Research Article

Spatiotemporal Patterns Induced by Hopf Bifurcations in a Homogeneous Diffusive Predator-Prey System

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Received 24 February 2019; Revised 18 April 2019; Accepted 24 April 2019; Published 13 June 2019

Academic Editor: Salvatore Alfonzetti

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In this paper, we consider a diffusive predator-prey system where the prey exhibits the herd behavior in terms of the square root of the prey population. The model is supposed to impose on homogeneous Neumann boundary conditions in the bounded spatial domain. By using the abstract Hopf bifurcation theory in infinite dimensional dynamical system, we are capable of proving the existence of both spatial homogeneous and nonhomogeneous periodic solutions driven by Hopf bifurcations bifurcating from the positive constant steady state solutions. Our results allow for the clearer understanding of the mechanism of the spatiotemporal pattern formations of the predator-prey interactions in ecology.

1. Introduction

Spatiotemporal pattern formations are one of the core problems arising in biology and ecology. Hence, it has attracted the extensive attentions from a huge number of investigations, including biologists, chemists, and mathematicians. It is well known that diffusion not only can induce spatial patterns, but also can drive spatiotemporal patterns, including oscillatory patterns. In this paper, we are mainly concerned with the spatiotemporal oscillatory patterns of a particular reaction-diffusion predator-prey system arising from ecology. In the model of our concern, the prey is assumed to exhibit herd behavior, so that the predator interacts with the prey along the outer corridor of the herd of prey, and the interaction terms are supposed to use the square root of the prey population rather than simply the prey population, where the use of the square root properly represents the assumption that the interactions occur along the boundary of the population (see [1–15]). The reaction-diffusion system takes in the following form:

$$\frac{\partial u}{\partial t} - d_1 \Delta u = u(1 - u) - \frac{\sqrt{uv}}{1 + \alpha \sqrt{u}},$$

$$x \in (0, \ell\pi), \quad t > 0$$

$$\frac{\partial v}{\partial t} - d_2 \Delta v = \beta v \left(-\gamma + \frac{\sqrt{u}}{1 + \alpha \sqrt{u}} \right),$$

$$x \in (0, \ell\pi), \quad t > 0$$

$$u_x(0, t) = v_x(0, t) = u_x(\ell\pi, t) = v_x(\ell\pi, t) = 0,$$

$$t > 0$$

$$u(x, 0) = u_0(x) \geq 0,$$

$$v(x, 0) = v_0(x) \geq 0,$$

$$x \in (0, \ell\pi)$$

(1)

where $u(x, t)$ and $v(x, t)$ stand for the population densities of the prey and predator at time t and the position x , respectively; β represents the interaction strength between the predator and the prey; d_1 and d_2 are the diffusion rates of u and v , respectively. The spatial domain is supposed to be one dimensional, $\Omega = (0, \ell\pi)$, with $\ell \in \mathbf{R}^+$; $\beta\gamma$ stands for the death rate of v ; α is the saturation rate. The boundary conditions are supposed to be the no flux boundary conditions.

System (1) has been studied extensively in the past few years (see [16–25]). For example, in [16], by using $s := \beta\gamma$ as the bifurcation parameter, the authors showed that, as the parameter s changes, the stability of the constant positive steady state varies, which induces the occurrence of Hopf bifurcating periodic solutions; however, their works tend to be numerical, rather than analytical. In [22], in the special case when $\alpha = 0$, Tang and Song introduced the time delay into the system. By using the stability analysis, center manifold theory, and normal form methods, they were capable of showing the existence of Hopf bifurcations, where the time delay is used to be the bifurcation parameter. In [25], in the special case when $\alpha = 0$, Xu and Song introduced the time delay into the system. By using the stability analysis, center manifold theory, and normal form methods, they were capable of showing the existence of Hopf bifurcations, where the parameter θ is chosen as the bifurcation parameter. In [17], G. Gimmelli, Bob W. Kooi, and E. Venturino considered the ecoepidemic models with prey group defense and feeding saturation in the form of ordinary differential equations. The Hopf bifurcation, zero-Hopf bifurcation, and global bifurcations are considered.

