}{1 + \alpha I} - (\delta + \gamma + \mu + \mu_1) - \frac{1}{2} \sigma_2^2 \right] dt \\ &+ \sigma_2 dB_2(t) \\ &\leq \left[\beta S - (\delta + \gamma + \mu + \mu_1) - \frac{1}{2} \sigma_2^2 \right] dt \\ &+ \sigma_2 dB_2(t). \end{aligned} \quad (73)$$

Integrating both sides of the above inequality from 0 to t and dividing t , we obtain

$$\begin{aligned} \frac{\ln I(t) - \ln I(0)}{t} &\leq \beta \langle S(t) \rangle - (\delta + \gamma + \mu + \mu_1) - \frac{\sigma_2^2}{2} \\ &+ \frac{\sigma_2 B_2(t)}{t}. \end{aligned} \quad (74)$$

Substituting (70) into (74) leads to

$$\begin{aligned} \frac{\ln I(t)}{t} &\leq \frac{\beta\Lambda}{\mu} - (\delta + \gamma + \mu + \mu_1) - \frac{\sigma_2^2}{2} + \beta\varphi(t) \\ &\quad + \frac{\sigma_2 B_2(t)}{t} + \frac{\ln I(0)}{t} \\ &= (\delta + \gamma + \mu + \mu_1) (\hat{R}_0^S - 1) + \beta\varphi(t) \\ &\quad + \frac{\sigma_2 B_2(t)}{t} + \frac{\ln I(0)}{t}. \end{aligned} \quad (75)$$

Due to the fact that $\lim_{t \rightarrow \infty} (B_2(t)/t) = 0$ a.s. and (72), by taking the superior limit on the both sides of (75), if $\hat{R}_0^S < 1$, then

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq (\delta + \gamma + \mu + \mu_1) (\hat{R}_0^S - 1) < 0 \quad (76)$$

a.s.,

which infers that

$$\lim_{t \rightarrow \infty} I(t) = 0 \quad a.s. \quad (77)$$

Hence, from (77) and the last two equations of (2), we have

$$\begin{aligned} \lim_{t \rightarrow \infty} Q(t) &= 0, \\ \lim_{t \rightarrow \infty} R(t) &= 0 \end{aligned} \quad (78)$$

a.s.

At last, by (72), (77), and (78) and from (70), we obtain

$$\lim_{t \rightarrow \infty} \langle S(t) \rangle = \frac{\beta\Lambda}{\mu} \quad a.s. \quad (79)$$

This completes the proof. $\square$

6. Numerical Simulations

In this section, we give some numerical simulations in order to illustrate the theoretical results. For numerical simulation, by using Milstein's method [33], we obtain the following discretization transformation of system (2):

$$\begin{aligned} S_{j+1} &= S_j + \left[\Lambda - \frac{\beta S_j I_j}{1 + \alpha I_j} - \mu S_j \right] \Delta t + \sigma_1 S_j \sqrt{\Delta t} \varepsilon_{1,j} \\ &\quad + \frac{\sigma_1^2}{2} S_j (\varepsilon_{1,j}^2 - 1) \Delta t, \\ I_{j+1} &= I_j + \left[\frac{\beta S_j I_j}{1 + \alpha I_j} - (\delta + \gamma + \mu + \mu_1) I_j \right] \Delta t \\ &\quad + \sigma_2 I_j \sqrt{\Delta t} \varepsilon_{2,j} + \frac{\sigma_2^2}{2} I_j (\varepsilon_{2,j}^2 - 1) \Delta t, \\ Q_{j+1} &= Q_j + (\delta I_j - (\kappa + \mu + \mu_2) Q_j) \Delta t \\ &\quad + \sigma_3 Q_j \sqrt{\Delta t} \varepsilon_{3,j} + \frac{\sigma_3^2}{2} Q_j (\varepsilon_{3,j}^2 - 1) \Delta t, \\ R_{j+1} &= R_j + (\gamma I_j + \kappa Q_j - \mu R_j) \Delta t + \sigma_4 R_j \sqrt{\Delta t} \varepsilon_{4,j} \\ &\quad + \frac{\sigma_4^2}{2} R_j (\varepsilon_{4,j}^2 - 1) \Delta t, \end{aligned} \quad (80)$$

where $\Delta t > 0$, $\sigma_i^2 > 0$ ($i = 1, 2, 3, 4$) are the intensities of the white noise, $\varepsilon_{i,j}$ ($i = 1, 2, 3, 4$, $j = 1, 2, \dots, n$) are independent Gaussian random variables which follow $N(0, 1)$. In order to verify the persistence and extinction results, we choose the parameters in system (80) by $\beta = 0.3$, $\mu = \mu_1 = 0.1$, $\mu_2 = \gamma = \delta = 0.2$, $\kappa = 0.1$, $\alpha = 0.5$, $\sigma_1 = \sigma_3 = 0.05$, and $\sigma_4 = 0.1$.

Firstly, we start our numerical simulation with $\Lambda = 0.3$, $\sigma_2 = 0.1$ and starting from the initial point $(2, 0.5, 0.5, 1)$. Direct calculation leads to $R_0^S := \beta\Lambda/(\mu + \sigma_1^2/2)(\gamma + \delta + \mu + \mu_1 + \sigma_2^2/2) \approx 1.4692 > 1$. Thus, from Theorem 6, we can conclude that system (2) has a unique ergodic stationary distribution which leads to the stochastically persistence of the disease. Figure 1 confirms this.

Next, we increase the stochastic intensity of I to $\sigma_2 = 0.4$ and choose $\Lambda = 0.22$. In this case, $\mu > (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)/2$ and $\hat{R}_0^S := R_0 - \sigma_2^2/2(\delta + \gamma + \mu + \mu_1) \approx 0.9667 < 1 < R_0$. According to Theorem 9, we obtain that system (2) has disease dies out with probability 1, and Figure 2 illustrates this.

7. Conclusion and Discussion

In this paper, we consider a stochastic SIQR epidemic model with saturated incidence rate. By constructing proper Lyapunov function, we establish the existence and uniqueness of solutions of system (2). Furthermore, we obtain the existence of ergodic stationary distribution of the solution to (2) when $R_0^S > 1$. Meanwhile, we establish sufficient conditions for the extinction of the disease as well, that is, $\mu > (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)/2$ and $\hat{R}_0^S < 1$.

It is worth further studying the following two topics. On the one hand, noting that $R_0^S \neq \hat{R}_0^S$, it is of great interest

FIGURE 1: The left column shows the solutions of stochastic system (2) under the noise intensities $\sigma_1 = \sigma_3 = 0.05$, $\sigma_2 = 0.4$, $\sigma_4 = 0.1$. The right column displays the histogram of the probability density function of $S(t)$, $I(t)$, $Q(t)$, $R(t)$.

that whether can we promote the sufficient conditions for the extinction of the disease and get the threshold value for the extinction and persistence of the disease? On the other hand, we can propose SIQR epidemic model with some other types of random perturbations, such as telephone noise which can cause the population system to switch among different environmental regimes (see, e.g., [34–37]). In the literatures, it is common to use the continuous-time Markov chain with

values in a finite state space to describe the random switching of environmental regimes. We will carry out these studies in the future.

Data Availability

No data were used to support this study.

FIGURE 2: Numerical simulation for system (2) with $\sigma_1 = \sigma_3 = 0.05$, $\sigma_2 = \sigma_4 = 0.1$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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