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Research Article

Homoclinic Solutions for a Higher Order ϕ_c-Laplacian Difference Equation Containing Both Advance and Retardation

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We consider a $2m$th-order nonlinear difference equation containing both advance and retardation with ϕ_c-Laplacian. Using the critical point theory, some new and concrete criteria for the existence of homoclinic solutions with mixed nonlinearities are obtained.

1. Introduction

Consider the following $2m$th-order nonlinear difference equation containing both advance and retardation with ϕ_c-Laplacian:

$$(-1)^m \Delta^m \left(\phi_c\left(\Delta^m u_{n-m}\right)\right) + \omega_n u_n = f\left(n, u_{n+1}, u_n, u_{n-1}\right), \quad n \in \mathbb{Z}, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$ and $\Delta^j u_k = \Delta(\Delta^{j-1} u_k)$ for $j \geq 2$, $\omega_n > 0$ is real valued for each $n \in \mathbb{Z}$, $f \in C(\mathbb{Z} \times \mathbb{R}^3, \mathbb{R})$, $\{\omega_n\}$, and $f(n)$ are T-periodic in n for a given positive integer T, and $\phi_c(s)$ is a special ϕ-Laplacian operator [1] defined by $\phi_c(s) = (s/\sqrt{1+|s|^2})$, $s \in \mathbb{R}$.

It is well known that $\phi_c(s)$ is the mean curvature operator given by the corresponding potential $\Phi_c(s) = \sqrt{1+|s|^2} - 1$. And it is such that

$$\phi_c^{-1}(v) = \frac{v}{\sqrt{1-|v|^2}} = \nabla \Phi_R(v), \quad v \in (-1, 1), \tag{2}$$

where $\Phi_R(v) = 1 - \sqrt{1-|v|^2}$ may be seen as the acceleration in classical relativity. Actually, it is important to determine the existence and regularity properties of maximal and constant mean curvature hypersurfaces, as they provide Riemannian submanifolds that reflect spatiotemporal properties. There have been extensive study and application of nonlinear difference equations with ϕ-Laplacian. For example, scalar equations were considered in [2, 3] using topological methods, and T-periodic solutions have been studied in [1] by variational methods.

Bonheure et al. [4] in 2007 studied classical and nonclassical solutions of a prescribed curvature equation:

$$-\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda f(t, u), \quad t \in \mathbb{R}, \text{ with } u(0) = u(1) = 0. \tag{3}$$

This problem depends on the behavior at the origin and at infinity of the potential $\int_0^u f(t,s)ds$. Their method is essentially variational and is based on a regularization of the action functional associated with the curvature problem.

Due to their wide range of applications, there has been tremendous interest in studying some properties of the nonlinear discrete ϕ_c-Laplacian equations, including periodic solutions [1], boundary value problems [5, 6], and homoclinic orbits [7, 8]. For instance, using the mountain pass lemma, Zhou and Su [5] obtained some sufficient conditions for the first time on the existence of solutions of the $2n$th-order ϕ_c-Laplacian difference equation with the boundary value

conditions, and by the critical point theory, Zhou and Ling [6] obtained some sufficient conditions on the existence of infinitely many positive solutions of the boundary value problems for a second-order ϕ_c-Laplacian difference equation. The importance of nonlinear difference equations is described in [9, 10], with applications involving statistics, computing, electrical circuit analysis, dynamical systems, and economics. And they appear naturally as discrete analogue of differential equations which model various diverse phenomena.

We assume that $f(n,0,0,0)=0$ for each $n\in\mathbb{Z}$, then $\{u_n\}=\{0\}$ is a solution of (1), which is called the trivial solution. As usual, we say that a solution $u=\{u_n\}$ of (1) is homoclinic (to 0) if $\lim_{|n|\longrightarrow\infty}u_n=0$. In addition, if $\{u_n\}\neq\{0\}$, then u is called a nontrivial homoclinic solution.

In recent years, the existence and multiplicity of homoclinic solutions for various discrete systems have been investigated by many researchers. For some recent work, the reader can refer to [11–22] and the reference therein. For example, by using the symmetric mountain pass theorem, Chen and Tang [12] studied the existence of infinitely many homoclinic orbits for the fourth-order difference systems containing both advance and retardation:

$$\Delta^4u(n-2)+q(n)u(n)=f(n,u(n+1),u(n),u(n-1)),\quad n\in\mathbb{Z}. \tag{4}$$

Kong [20] applied the critical point theory to study the existence of at least three homoclinic solutions for a p-Laplacian difference equation of higher order with both advance and retardation:

$$\begin{aligned}(-1)^n\Delta^n\left(a(k-n)\phi_p\left(\Delta^nu(k-n)\right)\right)+b(k)\phi_p(u(k))\\=\lambda f(k,u(k+1),u(k),u(k-1)),\quad k\in\mathbb{Z}.\end{aligned} \tag{5}$$

However, to the best of our knowledge, there is no work on the existence of homoclinic solutions for ϕ_c-Laplacian difference equations containing both advance and retardation which have important background and significance in the field of cybernetics and biological mathematics [23, 24]. Motivated by the reasons aforementioned, in this paper, we will establish some new sufficient conditions for the existence of homoclinic solutions to (1) in a very general case of the nonlinear functional.