In this paper, however, we do not impose the special case when $\alpha = 0$; instead, we consider the more general case when $\alpha > 0$. This is the first highlight of our paper. Secondly, in the existing literatures, the bifurcation parameter is to be chosen as either s , or the diffusion coefficient, while, in our paper, use μ , or equiv. the first component u_* of the positive equilibrium (u_*, v_*) , as the bifurcation parameter. The different choice of the bifurcation parameter makes our theoretical analysis quite different. This is the second highlight of our paper; finally, the Hopf bifurcation analysis in this paper is mainly based on the theoretical framework of the abstract Hopf bifurcations due to Yi, Wei, and Shi in [26]. In [26], the authors used the same methods to consider a kind of homogeneous diffusive predator-prey system with Holling type-II functional response subject to the homogeneous Neumann boundary conditions. They found that, under suitable conditions, for any eigenmode $\cos(nx/\ell)$, the predator-prey system in [26] has $2n$ Hopf bifurcation points. In this paper, however, we will show that the Hopf bifurcation points of the system are much more complex than those in [26]. This is the third highlight of our paper.

The main analytical tools we used in the discussions of the paper include center manifold theory, normal form methods, and the basic theory in reaction diffusion equations. To better illustrate and support our theoretical bifurcation analysis, we include some delicate numerical simulations in the paper.

The remaining parts of this paper are organized as follows: in Section 2, we not only prove the existence of Hopf bifurcations, but also consider the detailed properties of Hopf bifurcations by using center manifold methods and normal form methods; in Section 3, we include numerical simulations to support our theoretical analysis; in Section 4, we end up our discussions by drawing some conclusions.

2. Hopf Bifurcation Analysis

System (1) has three nonnegative constant steady state solutions: $(0, 0)$, $(1, 0)$, and $(u_*, v_*) := (\mu^2, v_\mu)$, where

$$\begin{aligned}\mu &:= \frac{\gamma}{1 - \alpha\gamma}, \\ v_\mu &:= \mu(1 - \mu^2)(1 + \alpha\mu) \\ &= \frac{\gamma(\alpha\gamma - \gamma - 1)(\alpha\gamma + \gamma - 1)}{(\alpha\gamma - 1)^4}.\end{aligned}\quad (2)$$

Clearly, (u_*, v_*) is a positive equilibrium solution if and only if $\alpha\gamma + \gamma < 1$. In the following, we will always assume that $\alpha\gamma + \gamma < 1$ holds and will choose μ as the main bifurcation parameter (or equivalently γ as a parameter) by fixing α and β .

The linearized operator of system (1) evaluated at (μ^2, v_μ) is given by

$$L(\mu) := \begin{bmatrix} d_1 \frac{\partial^2}{\partial x^2} + A(\mu) & B(\mu) \\ C(\mu) & d_2 \frac{\partial^2}{\partial x^2} \end{bmatrix}, \quad (3)$$

where

$$\begin{aligned}A(\mu) &:= \frac{-4\alpha\mu^3 + 2\alpha\mu - 3\mu^2 + 1}{2(\alpha\mu + 1)}, \\ B(\mu) &:= -\frac{\mu}{\alpha\mu + 1} \\ \text{and } C(\mu) &:= \frac{\beta(1 - \mu^2)}{2(\alpha\mu + 1)}.\end{aligned}\quad (4)$$

It follows from [26] that the eigenvalues of the operator $L(\mu)$ are determined by the eigenvalues of $L_n(\mu)$, with

$$L_n(\mu) := \begin{bmatrix} A(\mu) - \frac{d_1 n^2}{\ell^2} & B(\mu) \\ C(\mu) & -\frac{d_2 n^2}{\ell^2} \end{bmatrix}, \quad (5)$$

$$n \in \mathbf{N}_0 := \{0, 1, 2, \dots\}.$$

The characteristic equation of $L_n(\mu)$ is

$$\eta^2 - \eta T_n(\mu) + D_n(\mu) = 0, \quad n \in \mathbf{N}_0 := \{0, 1, 2, \dots\}, \quad (6)$$

where

$$\begin{aligned}T_n(\mu) &= \frac{-4\alpha\mu^3 + 2\alpha\mu - 3\mu^2 + 1}{2(\alpha\mu + 1)} - \frac{(d_1 + d_2)n^2}{\ell^2}, \\ D_n(\mu) &= \frac{\beta\mu(1 - \mu^2)}{2(\alpha\mu + 1)^2} \\ &\quad - d_2 \left[\frac{-4\alpha\mu^3 + 2\alpha\mu - 3\mu^2 + 1}{2(\alpha\mu + 1)} \right] \frac{n^2}{\ell^2} \\ &\quad + \frac{d_1 d_2 n^4}{\ell^4}.\end{aligned}\quad (7)$$

To consider the stability/instability of the positive constant steady state solution $(u_*, v_*) := (\mu^2, v_\mu)$, as well as the Hopf bifurcations bifurcating from $(u_*, v_*) := (\mu^2, v_\mu)$, we need to know the signs of $T_n(\mu)$ and $D_n(\mu)$.

