Research Article

Homoclinic Solutions for a Higher Order $\phi_c$-Laplacian Difference Equation Containing Both Advance and Retardation

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1. Introduction

Consider the following 2nth-order nonlinear difference equation containing both advance and retardation with $\phi_c$-Laplacian:

\[-(1)^m \Delta^m \left( \phi_c \left( \Delta^m u_{n-1}\right) \right) + \omega_n u_n = f(n, u_{n+1}, u_n, u_{n-1}), \quad n \in \mathbb{Z},\]

where $\Delta$ is the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$ and $\Delta^j u_k = \Delta (\Delta^{j-1} u_k)$ for $j \geq 2$, $\omega_n > 0$ is real valued for each $n \in \mathbb{Z}$, $f \in C(Z \times \mathbb{R}^3, \mathbb{R})$, $[\omega_n]$, and $f(n)$ are $T$-periodic in $n$ for a given positive integer $T$, and $\phi_c(s)$ is a special $\phi_c$-Laplacian operator [1] defined by $\phi_c(s) = (s/\sqrt{1 + |s|^2}), s \in \mathbb{R}$.

It is well known that $\phi_c(s)$ is the mean curvature operator given by the corresponding potential $\Phi_c(s) = \sqrt{1 + |s|^2} - 1$. And it is such that

\[\phi_c^{-1}(v) = \frac{v}{\sqrt{1 - |v|^2}} = \nabla \Phi_R(v), \quad v \in (-1, 1),\]

where $\Phi_R(v) = 1 - \sqrt{1 - |v|^2}$ may be seen as the acceleration in classical relativity. Actually, it is important to determine the existence and regularity properties of maximal and constant mean curvature hypersurfaces, as they provide Riemannian submanifolds that reflect spatiotemporal properties. There have been extensive study and application of nonlinear difference equations with $\phi_c$-Laplacian. For example, scalar equations were considered in [2, 3] using topological methods, and $T$-periodic solutions have been studied in [1] by variational methods.

Bonheure et al. [4] in 2007 studied classical and non-classical solutions of a prescribed curvature equation:

\[-\left( \frac{u'}{\sqrt{1 + |u'|^2}} \right)' = \lambda f(t, u), \quad t \in \mathbb{R}, \text{ with } u(0) = u(1) = 0.\]

(3)

This problem depends on the behavior at the origin and at infinity of the potential $\int_0^s f(t, s)ds$. Their method is essentially variational and is based on a regularization of the action functional associated with the curvature problem.

Due to their wide range of applications, there has been tremendous interest in studying some properties of the nonlinear discrete $\phi_c$-Laplacian equations, including periodic solutions [1], boundary value problems [5, 6], and homoclinic orbits [7, 8]. For instance, using the mountain pass lemma, Zhou and Su [5] obtained some sufficient conditions for the first time on the existence of solutions of the 2nth-order $\phi_c$-Laplacian difference equation with the boundary value
conditions, and by the critical point theory, Zhou and Ling [6] obtained some sufficient conditions on the existence of infinitely many positive solutions of the boundary value problems for a second-order $\phi_c$-Laplacian difference equation. The importance of nonlinear difference equations is described in [9, 10], with applications involving statistics, computing, electrical circuit analysis, dynamical systems, and economics. And they appear naturally as discrete analogue of differential equations which model various diverse phenomena.

We assume that $f(n, 0, 0, 0) = 0$ for each $n \in \mathbb{Z}$, then \{u_n\} = \{0\} is a solution of (1), which is called the trivial solution. As usual, we say that a solution $u = \{u_n\}$ of (1) is homoclinic (to 0) if $\lim_{\nu \to -\infty} u_n = 0$. In addition, if $\{u_n\} \neq \{0\}$, then $u$ is called a nontrivial homoclinic solution.