On the other hand, we notice that the nonlinear term f is only either superlinear or asymptotically linear at ∞ which plays an important role in the existence of homoclinic solutions when the similar arguments were considered in many references [25–28]. But in this paper, the nonlinearities can be mixed superlinear with asymptotically linear at ∞, see Remark 1 for details. In fact, our conditions on the potential are rather relaxed, and some existing results in the literature are improved (see Remark 2). Moreover, we give an example to illustrate the main result.

Now, we establish the main result.

Theorem 1. *Assume that the following conditions hold:*

(T_1) there exists a functional $F(n,v_1,v_2)\in C^1(\mathbb{Z}\times\mathbb{R}^2,\mathbb{R})$ with $F(n+T,v_1,v_2)=F(n,v_1,v_2)$ and it satisfies

$$\frac{\partial F(n-1,v_2,v_3)}{\partial v_2}+\frac{\partial F(n,v_1,v_2)}{\partial v_2}=f(n,v_1,v_2,v_3). \tag{6}$$

(T_2)

$$\begin{aligned}\limsup_{v_1^2+v_2^2\longrightarrow 0}\frac{F(n,v_1,v_2)}{v_1^2+v_2^2}=0,\\ \limsup_{v_1^2+v_2^2+v_3^2\longrightarrow 0}\frac{f(n,v_1,v_2,v_3)}{\sqrt{v_1^2+v_2^2+v_3^2}}=0.\end{aligned} \tag{7}$$

(T_3) there exists a real sequence $\{a_n\}$ such that

$$\liminf_{v_1^2+v_2^2\longrightarrow\infty}\frac{F(n,v_1,v_2)}{v_1^2+v_2^2}=a_n\le\infty. \tag{8}$$

(T_4) $(\partial F(n,v_1,v_2)/\partial v_1)v_1+(\partial F(n,v_1,v_2)/\partial v_2)v_2-2F(n,v_1,v_2)>0$ for $n\in\mathbb{Z}$, $v_1,v_2\in\mathbb{R}\{0\}$.

(T_5) $(\partial F(n,v_1,v_2)/\partial v_1)v_1+(\partial F(n,v_1,v_2)/\partial v_2)v_2-2F(n,v_1,v_2)\longrightarrow+\infty$ as $v_1^2+v_2^2\longrightarrow+\infty$.

If $2a_n>\omega_n$ for each $n\in\mathbb{Z}$, then (1) has at least a nontrivial solution u in l^2.

Remark 1. The condition (T_3) shows that the nonlinear terms mix superlinear nonlinearities with asymptotically linear ones at ∞.

Remark 2. If $m=1$ and $f(n,u_{n+1},u_n,u_{n-1})=g(n,u_n)$, Theorem 1 can reduce to Theorem 2 when ϕ-Laplacian is ϕ_c-Laplacian in [8]. And our sufficient conditions are based on the limit superior and limit inferior, which are more applicable.

This paper is organized as follows: In Section 2, we establish the variational framework associated with (1) and recall some related fundamental results for convenience. In Section 3, some lemmas are proved, and then we complete the proof of our results by the mountain pass lemma. Finally, we illustrate our main result with an example in Section 4.

2. Preliminaries

This section is to establish the corresponding variational framework for (1) and cite some basic conclusions for the coming discussion.

Let

$$l^p=\left\{u=\{u_n\}:n\in\mathbb{Z},u_n\in\mathbb{R},\|u\|_{l^p}=\left(\sum_{n\in\mathbb{Z}}|u_n|^p\right)^{1/p}<\infty\right\}, \tag{9}$$

then, the corresponding inner product denoted by $(\cdot,\cdot)$ in l^2 and the corresponding norm in l^2 is denoted by $\|\cdot\|=\|\cdot\|_{l^2}$.