Lemma 1. *Define*

$$f_1(\mu) := -4\alpha\mu^3 + 2\alpha\mu - 3\mu^2 + 1. \quad (8)$$

Then, $f_1(\mu) = 0$ has three zero points, denoted by μ_1, μ_2, μ_* , satisfying $\mu_1 < \mu_2 < 0 < \mu_*$, such that

$$\begin{aligned} f_1(\mu_1) &= f_1(\mu_2) = f_1(\mu_*) = 0, \\ f_1'(\mu_1) &< 0, \\ f_1'(\mu_2) &> 0, \\ f_1'(\mu_*) &< 0 \end{aligned} \quad (9)$$

and

$$f_1(\mu) \begin{cases} > 0, & \mu \in (-\infty, \mu_1) \cup (\mu_2, \mu_*), \\ < 0, & \mu \in (\mu_1, \mu_2) \cup (\mu_*, \infty). \end{cases} \quad (10)$$

And in particular, for the largest zero point μ_* , it satisfies that $\mu_* \in ((-3 + \sqrt{9 + 32\alpha^2})/8\alpha, +\infty)$.

Proof. By (8), we have $f_1'(\mu) = 2\alpha - 6\mu - 12\alpha\mu^2$, $f_1''(\mu) = -6 - 24\alpha\mu$. Solving $f_1'(\mu) = 0$, we have

$$\begin{aligned} \mu_1' &= \frac{-3 - \sqrt{3}\sqrt{3 + 8\alpha^2}}{12\alpha} < 0, \\ \mu_2' &= \frac{-3 + \sqrt{3}\sqrt{3 + 8\alpha^2}}{12\alpha} > 0. \end{aligned} \quad (11)$$

At $\mu = \mu_1'$, we have

$$\begin{aligned} f_1''(\mu_1') &= 2\sqrt{9 + 24\alpha^2} > 0, \\ f_1(\mu_1') &= \frac{36\alpha^2 - 9 - (8\alpha^2 + 3)\sqrt{9 + 24\alpha^2}}{72\alpha^2}. \end{aligned} \quad (12)$$

We argue that $f_1(\mu_1') < 0$. In fact, letting $\sqrt{9 + 24\alpha^2} = t$, we have $f_1(\mu_1') = (-2t^2 + 15t - 45)/18(t - 3)$. Since $t > 3$ and $-2t^2 + 15t - 45 < 0$, we have $f_1(\mu_1') < 0$. Since $f_1(\mu) = \mu(-4\alpha\mu^2 - 3\mu + 2\alpha) + 1$, we have $f_1(-\infty) = +\infty$. Then, there exists a $\mu_1 \in (-\infty, \mu_1')$, such that $f_1(\mu_1) = 0$ and $f_1'(\mu_1) < 0$.

On the other hand, by $f_1(0) = 1 > 0$ and $f_1(\mu_1') < 0$, there exists a $\mu_2 \in (\mu_1', 0)$ such that $f_1(\mu_2) = 0$ and $f_1'(\mu_2) > 0$. Clearly, $\mu_1 < \mu_2 < 0$.

Since $f_1((-3 + \sqrt{9 + 32\alpha^2})/8\alpha) = 1 = f_1(0)$, we can conclude that $f_1'(\mu) = 0$ at a critical point in $(0, (-3 + \sqrt{9 + 32\alpha^2})/8\alpha)$. Thus, we have $\mu_2' \in (0, (-3 + \sqrt{9 + 32\alpha^2})/8\alpha)$.

At $\mu = \mu_2'$, we have

$$\begin{aligned} f_1''(\mu_2') &= -2\sqrt{9 + 24\alpha^2} < 0, \\ f_1(\mu_2') &= \frac{-9 + 3\sqrt{9 + 24\alpha^2} + 4\alpha^2(9 + 2\sqrt{9 + 24\alpha^2})}{72\alpha^2}. \end{aligned} \quad (13)$$