In recent years, the existence and multiplicity of homoclinic solutions for various discrete systems have been investigated by many researchers. For some recent work, the reader can refer to [11–22] and the reference therein. For example, by using the symmetric mountain pass theorem, Chen and Tang [12] studied the existence of infinitely many homoclinic orbits for the fourth-order difference systems containing both advance and retardation:

$$\Delta^4 u(n-2) + q(n) u(n) = f(n, u(n+1), u(n), u(n-1)), \quad n \in \mathbb{Z}. \tag{4}$$

Kong [20] applied the critical point theory to study the existence of at least three homoclinic solutions for a $p$-Laplacian difference equation of higher order with both advance and retardation:

$$(-1)^n \Delta^n \left[(a(k-n)\phi_p'(\Delta^n u(k-n)) + b(k)\phi_p'(u(k)) = \lambda f(k, u(k+1), u(k), u(k-1)), \quad k \in \mathbb{Z}. \tag{5}$$

However, to the best of our knowledge, there is no work on the existence of homoclinic solutions for $\phi_c$-Laplacian difference equations containing both advance and retardation which have important background and significance in the field of cybernetics and biological mathematics [23, 24]. Motivated by the reasons aforementioned, in this paper, we will establish some new sufficient conditions for the existence of homoclinic solutions to (1) in a very general case of the nonlinear functional.

On the other hand, we notice that the nonlinear term $f$ is only either superlinear or asymptotically linear at $\infty$ which plays an important role in the existence of homoclinic solutions when the similar arguments were considered in many references [25–28]. But in this paper, the nonlinearities can be mixed superlinear with asymptotically linear at $\infty$, see Remark 1 for details. In fact, our conditions on the potential are rather relaxed, and some existing results in the literature are improved (see Remark 2). Moreover, we give an example to illustrate the main result.

Now, we establish the main result.

**Theorem 1.** Assume that the following conditions hold:

- $(T_1)$ there exists a functional $F(n, v_1, v_2) \in C^1(\mathbb{Z} \times \mathbb{R}^2, \mathbb{R})$ with $F(n+T, v_1, v_2) = F(n, v_1, v_2)$ and it satisfies

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3). \tag{6}$$

- $(T_2)$

$$\lim_{v_1^2 + v_2^2 \to 0} \frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} = 0,$$

$$\lim_{v_1^2 + v_2^2 + v_3^2 \to 0} \frac{F(n, v_1, v_2, v_3)}{v_1^2 + v_2^2 + v_3^2} = 0. \tag{7}$$

- $(T_3)$ there exists a real sequence $\{a_n\}$ such that

$$\limsup_{v_1^2 + v_2^2 \to \infty} \frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} = a_n \leq \infty. \tag{8}$$

**Remark 1.** The condition $(T_3)$ shows that the nonlinear term mix superlinear nonlinearities with asymptotically linear ones at $\infty$.

**Remark 2.** If $m = 1$ and $f(n, u_{n+1}, u_n, u_{n-1}) = g(n, u_n)$, Theorem 1 can reduce to Theorem 2 when $\phi_c$-Laplacian is $\phi_c$-Laplacian in [8]. And our sufficient conditions are based on the limit superior and limit inferior, which are more applicable.

This paper is organized as follows: In Section 2, we establish the variational framework associated with (1) and recall some related fundamental results for convenience. In Section 3, some lemmas are proved, and then we complete the proof of our results by the mountain pass lemma. Finally, we illustrate our main result with an example in Section 4.

## 2. Preliminaries

This section is to establish the corresponding variational framework for (1) and cite some basic conclusions for the coming discussion.

Let

$$l^p = \left\{ u = \{u_n\} : n \in \mathbb{Z}, u_n \in \mathbb{R}, \|u\|_{l^p} = \left( \sum_{n \in \mathbb{Z}} |u_n|^p \right)^{1/p} < \infty \right\}, \tag{9}$$

then, the corresponding inner product denoted by $\langle \cdot, \cdot \rangle$ in $l^2$ and the corresponding norm in $l^2$ is denoted by $\| \cdot \| = \| \cdot \|_{l^2}$. 
Let $S$ be the set of all two-sided sequences, that is,
\[ S = \{u = [u_n] | u_n \in \mathbb{R}, n \in \mathbb{Z}\}. \tag{10} \]

Then, $S$ is a vector space with $au + bv = [au_n + bv_n]$ for $u, v \in S$ and $a, b \in \mathbb{R}$. For any fixed positive integer $k$, we define the subspace $E_k$ of $S$ as
\[ E_k = \{u = [u_n] \in S | u_{n+2kT} = u_n, n \in \mathbb{Z}\}. \tag{11} \]