Let S be the set of all two-sided sequences, that is,

$$S = \{u = \{u_n\} \mid u_n \in \mathbb{R}, n \in \mathbb{Z}\}. \tag{10}$$

Then, S is a vector space with $au + bv = \{au_n + bv_n\}$ for $u, v \in S$ and $a, b \in \mathbb{R}$. For any fixed positive integer k, we define the subspace E_k of S as

$$E_k = \{u = \{u_n\} \in S \mid u_{n+2kT} = u_n, n \in \mathbb{Z}\}. \tag{11}$$

Obviously, E_k is isomorphic to $\mathbb{R}^{2kT}$, and hence E_k can be equipped with the inner product $(\cdot,\cdot)_k$ and norm $\|\cdot\|_k$ as

$$\begin{aligned} (u, v)_k &= \sum_{n=-kT}^{kT-1} u_n v_n, \quad u, v \in E_k, \\ \|u\|_k &= \left(\sum_{n=-kT}^{kT-1} u_n^2\right)^{1/2}, \quad u \in E_k, \end{aligned} \tag{12}$$

respectively. We also define a norm $\|\cdot\|_{k\infty}$ in E_k by

$$\|u\|_{k\infty} = \max\{|u_n| : -kT \le n \le kT - 1\}, \quad u \in E_k. \tag{13}$$

Consider the functional J_k in E_k defined by

$$J_k(u) = \sum_{n=-kT}^{kT-1} \left[\Phi_c(\Delta^m u_n) + \frac{1}{2}\omega_n u_n^2 - F(n, u_{n+1}, u_n)\right], \tag{14}$$

and the Fréchet derivative is given by

$$\begin{aligned} \langle J_k'(u), v\rangle = \sum_{n=-kT}^{kT-1} &[\phi_c(\Delta^m u_n)\Delta^m v_n + \omega_n u_n v_n \\ &- f(n, u_{n+1}, u_n, u_{n-1}) v_n], \quad u, v \in E_k. \end{aligned} \tag{15}$$

Equation (15) implies that (1) is the corresponding Euler–Lagrange equation for J_k. Therefore, we have reduced the problem of finding a nontrivial solution of (1) to that of seeking a nonzero critical point of the functional J_k.

It is easy to see that the critical points of J_k in E_k are exactly $2kT$-periodic solutions of system (1).

Let H be a Hilbert space, and $C^1(H, \mathbb{R})$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on H. We conclude this section with some known results.

Definition 1. Let $J \in C^1(H, \mathbb{R})$. A sequence $\{x_j\} \subset H$ is called a Cerami sequence ((C) sequence for short) for J, if $J(x_j) \longrightarrow c$ for some $c \in \mathbb{R}$ and $(1 + \|x_j\|)J'(x_j) \longrightarrow 0$ as $j \longrightarrow \infty$. We say, J satisfies the Cerami condition ((C) condition for short) if any (C) sequence for J possesses a convergent subsequence.

Let B_r be the open ball in H with radius r and center 0, and let ∂B_r denote its boundary.

Lemma 1. *(mountain pass lemma [29]). If $J \in C^1(H, \mathbb{R})$ and satisfies the following conditions: there exist $e \in H\backslash\{0\}$ and $r \in (0, ||e||)$ such that $\max\{J(0), J(e)\} < \inf_{\|u\|=r} J(u)$. Then, there exists a (C) sequence $\{u_n\}$ for the mountain pass level c which is defined by*

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s)), \tag{16}$$

where

$$\Gamma = \{h \in C([0, 1], H) : h(0) = 0, h(1) = e\}. \tag{17}$$

3. Proof of Main Result

In order to prove Theorem 1, we need several lemmas.

Let

$$\omega_* = \min_{n\in\mathbb{Z}}\{\omega_n\}. \tag{18}$$

Lemma 2. *Under the assumptions of Theorem 1, the functional J_k satisfies the (C) condition.*

Proof 1. Let $\{u^{(j)}\} \subset E_k$ be a (C) sequence for J_k. We need to show that $\{u^{(j)}\}$ has a convergent subsequence. Since E_k is finite dimensional, it suffices to show that $\|u^{(j)}\|_k$ is bounded. Since $J_k(u^{(j)}) \longrightarrow c$ and $(1 + \|u^{(j)}\|_k)J_k'(u^{(j)}) \longrightarrow 0$, then there exists $M > 2c + 1$ such that $|J_k(u^{(j)})| \le ((M-1)/2)$ and $\|(1 + \|u^{(j)}\|_k)J_k'(u^{(j)})\| < 1$ for $j \in \mathbb{N}$. So, we have $\|u^{(j)}\|_k\|J_k'(u^{(j)})\| \le \|(1 + \|u^{(j)}\|_k)J_k'(u^{(j)})\| < 1$ for $j \in \mathbb{N}$. Then, by (14), (15), and

$$2\Phi_c(u) = 2\left(\sqrt{1+u^2} - 1\right) \ge \frac{u}{\sqrt{1+u^2}} u = \phi_c(u)u \ge 0, \quad \text{for } u \in \mathbb{R}, \tag{19}$$