We argue that $f_1(\mu_2') > 0$. In fact, letting $\sqrt{9 + 24\alpha^2} = t$, we have $f_1(\mu_2') = (2t^2 + 15t + 45)/18(t + 3) > 0$. Noticing that $f_1(+\infty) = -\infty$, we can conclude that there exists a $\mu_* \in (\mu_2', \infty)$, such that $f_1(\mu_*) = 0$. Since $f_1((-3 + \sqrt{9 + 32\alpha^2})/8\alpha) = 1 > 0$, we have $\mu_* \in ((-3 + \sqrt{9 + 32\alpha^2})/8\alpha, +\infty)$. In particular,

$$f_1(\mu) \begin{cases} > 0, & \mu \in (-\infty, \mu_1) \cup (\mu_2, \mu_*), \\ < 0, & \mu \in (\mu_1, \mu_2) \cup (\mu_*, \infty). \end{cases} \quad (14)$$

So far, we have completed the proof. $\square$

We are now identifying Hopf bifurcation values μ^H which satisfy the Hopf bifurcation condition (H_1) (due to [26]), taking the following form: there exists a $n \in \mathbf{N}_0 := \{0, 1, 2, \dots\}$, such that

$$\begin{aligned} T_n(\mu^H) &= 0, \\ D_n(\mu^H) &> 0, \\ \text{and } T_j(\mu^H) &\neq 0, \\ D_j(\mu^H) &\neq 0 \end{aligned} \quad (15)$$

for $j \neq n$

and for the unique pair of complex eigenvalues near the imaginary axis $\lambda(\mu) \pm i\omega(\mu)$,

$$\lambda'(\mu^H) \neq 0. \quad (16)$$

From (7), it follows that $T_n(\mu) < 0$ for $\mu \in (\mu_*, +\infty)$. Hence any potential bifurcation point μ^H must be in the interval $(0, \mu_*)$. Let $\lambda(\mu) \pm i\omega(\mu)$ be the eigenvalues of $L_n(\mu)$ for μ close to the Hopf bifurcation value μ_H . Then,

$$\lambda(\mu) = \frac{A(\mu)}{2} - \frac{(d_1 + d_2)n^2}{2\ell^2}, \quad (17)$$

$$\omega(\mu) = \sqrt{D_n(\mu) - \lambda^2(\mu)},$$

and

$$\lambda'(\mu^H) = \frac{A'(\mu)}{2} \Big|_{\mu=\mu^H} = \frac{g_1(\mu^H)}{2(1 + \alpha\mu^H)^2}, \quad (18)$$

where

$$g_1(\mu) := \alpha - 6\mu - 15\alpha\mu^2 - 8\alpha^2\mu^3. \quad (19)$$

Lemma 2. Let $g_1(\mu)$ be defined in (19). Then for any $\mu > 0$, $g_1'(\mu) > 0$ and there exists a unique $\mu_\# \in (0, (-3 + \sqrt{9 + 32\alpha^2})/8\alpha)$, such that $g_1(\mu_\#) = 0$.

FIGURE 1: The graphs of the functions $A(\mu)$ and $A'(\mu)$.

Proof. By $g_1(\mu) = \alpha - 6\mu - 15\alpha\mu^2 - 8\alpha^2\mu^3$, we have $g'_1(\mu) = -6 - 30\alpha\mu - 24\alpha^2\mu^2 < 0$, where $\mu > 0$. At $\mu = (-3 + \sqrt{9 + 32\alpha^2})/8\alpha$, we have

$$g_1\left(\frac{-3 + \sqrt{9 + 32\alpha^2}}{8\alpha}\right) = -\frac{9 + 64\alpha^2 - 3\sqrt{9 + 32\alpha^2} + 16\alpha^2\sqrt{9 + 32\alpha^2}}{32\alpha}. \quad (20)$$

Again, letting $t = \sqrt{9 + 32\alpha^2}$, we have

$$g_1\left(\frac{-3 + \sqrt{9 + 32\alpha^2}}{8\alpha}\right) = -\frac{(-3 + t)(2 + t)(9 + t)}{8\sqrt{2}\sqrt{-9 + t^2}} < 0, \quad (21)$$

where we use the fact that $t > 3$. By $g_1(0) = \alpha > 0$, we conclude that there exists a $\mu_{\#} \in (0, (-3 + \sqrt{9 + 32\alpha^2})/8\alpha)$ such that $g_1(\mu_{\#}) = 0$. We thus complete the proof. $\square$