Obviously, $E_k$ is isomorphic to $\mathbb{R}^{2kT}$, and hence $E_k$ can be equipped with the inner product $\langle \cdot, \cdot \rangle_k$ and norm $\| \cdot \|_k$ as
\[
(\langle u, v \rangle)_k = \sum_{n=-kT}^{kT-1} u_n v_n, \quad u, v \in E_k,
\]
\[
\|u\|_k = \left( \sum_{n=-kT}^{kT-1} u_n^2 \right)^{1/2}, \quad u \in E_k,
\]
respectively. We also define a norm $\| \cdot \|_{kco}$ in $E_k$ by
\[
\|u\|_{kco} = \max\{\|u\| : -kT \leq n \leq kT - 1\}, \quad u \in E_k. \tag{13}
\]

Consider the functional $J_k$ in $E_k$ defined by
\[
J_k(u) = \sum_{n=-kT}^{kT-1} \left[ \Phi_k(\Delta^n u_n) + \frac{1}{2} \omega_n u_n^2 - F(n, u_{n+1}, u_n) \right],
\]
and the Fréchet derivative is given by
\[
\langle J'_k(u), v \rangle = \sum_{n=-kT}^{kT-1} \left[ \Phi_k(\Delta^n u_n) \Delta^n v_n + \omega_n u_n v_n - f(n, u_{n+1}, u_n, u_{n-1}) v_n \right], \quad u, v \in E_k. \tag{15}
\]

Equation (15) implies that (1) is the corresponding Euler–Lagrange equation for $J_k$. Therefore, we have reduced the problem of finding a nontrivial solution of (1) to that of seeking a nonzero critical point of the functional $J_k$.

It is easy to see that the critical points of $J_k$ in $E_k$ are exactly $2kT$-periodic solutions of system (1).

Let $H$ be a Hilbert space, and $C^1(H, \mathbb{R})$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on $H$. We conclude this section with some known results.

**Definition 1.** Let $J \in C^1(H, \mathbb{R})$. A sequence $\{x_j\} \subset H$ is called a Cerami sequence ((C) sequence for short) for $J$, if $J(x_j) \to c$ for some $c \in \mathbb{R}$ and $(1 + \|x_j\|)J'(x_j) \to 0$ as $j \to \infty$. We say, $J$ satisfies the Cerami condition ((C) condition for short) if any (C) sequence for $J$ possesses a convergent subsequence.

Let $B_r$ be the open ball in $H$ with radius $r$ and center 0, and let $\partial B_r$ denote its boundary.

**Lemma 1.** (Mountain pass lemma [29]). If $J \in C^1(H, \mathbb{R})$ and satisfies the following conditions: there exist $e \in H \setminus \{0\}$ and $r \in (0, \|e\|)$ such that $\max\{J(0), J(e)\} < \inf_{\|u\| = r} J(u)$, then, there exists a $(C)$ sequence $\{u_n\}$ for the mountain pass level $c$ which is defined by
\[
c = \inf_{h \in \mathcal{S}} \max_{s \in [0, 1]} J(h(s)), \tag{16}
\]
where
\[
\mathcal{S} = \{h \in C([0, 1], H) : h(0) = 0, h(1) = e\}. \tag{17}
\]

**3. Proof of Main Result**

In order to prove Theorem 1, we need several lemmas. Let
\[
\omega_* = \min_{n \in \mathbb{Z}} \{\omega_n\}. \tag{18}
\]