we have

$$\begin{aligned} &\sum_{n=-kT}^{kT-1}\left(u_{n+1}^{(j)} + \frac{\partial F\left(n, u_{n+1}^{(j)}, u_n^{(j)}\right)}{\partial u_{n+1}^{(j)}} u_{n+1}^{(j)} - 2F\left(n, u_{n+1}^{(j)}, u_n^{(j)}\right)\right) \\ &\le 2J_k\left(u^{(j)}\right) - \left\langle J_k'\left(u^{(j)}\right), u^{(j)}\right\rangle \\ &\le 2\left|J_k\left(u^{(j)}\right)\right| + \left\|u^{(j)}\right\|_k \left\|J_k'\left(u^{(j)}\right)\right\| \\ &\le M. \end{aligned} \tag{20}$$

From (T_4) and (T_5), there exists $\delta > 0$ such that

$$\frac{\partial F(n, u_1, u_2)}{\partial u_1} u_1 + \frac{\partial F(n, u_1, u_2)}{\partial u_2} u_2 - 2F(n, u_1, u_2) > M, \quad \text{for } n \in \mathbb{Z} \text{ and } u_1^2 + u_2^2 > \delta^2, \tag{21}$$

then, (20) implies that $|u_n^{(j)}| \le \delta$ for $n \in \mathbb{Z}$, that is,

$$\left\|u^{(j)}\right\|_{k\infty} \le \delta. \tag{22}$$

Since E_k is finite dimensional, $\|\cdot\|_k$ and $\|\cdot\|_{k\infty}$ are equivalent. Then, (22) implies that $\{\|u^{(j)}\|_k\}$ is bounded. The proof is completed.

Lemma 3. *Under the assumptions of Theorem 1, there exists $n_0 \in \mathbb{N}$ such that J_k has at least a nonzero critical point $u^{(k)}$ in E_k for each $k \geq n_0$.*

Proof 2. We first show that J_k satisfies the conditions of Lemma 1. From (T_2), there exists $r > 0$ such that

$$F(n, u_1, u_2) \leq \frac{1}{8}\omega_*\left(u_1^2 + u_2^2\right), \quad \text{for } u_1^2 + u_2^2 \leq r^2. \tag{23}$$

Then, for $u \in E_k$ with $u_k \leq r$,

$$\begin{aligned} J_k(u) &\geq \frac{1}{2}\sum_{n=-kT}^{kT-1}\omega_n u_n^2 - \sum_{n=-kT}^{kT-1} F(n, u_{n+1}, u_n) \\ &\geq \frac{1}{2}\sum_{n=-kT}^{kT-1}\omega_n u_n^2 - \sum_{n=-kT}^{kT-1}\frac{1}{8}\omega_*\left(u_{n+1}^2 + u_n^2\right) \\ &\geq \frac{1}{4}\omega_*\|u\|_k^2. \end{aligned} \tag{24}$$

Taking $a = (1/4)\omega_* r^2$, then $J_k|_{\partial B_r} \geq a > 0$.
Since $2a_n > \omega_n$, there exists $d > 0$ such that

$$a_n - \frac{\omega_n}{2} > d, \quad \text{for } n \in \mathbb{Z}. \tag{25}$$

Let $\varepsilon \in (0, 1)$ satisfy

$$2\varepsilon - d < 0. \tag{26}$$

Since $-\Delta^2$ is a bounded semipositive self-adjoint linear operator in l^2 and $0 \in \sigma(-\Delta^2)$, there exists $e = \{e_n\} \in l^2$ with $\sum_{n=-\infty}^{+\infty}|e_n|^2 = 1$ such that $\sum_{n=-\infty}^{+\infty}|\Delta^m e_n|^2 < \varepsilon$. Let n_0 be large enough such that

$$\begin{gathered} \sum_{n=-n_0T}^{n_0T-1}\left|\Delta^m e_n\right|^2 < \varepsilon, \\ \frac{1}{2} \leq \sum_{n=-n_0T}^{n_0T-1} e_n^2 \leq 1. \end{gathered} \tag{27}$$

For $k \geq n_0$, define $e^{(k)} \in E_k$ by

$$e_n^{(k)} = \begin{cases} e_n, & -n_0T \leq n \leq n_0T - 1, \\ 0, & -kT \leq n \leq -n_0T - 1 \text{ or } n_0T \leq n \leq kT - 1. \end{cases} \tag{28}$$

By (T_3), there exists $\mu_0 > \max\{r, 1\}$, such that

$$\begin{gathered} F(n, \mu e_{n+1}, \mu e_n) \geq (a_n - \varepsilon)\mu^2\left(e_{n+1}^2 + e_n^2\right), \\ \text{for } -n_0T \leq n \leq n_0T - 1,\ \mu \geq \mu_0. \end{gathered} \tag{29}$$