According to Lemmas 1 and 2, we can draw the graphs of the functions $A(\mu)$ and $A'(\mu)$ in a coordinate system (Figure 1). Thus, the transversality condition (16) is always satisfied as long as $\mu^H \neq \mu_{\#}$. Moreover when $\mu_{\#} < \mu^H < (-3 + \sqrt{9 + 32\alpha^2})/8\alpha$, the real part of one pair of complex eigenvalues of $\lambda(\mu)$ becomes positive when μ decreases crossing μ^H , and when $0 < \mu^H < \mu_{\#}$, the real part of one pair of complex eigenvalues of $\lambda(\mu)$ becomes negative when μ decreases crossing μ^H . That is, in $(\mu_{\#}, (-3 + \sqrt{9 + 32\alpha^2})/8\alpha)$, the constant steady state loses stability when μ decreases across a bifurcation point, but, in $(0, \mu_{\#})$, it regains the stability when μ decreases across a bifurcation point.

From the aforementioned discussions, the determination of Hopf bifurcation points reduces to describing the set

$$\Lambda := \{\mu \in (0, \mu_{\#}] \setminus \{\mu_{\#}\} : \text{for some } n \in \mathbf{N}_0 \text{ is satisfied}\} \quad (22)$$

when a set of parameters $(\ell, d_1, d_2, \alpha, \beta)$ is given. In the following we fix $d_1, d_2, \alpha > 0$ and $0 < \gamma/(1 - \alpha\gamma) < 1$, but we choose ℓ appropriately.

On one hand, it is clear that $\mu_0^H := \mu_{\#}$ is always an element of Λ for any $\ell > 0$ since $T_0(\mu_0^H) = 0, T_j(\mu_0^H) < 0$ for any $j \geq 1$, and $D_m(\mu_0^H) > 0$ for any $m \in \mathbf{N}_0$. This corresponds to the Hopf bifurcation of spatially homogeneous periodic solution which has been known from the studies of ODE model. Clearly, μ_0^H is also the unique value μ for the Hopf bifurcation of spatially homogeneous periodic solution for any $\ell > 0$.

In the following, we will look for the spatially nonhomogeneous Hopf bifurcation for $n \geq 1$. By the properties of $f_1(u)$ (stated in Lemma 1), it follows that $A(\mu_1) = A(\mu_2) = A(\mu_{\#}) = 0$, and in particular, $A(\mu)$ has a unique critical point $\mu = \mu_{\#}$, with $\mu_2 < \mu_{\#} < \mu_{\#}$, at which $A(\mu)$ achieves a local maximum $A(\mu_{\#}) := M_{\#} > 0$. In particular, by Lemma 2, $\mu_{\#} \in (0, (-3 + \sqrt{9 + 32\alpha^2})/8\alpha)$.

Theorem 3. Let $\mu_{\#}$ be the critical point of $A(\mu)$, with $\mu_2 < \mu_{\#} < \mu_{\#}$, at which $A(\mu)$ achieves a local maximum $A(\mu_{\#}) := M_{\#} > 0$.

- (1) Suppose that $\mu_{\#} \leq 0$ and that, for sufficiently small $\epsilon > 0$,

$$\frac{d_2 A(0)^2}{2\beta d_1} < \frac{\epsilon - \epsilon^3}{(\alpha\epsilon + 1)^2} \quad (23)$$

holds. Define

$$\bar{\ell}_n = n\sqrt{\frac{d_1 + d_2}{A(0)}}, \quad n \in \mathbf{N}. \quad (24)$$

For $\bar{\ell}_n < \ell \leq \bar{\ell}_{n+1}$ and $1 \leq j \leq n$, there exist n points μ_j^H , with $1 \leq j \leq n$, satisfying

$$0 < \mu_n^H < \dots < \mu_2^H < \mu_1^H < \mu_0^H := \mu_{\#}, \quad (25)$$

such that the system (1) undergoes a Hopf bifurcation at $\mu = \mu_j^H$ or $\mu = \mu_0^H := \mu_{\#}$.

- (2) Suppose that $\mu_{\#} > 0$ and that, for sufficiently small $\epsilon > 0$,

$$\frac{d_2 A(0)^2}{2\beta d_1} < \frac{\epsilon - \epsilon^3}{(\alpha\epsilon + 1)^2} \quad (26)$$

holds. Define

$$\underline{\ell}_n = n\sqrt{\frac{d_1 + d_2}{M_{\#}}}, \quad n \in \mathbf{N}. \quad (27)$$

- (a) Then for $\underline{\ell}_n < \ell \leq \underline{\ell}_{n+1}$ and $\bar{\ell}_{n-k} < \ell < \bar{\ell}_{n-k+1}$, with $k \leq j \leq n$ for some $k \in \mathbf{N}_0$, there exist $2k$ points $\mu_{j,\pm}^H$, with $n - k \leq j \leq n$ satisfying

$$0 < \mu_{1,-}^H < \mu_{2,-}^H < \dots < \mu_{k,-}^H < \mu_{\#} < \mu_{n,+}^H < \dots < \mu_{2,+}^H < \mu_{k,+}^H < \mu_0^H := \mu_{\#} \quad (28)$$

such that the system (1) undergoes a Hopf bifurcation at $\mu = \mu_j^H$ or $\mu = \mu_0^H := \mu_*$.