**Lemma 2.** Under the assumptions of Theorem 1, the functional $J_k$ satisfies the $(C)$ condition.

**Proof 1.** Let $\{u^{(j)}\} \subset E_k$ be a $(C)$ sequence for $J_k$. We need to show that $\{u^{(j)}\}$ has a convergent subsequence. Since $E_k$ is finite dimensional, it suffices to show that $\|u^{(j)}\|_k$ is bounded. Since $J_k(u^{(j)}) \to c$ and $(1 + \|u^{(j)}\|)J'_k(u^{(j)}) \to 0$, then there exists $M > 2c + 1$ such that $|J_k(u^{(j)})| \leq ((M - 1)/2)$ and $(1 + \|u^{(j)}\|)J'_k(u^{(j)}) < 1$ for $j \in \mathbb{N}$. So, we have $\|u^{(j)}\|_k J'_k(u^{(j)}) \leq (1 + \|u^{(j)}\|)J'_k(u^{(j)}) < 1$ for $j \in \mathbb{N}$. Then, by (14), (15), and
\[
2\Phi_k(u) = 2(\sqrt{1 + u^2} - 1) \geq \frac{u}{\sqrt{1 + u^2}} u = \Phi_k(u) \geq 0,
\]
for $u \in \mathbb{R}$,
we have
\[
\sum_{n=-kT}^{kT-1} \left( u^{(j)}_{n+1} + \frac{\partial F(n, u^{(j)}_{n+1}, u^{(j)}_n)}{\partial u^{(j)}_{n+1}} u^{(j)}_{n+1} - 2F(n, u^{(j)}_{n+1}, u^{(j)}_n) \right)
\leq 2J_k(u^{(j)}) - \left( f'_k(u^{(j)}), u^{(j)} \right)
\leq 2J_k(u^{(j)}) + \|u^{(j)}\|_k \left\| J'_k(u^{(j)}) \right\|
\leq M. \tag{20}
\]

From $(T_4)$ and $(T_3)$, there exists $\delta > 0$ such that
\[
\frac{\partial F(n, u_1, u_2)}{\partial u_1} u_1 + \frac{\partial F(n, u_1, u_2)}{\partial u_2} u_2 - 2F(n, u_1, u_2) > M, \tag{21}
\]
for $n \in \mathbb{Z}$ and $u_1^2 + u_2^2 > \delta^2$,
then, (20) implies that $\|u^{(j)}\|_k \leq \delta$ for $n \in \mathbb{Z}$, that is,
\[
\left\| u^{(j)} \right\|_{kco} \leq \delta. \tag{22}
\]
Since $E_k$ is finite dimensional, $\| \cdot \|_k$ and $\| \cdot \|_{k_{\infty}}$ are equivalent. Then, (22) implies that $\| u^{(k)}_m \|_k$ is bounded. The proof is completed.

**Lemma 3.** Under the assumptions of Theorem 1, there exists $n_0 \in \mathbb{N}$ such that $J_k$ has at least a nonzero critical point $u^{(k)}$ in $E_k$ for each $k \geq n_0$.

**Proof 2.** We first show that $J_k$ satisfies the conditions of Lemma 1. From (T2), there exists $r > 0$ such that

$$F(n, u_1, u_2) \leq \frac{1}{8} \omega_n(u_1^2 + u_2^2), \quad \text{for } u_1^2 + u_2^2 \leq r^2. \quad (23)$$

Then, for $u \in E_k$ with $u_k \leq r$,

$$J_k(u) \geq \frac{1}{2} \sum_{n=1}^{kT-1} \omega_n u_n^2 - \sum_{n=1}^{kT-1} F(n, u_{n+1}, u_n) \geq \frac{1}{2} \sum_{n=1}^{kT-1} \omega_n u_n^2 - \sum_{n=1}^{kT-1} \frac{1}{8} \omega_n (u_{n+1}^2 + u_n^2) \geq \frac{1}{4} \omega_n \| u \|_{E_k}^2. \quad (24)$$

Taking $a = (1/4) \omega_n r^2$, then $J_k|_{\| u \| \geq a} > 0$. Since $2a_n > \omega_n$, there exists $d > 0$ such that

$$a_n - \frac{\alpha_n}{2} > d, \quad \text{for } n \in \mathbb{Z}. \quad (25)$$

Let $\varepsilon \in (0, 1)$ satisfy

$$2 \varepsilon - d < 0. \quad (26)$$

Since $-\Delta^2$ is a bounded semipositive self-adjoint linear operator in $L^2$ and $0 \in \sigma(-\Delta^2)$, there exists $e = \{e_n\} \in L^2$ with $\sum_{n=-\infty}^{\infty} |e_n|^2 = 1$ such that $\sum_{n=-\infty}^{\infty} |\Delta^m e_n|^2 \leq \varepsilon$. Let $n_0$ be large enough such that

$$\sum_{n=-n_0T}^{n_0T-1} |\Delta^m e_n|^2 \leq \varepsilon, \quad (27)$$

$$\frac{n_0T-1}{2} \leq \sum_{n=-n_0T}^{n_0T-1} e_n^2 \leq 1.$$

For $k \geq n_0$, define $e^{(k)}_n$ by

$$e^{(k)}_n = \begin{cases} e_n, & -n_0T \leq n \leq n_0T - 1, \\ 0, & -kT \leq n \leq -n_0T - 1 \text{ or } n_0T \leq n \leq kT - 1. \end{cases} \quad (28)$$