Then, for $\mu \geq \mu_0$,

$$\begin{aligned} J_k\left(\mu e^{(k)}\right) &= \sum_{n=-kT}^{kT-1}\left(\Phi_c\left(\mu\Delta^m e_n^{(k)}\right) + \frac{\omega_n}{2}\left|\mu e_n^{(k)}\right|^2 - F\left(n, \mu e_{n+1}^{(k)}, \mu e_n^{(k)}\right)\right) \\ &\leq \sum_{n=-kT}^{kT-1}\left(\left[\mu\Delta^m e_n^{(k)}\right] + \frac{\omega_n}{2}\left|\mu e_n^{(k)}\right|^2 + (\varepsilon - a_n)\mu^2\right. \\ &\quad \left.\cdot\left(\left|e_{n+1}^{(k)}\right|^2 + \left|e_n^{(k)}\right|^2\right)\right) \\ &\leq \varepsilon\mu + \frac{\omega_n}{2}\mu^2 + (\varepsilon - a_n)\mu^2 \\ &\leq (2\varepsilon - d)\mu^2. \end{aligned} \tag{30}$$

Thus,

$$J_k\left(\mu_0 e^{(k)}\right) \leq (2\varepsilon - d)\mu_0^2 < 0. \tag{31}$$

It can easily be seen that $J_k(0) = 0$ for each $k \in \mathbb{Z}$. Then, we have $r \in (0, \|\mu_0 e^{(k)}\|_k)$ and

$$\max\left\{J_k(0), J_k\left(\mu_0 e^{(k)}\right)\right\} = 0 < a \leq \inf_{\|u\|_k = r} J_k(u). \tag{32}$$

Now that we have verified all assumptions of Lemma 1, and then we know J_k possesses a (C) sequence $\left\{u_j^{(k)}\right\}$ for the mountain pass level $\alpha_k \geq a$ with

$$\alpha_k = \inf_{h \in \Gamma_k}\max_{s \in [0,1]} J_k(h(s)), \tag{33}$$

where

$$\Gamma_k = \left\{h \in C([0, 1], E_k) : h(0) = 0, h(1) = \mu_0 e^{(k)}\right\}. \tag{34}$$

According to Lemma 2, $\left\{u_j^{(k)}\right\}$ has a convergent subsequence $\left\{u_{j_m}^{(k)}\right\}$ such that $u_{j_m}^{(k)} \longrightarrow u^{(k)}$ as $j_m \longrightarrow +\infty$ for some $u^{(k)} \in E_k$. Since $J_k \in C^1(E_k, \mathbb{R})$, we have

$$\begin{gathered} J_k\left(u_{j_m}^{(k)}\right) \longrightarrow J_k\left(u^{(k)}\right), \\ \left(1 + \left\|u_{j_m}^{(k)}\right\|_k\right)J'\left(u_{j_m}^{(k)}\right) \longrightarrow \left(1 + \left\|u^{(k)}\right\|_k\right)J'\left(u^{(k)}\right), \end{gathered} \tag{35}$$

as $j_m \longrightarrow +\infty$. By the uniqueness of the limit, we obtain that $u^{(k)}$ is a critical point of J_k corresponding to α_k. Moreover, $u^{(k)}$ is nonzero as $\alpha_k \geq a > 0$.

Lemma 4. *There exist $p, q > 0$ such that*

$$p \leq \left\|u^{(k)}\right\|_{k\infty} \leq q, \tag{36}$$

holds for every critical point $u^{(k)}$ obtained by Lemma 3, of J_k in E_k with $k \geq n_0$, where n_0 is defined in Lemma 3.

Proof 3. For $k \geq n_0$ and $s \in [0, 1]$, we define $h_k \in \Gamma_k$ as $h_k(s) = s\mu_0 e^{(k)}$; similarly to the derivation of [5], we can find

$$\begin{aligned} J_k\left(u^{(k)}\right) &\le \max_{s\in[0,1]}\left\{J_k\left(s\mu_0 e^{(k)}\right)\right\} \\ &\le \max_{s\in[0,1]}\left\{\sum_{n=-n_0T}^{n_0T-1}\left(\left|\Delta^m\left(s\mu_0 e_n\right)\right| + \frac{\omega_n}{2}\left(s\mu_0 e_n\right)^2 - F\left(n, s\mu_0 e_{n+1}, s\mu_0 e_n\right)\right)\right\} \\ &\le \max_{s\in[0,1]}\left\{2^m\sqrt{2n_0T}\left(\sum_{n=-n_0T}^{n_0T-1}\left(s\mu_0 e_n\right)^2\right)^{1/2} + \sum_{n=-n_0T}^{n_0T-1}\left(\frac{\omega_n}{2}\left(s\mu_0 e_n\right)^2 - F\left(n, s\mu_0 e_{n+1}, s\mu_0 e_n\right)\right)\right\}. \end{aligned} \tag{37}$$