- (b) For $\underline{\ell}_n < \ell \leq \underline{\ell}_{n+1}$ and $1 \leq j \leq n$, there exist n points μ_j^H , with $1 \leq j \leq n$, satisfying

$$0 < \mu_n^H < \dots < \mu_2^H < \mu_1^H < \mu_0^H := \mu_*. \quad (29)$$

Then for any $\ell \in (\underline{\ell}_n, \underline{\ell}_{n+1}]$, there exist n points $\mu_j^H(\ell)$, $1 \leq j \leq n$, satisfying $0 < \mu_n^H < \dots < \mu_2^H < \mu_1^H < \mu_0^H$, such that the system (1) undergoes a Hopf bifurcation at $\mu = \mu_j^H$ or $\mu = \mu_0^H := \mu_*$.

Moreover:

(1) The bifurcating periodic solutions from $\mu = \mu_0^H := \mu_*$ are spatially homogeneous, which coincides with the periodic solution of the corresponding ODE system.

(2) The bifurcating periodic solutions from $\mu = \mu_{j,\pm}^H$ (or μ_j^H) are spatially nonhomogeneous.

Proof.

- (1) Suppose that $\mu_* \leq 0$ holds. Then, $A(0) = 1 = \max_{\mu \in [0, +\infty)} A(\mu)$ and $A'(\mu) < 0$ for any $\mu \in [0, +\infty)$. In this case, define

$$\bar{\ell}_n = n \sqrt{\frac{d_1 + d_2}{A(0)}}, \quad n \in \mathbf{N}. \quad (30)$$

For $\bar{\ell}_n < \ell \leq \bar{\ell}_{n+1}$ and $1 \leq j \leq n$, we define μ_j^H to be the unique root of $A(\mu) = (d_1 + d_2)j^2/\ell^2$ satisfying $0 < \mu_j^H < \mu_* < \mu_0^H := \mu_*$. These points satisfy

$$0 < \mu_n^H < \dots < \mu_2^H < \mu_1^H < \mu_0^H. \quad (31)$$

Clearly $T_j(\mu_j^H) = 0$ and $T_i(\mu_j^H) \neq 0$ for $i \neq j$. In this case, the transversality condition always holds true. Since $\mu = \gamma/(1 - \alpha\gamma)$ and $\alpha\gamma + \gamma < 1$, we can assume that $\mu > \epsilon$, where ϵ is an arbitrary positive number. Indeed if $\mu \in (0, \mu_0^H]$, then we have

$$\begin{aligned} D_i(\mu) &\geq \frac{\beta(\epsilon - \epsilon^3)}{2(\alpha\epsilon + 1)^2} - d_2 A(0) \frac{i^2}{\ell^2} + d_1 d_2 \frac{i^4}{\ell^4} \\ &:= h\left(\frac{i^2}{\ell^2}\right). \end{aligned} \quad (32)$$

The quadratic function $h(y) := d_1 d_2 y^2 - d_2 A(0)y + \beta(\epsilon - \epsilon^3)/2(\alpha\epsilon + 1)^2$ is positive for all $y \in \mathbf{R}$ if

$$\frac{d_2 A(0)^2}{2\beta d_1} < \frac{\epsilon - \epsilon^3}{(\alpha\epsilon + 1)^2}. \quad (33)$$

- (2) Suppose that $\mu_* > 0$ holds. Then, $A'(\mu) > 0$ for any $\mu \in [0, +\mu_{**})$ and $A'(\mu) < 0$ for any $\mu \in (\mu_{**}, +\infty)$.