By (T3), there exists $\mu_0 > \max\{r, 1\}$, such that

$$F(n, u_n, \mu e_n) \geq (a_n - \varepsilon) \mu^2 (e_{n+1}^2 + e_n^2), \quad \text{for } -n_0T \leq n \leq n_0T - 1, \mu \geq \mu_0. \quad (29)$$

Then, for $\mu \geq \mu_0$,

$$J_k(\mu e^{(k)}) = \sum_{n=-kT}^{kT-1} \left( \Phi_n(\mu a_n e^{(k)}_n) + \frac{\alpha_n}{2} \mu^2 e_n^{(k)} - F(n, u_n^{(k)}, \mu e_n^{(k)}) \right) \leq \sum_{n=-kT}^{kT-1} \left( \mu a_n \mu^2 e_n^{(k)} + \frac{\alpha_n}{2} \mu^2 \right) + (\varepsilon - a_n) \mu^2 \leq (2 \varepsilon - d) \mu^2. \quad (30)$$

Thus,

$$J_k(\mu e^{(k)}) \leq (2 \varepsilon - d) \mu^2 < 0. \quad (31)$$

It can easily be seen that $J_k(0) = 0$ for each $k \in \mathbb{Z}$. Then, we have $r \in (0, \| \mu e^{(k)} \|_k)$ and

$$\max \{ J_k(0), J_k(\mu e^{(k)}) \} = 0 < a \leq \inf_{h \in \Gamma_k, \varepsilon \in (0, 1)} J_k(u). \quad (32)$$

Now that we have verified all assumptions of Lemma 1, and then we know $J_k$ possesses a $(C)$ sequence $\{ u^{(k)} \}$ for the mountain pass level $\alpha_k \geq a$ with

$$\alpha_k = \inf_{h \in \Gamma_k, \varepsilon \in (0, 1)} \max J_k(h(s)), \quad (33)$$

where

$$\Gamma_k = \{ h \in C([0,1], E_k) : h(0) = 0, h(1) = \mu e^{(k)} \}. \quad (34)$$

According to Lemma 2, $\{ u^{(k)} \}$ has a convergent subsequence $\{ u^{(k)}_{j_m} \}$ such that $u^{(k)}_{j_m} \rightarrow u^{(k)}$ as $j_m \rightarrow +\infty$ for some $u^{(k)} \in E_k$. Since $J_k \in C_k(E_k, \mathbb{R})$, we have

$$J_k(u^{(k)}_{j_m}) \rightarrow J_k(u^{(k)}), \quad (35)$$

$$\left( 1 + \| u^{(k)}_{j_m} \|_k \right) f'(u^{(k)}_{j_m}) \rightarrow \left( 1 + \| u^{(k)} \|_k \right) f'(u^{(k)}),$$

as $j_m \rightarrow +\infty$. By the uniqueness of the limit, we obtain that $u^{(k)}$ is a critical point of $J_k$ corresponding to $\alpha_k$. Moreover, $u^{(k)}$ is nonzero as $\alpha_k \geq a > 0$.

**Lemma 4.** There exist $p, q > 0$ such that

$$p \leq \| u^{(k)} \|_{k_{\infty}} \leq q.\quad (36)$$

holds for every critical point $u^{(k)}$ obtained by Lemma 3, of $J_k$ in $E_k$ with $k \geq n_0$, where $n_0$ is defined in Lemma 3.