Let

$$M_0 = \max_{s\in[0,1]}\left\{2^m\sqrt{2n_0T}\left(\sum_{n=-n_0T}^{n_0T-1}\left(s\mu_0 e_n\right)^2\right)^{1/2} + \sum_{n=-n_0T}^{n_0T-1}\left(\frac{\omega_n}{2}\left(s\mu_0 e_n\right)^2 - F\left(n, s\mu_0 e_{n+1}, s\mu_0 e_n\right)\right)\right\}, \tag{38}$$

then,

$$J_k\left(u^{(k)}\right) \le M_0. \tag{39}$$

Obviously, $M_0 > 0$ is independent of k.

Since $u^{(k)}$ is a critical point of J_k, by (14), (15), and (39), we have

$$\begin{aligned} &\sum_{n=-kT}^{kT-1}\left(\frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial u_{n+1}^{(k)}}u_{n+1}^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial u_n^{(k)}}u_n^{(k)}\right. \\ &\left. - 2F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)\right) \le 2M_0. \end{aligned} \tag{40}$$

From (T_5), there exists $q > 0$ such that

$$\begin{aligned} &\frac{\partial F\left(n, u_1, u_2\right)}{\partial u_1}u_1 + \frac{\partial F\left(n, u_1, u_2\right)}{\partial u_2}u_2 \\ &- 2F\left(n, u_1, u_2\right) > 2M_0, \quad \text{for } n \in \mathbb{Z},\ u_1^2 + u_2^2 > q^2. \end{aligned} \tag{41}$$

By combining (40), it implies that $|u_n^{(k)}| \le q$ for each $n \in \mathbb{Z}$, that is,

$$\left\|u^{(k)}\right\|_{k\infty} \le q. \tag{42}$$

From (15), we have

$$\sum_{n=-kT}^{kT-1}\omega_n\left(u_n^{(k)}\right)^2 \le \sum_{n=-kT}^{kT-1} f\left(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)}\right)u_n^{(k)}. \tag{43}$$

By (T_2), there exists $p > 0$ such that

$$\begin{aligned} &f\left(n, v_1, v_2, v_3\right) \le \frac{\sqrt{3}}{6}\omega_*\sqrt{v_1^2 + v_2^2 + v_3^2}, \\ &\text{for } v_1^2 + v_2^2 + v_3^2 < 3p^2, \end{aligned} \tag{44}$$

which combining with (43) gives

$$\begin{aligned} \sum_{n=-kT}^{kT-1}\omega_n\left(u_n^{(k)}\right)^2 &\le \sum_{n=-kT}^{kT-1}\frac{\sqrt{3}}{6}\omega_*\sqrt{\left(u_{n+1}^{(k)}\right)^2 + \left(u_n^{(k)}\right)^2 + \left(u_{n-1}^{(k)}\right)^2 u_n^{(k)}} \\ &\le \frac{\sqrt{3}}{6}\omega_*\left(\sum_{n=-kT}^{kT-1}\left(\left(u_{n+1}^{(k)}\right)^2 + \left(u_n^{(k)}\right)^2 + \left(u_{n-1}^{(k)}\right)^2\right)\right)^{1/2}\left\|u^{(k)}\right\|_k \\ &= \frac{1}{2}\omega_*\left\|u^{(k)}\right\|_k^2. \end{aligned} \tag{45}$$

Arguing by a contradiction $\omega_* = \min_{n\in\mathbb{Z}}\{\omega_n\}$, we have

$$\left\|u^{(k)}\right\|_{k\infty} \ge p. \tag{46}$$

The proof is complete.

With the help of the preceding lemmas we can prove Theorem 1.

Proof of Theorem 1. For every $k > n_0$, J_k has a critical point $u^{(k)} = \{u_n^{(k)}\} \in E_k$, which is obtained by Lemma 3. Moreover, there exists $n_k \in \mathbb{Z}$ such that

$$p \le \left|u_{n_k}^{(k)}\right| \le q. \tag{47}$$

Note that

$$(-1)^m\Delta^m\left(\phi_c\left(\Delta^m u_{n-m}^{(k)}\right)\right) + \omega_n u_n^{(k)} = f\left(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)}\right), \quad n \in \mathbb{Z}. \tag{48}$$