Define

$$\underline{\ell}_n = n \sqrt{\frac{d_1 + d_2}{M_*}}, \quad n \in \mathbf{N}. \quad (34)$$

Then for $\underline{\ell}_n < \ell \leq \underline{\ell}_{n+1}$ and $\bar{\ell}_{n-k} < \ell < \bar{\ell}_{n-k+1}$, with $k \leq j \leq n$ for some $k \in \mathbf{N}_0$, we define $\mu_{j,-}^H$ and $\mu_{j,+}^H$ to be the roots of $A(\mu) = (d_1 + d_2)j^2/\ell^2$ satisfying $0 < \mu_{j,-}^H < \mu_{**} < \mu_{j,+}^H < \mu_0^H = \mu_*$. These points satisfy

$$\begin{aligned} 0 < \mu_{1,-}^H < \mu_{2,-}^H < \dots < \mu_{k,-}^H < \mu_* < \mu_{n,+}^H < \dots < \mu_{2,+}^H \\ < \mu_{k,+}^H < \mu_0^H \end{aligned} \quad (35)$$

Clearly $T_j(\mu_{j,\pm}^H) = 0$ and $T_i(\mu_{j,\pm}^H) \neq 0$ for $i \neq j$. In this case, the transversality condition always holds true.

For $\underline{\ell}_n < \ell \leq \underline{\ell}_{n+1}$ and $1 \leq j \leq n$, we define μ_j^H to be the unique root of $A(\mu) = (d_1 + d_2)j^2/\ell^2$ satisfying $0 < \mu_j^H < \mu_0^H := \mu_*$. These points satisfy

$$0 < \mu_n^H < \dots < \mu_2^H < \mu_1^H < \mu_0^H. \quad (36)$$

Clearly $T_j(\mu_j^H) = 0$ and $T_i(\mu_j^H) \neq 0$ for $i \neq j$. In this case, the transversality condition always holds true.

Now we only need to verify whether $D_i(\mu^H) \neq 0$ for all $i \in \mathbf{N}_0$ for the aforementioned three cases, and, in particular, $D_i(\mu^H) > 0$. Here we derive a condition on the parameters so that $D_i(\mu) > 0$ for all $\mu \in (0, \mu_0^H]$ so that $D_i(\mu_{j,\pm}^H) > 0$. Since $\mu = \gamma/(\alpha\gamma - 1)$ and $\alpha\gamma > \gamma + 1$, we can assume that $\mu > \epsilon$, where ϵ is an arbitrary positive number. Indeed if $\mu \in (0, \mu_0^H]$, then we have

$$\begin{aligned} D_i(\mu) &\geq \frac{\beta(\epsilon - \epsilon^3)}{2(\alpha\epsilon + 1)^2} - d_2 M_* \frac{i^2}{\ell^2} + d_1 d_2 \frac{i^4}{\ell^4} \\ &:= h\left(\frac{i^2}{\ell^2}\right). \end{aligned} \quad (37)$$

The quadratic function $h(y) := d_1 d_2 y^2 - d_2 M_* y + \beta(\epsilon - \epsilon^3)/2(\alpha\epsilon + 1)^2$ is positive for all $y \in \mathbf{R}$ if

$$\frac{d_2 M_*^2}{2\beta d_1} < \frac{\epsilon - \epsilon^3}{(\alpha\epsilon + 1)^2}. \quad (38)$$

Then, applying Theorem 2.1 in [27], we have completed the proof. $\square$

In the following, we will consider the bifurcation direction and the stability of the bifurcating periodic solutions. We have the following results.

Theorem 4. For the Hopf bifurcation points $\mu_0^H = \mu_*$ obtained in Theorem 3, the following conclusions hold:

- (1) The Hopf bifurcation at μ_* is supercritical if $\Gamma_0(\mu_*) < 0$. It is subcritical if $\Gamma_0(\mu_*) > 0$.
- (2) The spatially homogeneous periodic solutions bifurcating from the constant positive steady state (u_*, v_*) at $\mu = \mu_*$ are locally orbitally stable if $\Gamma_0(\mu_*) < 0$, and they are unstable if $\Gamma_0(\mu_*) > 0$,

where $\Gamma_0(\mu_*)$ is precisely defined in (63).