**Proof 3.** For $k \geq n_0$ and $s \in [0,1]$, we define $h_k \in \Gamma_k$ as $h_k(s) = s \mu e^{(k)}$; similarly to the derivation of [5], we can find
\[ J_k(u^{(k)}) \leq \max_{n \in [0,1]} \{ J_k(s \mu_0 e^{(k)}) \} \]
\[ \leq \max_{n \in [0,1]} \left\{ \sum_{n=n_0}^{n=T} \left[ |\Delta^n (s \mu_0 e_n)| + \frac{\omega_n}{2} (s \mu_0 e_n)^2 - F(n, s \mu_0 e_{n+1}, s \mu_0 e_n) \right] \right\} \]
\[ \leq \max_{n \in [0,1]} \left\{ 2^n \sqrt{2n_0 T} \left( \sum_{n=n_0}^{n=T} (s \mu_0 e_n)^2 \right)^{1/2} + \sum_{n=n_0}^{n=T} \left( \frac{\omega_n}{2} (s \mu_0 e_n)^2 - F(n, s \mu_0 e_{n+1}, s \mu_0 e_n) \right) \right\} . \]

Let
\[ M_0 = \max_{n \in [0,1]} \left\{ 2^n \sqrt{2n_0 T} \left( \sum_{n=n_0}^{n=T} (s \mu_0 e_n)^2 \right)^{1/2} + \sum_{n=n_0}^{n=T} \left( \frac{\omega_n}{2} (s \mu_0 e_n)^2 - F(n, s \mu_0 e_{n+1}, s \mu_0 e_n) \right) \right\} , \]
then,
\[ J_k(u^{(k)}) \leq M_0. \]

Obviously, \( M_0 > 0 \) is independent of \( k \). Since \( u^{(k)} \) is a critical point of \( J_k \), by (14), (15), and (39), we have
\[ \sum_{n=kT}^{kT-1} \left( \frac{\partial F(n, u_n^{(k)}, u_n^{(k)})}{\partial u_{n+1}} u_{n+1} + \frac{\partial F(n, u_n^{(k)}, u_n^{(k)})}{\partial u_n} u_n \right) - 2F(n, u_n^{(k)}, u_n^{(k)}) \leq 2M_0. \]

From \( (T_3) \), there exists \( q > 0 \) such that
\[ \frac{\partial F(n, u_1, u_2)}{\partial u_1} u_1 + \frac{\partial F(n, u_1, u_2)}{\partial u_2} u_2 - 2F(n, u_1, u_2) > 2M_0, \] for \( n \in \mathbb{Z}, u_1^2 + u_2^2 > q^2 \).

By combining (40), it implies that \( |u_n^{(k)}| \leq q \) for each \( n \in \mathbb{Z} \), that is,
\[ \|u^{(k)}\|_{\infty} \leq q. \]

From (15), we have
\[ \sum_{n=kT}^{kT-1} \omega_n (u_n^{(k)})^2 \leq \sum_{n=kT}^{kT-1} f(n, u_{n+1}^{(k)}, u_n^{(k)}) u_n^{(k)}. \]

By \( (T_3) \), there exists \( p > 0 \) such that
\[ f(n, v_1, v_2, v_3) \leq \frac{\sqrt{3}}{6} \omega_n \sqrt{v_1^2 + v_2^2 + v_3^2}, \]
for \( v_1^2 + v_2^2 + v_3^2 < 3p^2 \),
which combining with (43) gives
\[ \sum_{n=kT}^{kT-1} \omega_n (u_n^{(k)})^2 \leq \sum_{n=kT}^{kT-1} \frac{\sqrt{3}}{6} \omega_n \sqrt{v_1^2 + v_2^2 + v_3^2}. \]

By the periodicity of \( \{\omega_n\} \) and \( f(n, u_{n+1}, u_n, u_{n+1}) \), we see that \( \{u^{(k)}\} \) is also a solution of (48). Without the loss of generality, we may assume that \( 0 \leq n_k \leq T - 1 \) in (47). Moreover, passing to a subsequence of \( \{u^{(k)}\} \), if necessary, we can also assume that \( n_k = n^* \) for \( k \geq k_0 \) and some integer \( n^* \) such that \( 0 \leq n^* \leq T - 1 \). It follows from (47) that we can choose a subsequence, still denoted by \( \{u^{(k)}\} \), such that
\[ u_n^{(k)} \rightarrow u_n \text{ as } k \rightarrow \infty, \quad n \in \mathbb{Z}. \]