By the periodicity of $\{\omega_n\}$ and $f(n, u_{n+1}, u_n, u_{n-1})$, we see that $\{u_{n+T}^{(k)}\}$ is also a solution of (48). Without the loss of generality, we may assume that $0 \le n_k \le T-1$ in (47). Moreover, passing to a subsequence of $\{u^{(k)}\}$, if necessary, we can also assume that $n_k = n^*$ for $k \ge n_0$ and some integer n^* such that $0 \le n^* \le T-1$. It follows from (47) that we can choose a subsequence, still denoted by $\{u^{(k)}\}$, such that

$$u_n^{(k)} \longrightarrow u_n \text{ as } k \longrightarrow \infty, \quad n \in \mathbb{Z}. \tag{49}$$

Then, $u = \{u_n\}$ is a nonzero sequence as (47) which implies $|u_{n^*}| \ge p$. It remains to show that $u = \{u_n\} \in l^2$, and it is a solution of (1).

Let

$$A_k = \left\{ n \in \mathbb{Z} : \left|u_{n+1}^{(k)}\right| < p, \left|u_n^{(k)}\right| < p, \left|u_{n-1}^{(k)}\right| < p, -kT \le n \le kT-1 \right\},$$
$$B_k = \left\{ n \in \mathbb{Z} : \left|u_{n+1}^{(k)}\right| \ge p \text{ or } \left|u_n^{(k)}\right| \ge p \text{ or } \left|u_{n-1}^{(k)}\right| \ge p, -kT \le n \le kT-1 \right\}. \tag{50}$$

Since $F(n, u_1, u_2)$ is continuously differentiable in the second and third variables and T-periodic in n, let

$$c_1 = \max\left\{ \frac{\partial F(n-1, u_2, u_3)}{\partial u_2} u_2 + \frac{\partial F(n, u_1, u_2)}{\partial u_2} u_2 : p^2 \le u_1^2 + u_2^2 + u_3^2 \le 3q^2, n \in \mathbb{Z} \right\},$$
$$c_2 = \min\left\{ \frac{1}{2}\left(\frac{\partial F(n-1, u_2, u_3)}{\partial u_2} u_2 + \frac{\partial F(n, u_1, u_2)}{\partial u_2} u_2 \right) - F(n, u_1, u_2) : p^2 \le u_1^2 + u_2^2 + u_3^2 \le 3q^2, n \in \mathbb{Z}, \right\} \tag{51}$$

it implies $c_1, c_2 > 0$. Thus, for $n \in B_k$,

$$\frac{\partial F\left(n-1, u_n^{(k)}, u_{n-1}^{(k)}\right)}{\partial u_n^{(k)}} u_n^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial u_n^{(k)}} u_n^{(k)} \le \frac{c_1}{c_2}\left(\frac{1}{2}\left(\frac{\partial F\left(n-1, u_n^{(k)}, u_{n-1}^{(k)}\right)}{\partial u_n^{(k)}} u_n^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial u_n^{(k)}} u_n^{(k)} \right) - F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) \right). \tag{52}$$

Then, combining (40), (43), and (45) gives us

$$\begin{aligned}
\sum_{n=-kT}^{kT-1} \omega_n \left(u_n^{(k)}\right)^2 &\le \sum_{n \in A_k} f\left(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)}\right) u_n^{(k)} + \sum_{n \in B_k} f\left(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)}\right) u_n^{(k)} \\
&\le \frac{1}{2}\omega_* \left\|u^{(k)}\right\|_k^2 + \sum_{n \in B_k} \left(\frac{\partial F\left(n-1, u_n^{(k)}, u_{n-1}^{(k)}\right)}{\partial u_n^{(k)}} u_n^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial u_n^{(k)}} u_n^{(k)} \right) \\
&\le \frac{1}{2}\omega_* \left\|u^{(k)}\right\|_k^2 \\
&\quad + \frac{c_1}{c_2} \sum_{n \in B_k} \left(\frac{1}{2}\left(\frac{\partial F\left(n-1, u_n^{(k)}, u_{n-1}^{(k)}\right)}{\partial u_n^{(k)}} u_n^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial u_n^{(k)}} u_n^{(k)} \right) - F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) \right) \\
&\le \frac{1}{2}\omega_* \left\|u^{(k)}\right\|_k^2 + \frac{c_1 M_0}{c_2}.
\end{aligned} \tag{53}$$

Thus, we have

$$\left\|u^{(k)}\right\|_k^2 \le \frac{2c_1 M_0}{c_2 \omega_*}. \tag{54}$$

For each $\varrho \in \mathbb{N}$, let $k > \max\{\varrho, n_0\}$, and it follows from (54) that

$$\sum_{n=-\varrho}^{\varrho} \left(u_n^{(k)}\right)^2 \le \left\|u^{(k)}\right\|_k^2 \le \frac{2c_1 M_0}{c_2 \omega_*}. \tag{55}$$

It is clear that $\sum_{n=-\varrho}^{\varrho} u_n^2 \le (2c_1 M_0 / c_2 \omega_*)$ as $k \longrightarrow \infty$ and $u = \{u_n\} \in l^2$ by the arbitrariness of ϱ.