Then, \( u = \{u_n\} \) is a nonzero sequence as (47) which implies \( |u_{n-1}| \geq p \). It remains to show that \( u = \{u_n\} \in \ell^2 \), and it is a solution of (1).
mathematical problem in engineering

\[ A_k = \{ n \in \mathbb{Z} : |u_n| < p, |u_{n+1}| < p, |u_{n-1}| < p, -kT \leq n \leq kT - 1 \}, \]

\[ B_k = \{ n \in \mathbb{Z} : |u_n| \geq p \text{ or } |u_{n+1}| \geq p \text{ or } |u_{n-1}| \geq p, -kT \leq n \leq kT - 1 \}. \]

\[ \frac{\partial F(n - 1, u_{n-1}, u_n)}{\partial u_n} \frac{\partial F(n, u_{n+1}, u_n)}{\partial u_n} \]

\[ \leq \frac{c_1}{c_2} \left( \frac{1}{2} \left( \frac{\partial F(n - 1, u_{n-1}, u_n)}{\partial u_n} \frac{\partial F(n, u_{n+1}, u_n)}{\partial u_n} \right) \right) \]

\[ \leq F(n, u_{n+1}, u_n) - F(n, u_{n-1}, u_n). \]

Then, combining (40), (43), and (45) gives us

\[ \sum_{n=-kT}^{kT-1} \omega_n |u_n| \leq \frac{2c_1 M_0}{c_2 \omega_s} \]

\[ \left\| u^{(k)} \right\|_k \leq \frac{2c_1 M_0}{c_2 \omega_s}. \]
where \( \theta > 2 \) and \( T \) is a given positive integer. Then,

\[
F(n, v_1, v_2) = \left[ 2 + \cos\left(\frac{nt}{T}\right) \right] (v_1^2 + v_2^2)^{(\theta/2)},
\]

\[
\frac{\partial F(n-1, v_2, v_2)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = \theta v_2 \left[ 2 + \cos\left(\frac{nt}{T}\right) \right] \left( v_1^2 + v_2^2 \right)^{(\theta/2)-1}
\]

\[
+ \left( 2 + \cos\left(\frac{(n-1)t}{T}\right) \right) \left( v_2^2 + v_3^2 \right)^{(\theta/2)-1}.
\]

(58)

We can see that

\[
\frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} = \left[ 2 + \cos\left(\frac{nt}{T}\right) \right] (v_1^2 + v_2^2)^{(\theta/2)-1},
\]

due to \((\theta/2) - 1 > 0\), it follows that

\[
\lim_{v_1^2 + v_2^2 \to 0} \frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} = 0,
\]

(60)

\[
\lim_{v_1^2 + v_2^2 \to \infty} \frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} = \infty.
\]

And we know

\[
\left| \frac{f(n, v_1, v_2, v_3)}{(v_1^2 + v_2^2 + v_3^2)^{(1/2)}} \right|
\]

\[
\leq \theta v_2 \left( v_1^2 + v_2^2 + v_3^2 \right)^{(1/2)} \left( 2 + \cos\left(\frac{nt}{T}\right) \right) \left( v_1^2 + v_2^2 \right)^{(\theta/2)-1} + \left( 2 + \cos\left(\frac{(n-1)t}{T}\right) \right) \left( v_2^2 + v_3^2 \right)^{(\theta/2)-1}
\]

\[
\leq 6 \theta \left( v_1^2 + v_2^2 + v_3^2 \right)^{(\theta/2)-1}.
\]

Then, it implies that

\[
\lim_{v_1^2 + v_2^2 + v_3^2 \to 0} \frac{f(n, v_1, v_2, v_3)}{\sqrt{v_1^2 + v_2^2 + v_3^2}} = 0.
\]

(62)

It is easy to verify that all 1.1 are satisfied. Consequently, (1) has at least a nontrivial solution \( u \) in \( I^2 \).

5. Conclusion

In this work, we have considered a higher order nonlinear difference equation containing both advance and retardation with \( \phi \)-Laplacian. To the best of our knowledge, this is the first time to discuss the existence of homoclinic solutions for such equations. Applying the variational method and mountain pass theorem with Cerami’s condition, we have derived some new sufficient conditions for the existence of homoclinic solutions to (1) in a very general case of the nonlinear functional. By setting up the variational framework with (1), we have reduced the problem of finding a nontrivial solution of (1) to that of seeking a nonzero critical point of the functional \( J_k \). It is worth mentioning that the nonlinear term \( f \) can be mixed superlinear with asymptotically linear at \( \infty \). In fact, our conditions on the potential are rather relaxed, and some existing results in the literature are improved. Finally, we have given an example to illustrate the main result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

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