Indeed, for each $n \in \mathbb{Z}$, letting $k \longrightarrow \infty$ in (48) gives us

$$(-1)^m \Delta^m \left(\phi_c\left(\Delta^m u_{n-m}\right)\right) + \omega_n u_n = f\left(n, u_{n+1}, u_n, u_{n-1}\right), \quad n \in \mathbb{Z}, \tag{56}$$

that is, $u = \{u_n\}$ satisfies (1).

Consequently, we infer that $u = \{u_n\}$ is a nontrivial solution of (1). We have thus proved Theorem 1.

4. An Example

As an application of Theorem 1, we give an example to illustrate our main result.

Example 1. Consider the difference equation (1), with

$$f(n, v_1, v_2, v_3) = \theta v_2 \left[\left(2 + \cos\left(\frac{n\pi}{T}\right)\right)\left(v_1^2 + v_2^2\right)^{(\theta/2)-1} + \left(2 + \cos\left(\frac{(n-1)\pi}{T}\right)\right)\left(v_2^2 + v_3^2\right)^{(\theta/2)-1} \right], \tag{57}$$

where $\theta > 2$ and T is a given positive integer. Then,

$$F(n, v_1, v_2) = \left[2 + \cos\left(\frac{n\pi}{T}\right)\right]\left(v_1^2 + v_2^2\right)^{\theta/2},$$

$$\begin{aligned}\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = \theta v_2 \Bigg[&\left(2 + \cos\left(\frac{n\pi}{T}\right)\right) \\ &\cdot \left(v_1^2 + v_2^2\right)^{(\theta/2)-1} \\ &+ \left(2 + \cos\left(\frac{(n-1)\pi}{T}\right)\right) \\ &\cdot \left(v_2^2 + v_3^2\right)^{(\theta/2)-1}\Bigg].\end{aligned} \tag{58}$$

We can see that

$$\frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} = \left[2 + \cos\left(\frac{n\pi}{T}\right)\right]\left(v_1^2 + v_2^2\right)^{(\theta/2)-1}, \tag{59}$$

due to $(\theta/2) - 1 > 0$, it follows that

$$\begin{aligned}\lim_{v_1^2+v_2^2 \longrightarrow 0} \frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} &= 0, \\ \lim_{v_1^2+v_2^2 \longrightarrow \infty} \frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} &= \infty.\end{aligned} \tag{60}$$

And we know

$$\begin{aligned}&\left|f(n, v_1, v_2, v_3)\left(v_1^2 + v_2^2 + v_3^2\right)^{-(1/2)}\right| \\ &\le \left|\theta v_2\right|\left(v_1^2 + v_2^2 + v_3^2\right)^{-(1/2)}\left|\left(2 + \cos\left(\frac{n\pi}{T}\right)\right)\left(v_1^2 + v_2^2\right)^{(\theta/2)-1} + \left(2 + \cos\left(\frac{(n-1)\pi}{T}\right)\right)\left(v_2^2 + v_3^2\right)^{(\theta/2)-1}\right| \\ &\le 6\theta\left(v_1^2 + v_2^2 + v_3^2\right)^{(\theta/2)-1}.\end{aligned} \tag{61}$$

Then, it implies that

$$\lim_{v_1^2+v_2^2+v_3^2 \longrightarrow 0} \frac{f(n, v_1, v_2, v_3)}{\sqrt{v_1^2 + v_2^2 + v_3^2}} = 0. \tag{62}$$

It is easy to verify that all 1.1 are satisfied. Consequently, (1) has at least a nontrivial solution u in l^2.

5. Conclusion

In this work, we have considered a higher order nonlinear difference equation containing both advance and retardation with ϕ_c-Laplacian. To the best of our knowledge, this is the first time to discuss the existence of homoclinic solutions for such equations. Applying the variational method and mountain pass theorem with Cerami's condition, we have derived some new sufficient conditions for the existence of homoclinic solutions to (1) in a very general case of the nonlinear functional. By setting up the variational framework with (1), we have reduced the problem of finding a nontrivial solution of (1) to that of seeking a nonzero critical point of the functional J_k. It is worth mentioning that the nonlinear term f can be mixed superlinear with asymptotically linear at ∞. In fact, our conditions on the potential are rather relaxed, and some existing results in the literature are improved. Finally, we have given an example to illustrate the main result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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