Proof. We rely on the abstract results obtained in [26] to determine the stability and bifurcation direction of the bifurcating periodic solution. To that end, we calculate $\text{Re}(c_1(\mu_*))$ in the context of [26]. We use μ_* to stand for μ_* . When $\mu = \mu_*$, we let

$$\begin{aligned} q &:= \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} -\frac{\mu_* i}{\omega_0(\alpha\mu_* + 1)} \\ 1 \end{bmatrix}, \\ q^* &:= \begin{bmatrix} a_0^* \\ b_0^* \end{bmatrix} = \begin{bmatrix} -\frac{\omega_0(\alpha\mu_* + 1)i}{2\ell\pi\mu_*} \\ \frac{1}{2\ell\pi} \end{bmatrix} \end{aligned} \quad (39)$$

be such that

$$\begin{aligned} L(\mu_*)q &= i\omega_0 q, \\ L^*(\mu_*)q^* &= -i\omega_0 q^*, \\ \langle q^*, q \rangle &= 1, \\ \langle q^*, \bar{q} \rangle &= 0, \end{aligned} \quad (40)$$

where $\omega_0 = \sqrt{\beta\mu_*(1 - \mu_*^2)}/\sqrt{2}(\alpha\mu_* + 1)$ ($\mu_* > 0$ and $-4\alpha\mu_*^3 + 2\alpha\mu_* - 3\mu_*^2 + 1 = 0$), and

$$L^*(\mu_*) := \begin{bmatrix} d_1 \frac{\partial^2}{\partial x^2} + \frac{-4\alpha\mu_*^3 + 2\alpha\mu_* - 3\mu_*^2 + 1}{2(\alpha\mu_* + 1)} & -\frac{\mu_*}{\alpha\mu_* + 1} \\ \frac{\beta(1 - \mu_*^2)}{2(\alpha\mu_* + 1)} & d_2 \frac{\partial^2}{\partial x^2} \end{bmatrix}. \quad (41)$$

As in [26], we define

$$\begin{aligned} f(\mu, u, v) &:= (u + u_*)(1 - u - u_*) \\ &\quad - \frac{\sqrt{u + u_*}(v + v_*)}{1 + \alpha\sqrt{u + u_*}}, \\ g(\mu, u, v) &:= \beta(v + v_*) \left(-\gamma + \frac{\sqrt{u + u_*}}{1 + \alpha\sqrt{u + u_*}} \right). \end{aligned} \quad (42)$$

We rewrite the original system in the following abstract equation:

$$\frac{dU}{dt} = L(\mu)U + F(\mu, U), \quad (43)$$

where

$$F(\mu, U) := \begin{bmatrix} f(\mu, u, v) - A(\mu)u - B(\mu)v \\ g(\mu, u, v) - C(\mu)u \end{bmatrix}, \quad (44)$$

with $U = (u, v)^T \in X$, where $A(\mu)$, $B(\mu)$ and $C(\mu)$ are shown in (4). At $\mu = \mu_*$, the system (43) reduces to

$$\frac{dU}{dt} = L(\mu_*)U + F_0(U), \quad (45)$$

where $F_0(U) := F(\mu, U)|_{\mu=\mu_*}$.

Let

$$\begin{aligned} X &= \{(u, v) \in H^2(0, \ell\pi) \times H^2(0, \ell\pi) \mid (u_x, v_x)|_{x=0, \ell\pi} \\ &= 0\} \end{aligned} \quad (46)$$

and decompose $X = X^c \oplus X^s$, with $X^c := \{zq + \bar{z}\bar{q} \mid z \in \mathbb{C}\}$, $X^s := \{u \in X \mid \langle q^*, u \rangle = 0\}$. For any $(u, v) \in X$, there exist $z \in \mathbb{C}$ and $w = (w_1, w_2) \in X^s$ such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = zq + \bar{z}\bar{q} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \quad (47)$$

Thus the system (45) is reduced to the following system in (z, w) coordinates:

$$\begin{aligned} \frac{dz}{dt} &= i\omega_0 z + \langle q^*, F_0 \rangle, \\ \frac{dw}{dt} &= L(\lambda_0)w + H(z, \bar{z}, w), \end{aligned} \quad (48)$$

where

$$\begin{aligned} H(z, \bar{z}, w) &:= F_0 - \langle q^*, F_0 \rangle q - \langle \bar{q}^*, F_0 \rangle \bar{q}, \\ &\text{and } F_0 := F_0(zq + \bar{z}\bar{q} + w). \end{aligned} \quad (49)$$

As in [26], we write F_0 in the following form:

$$F_0(U) := \frac{1}{2}Q(U, U) + \frac{1}{6}C(U, U, U) + O(|U|^4), \quad (50)$$

where $U = (u, v)$,

where Q, C are symmetric multilinear forms. For simplicity, we write $Q_{XY} = Q(X, Y)$ and $C_{XYZ} = C(X, Y, Z)$.

Then, the reaction-diffusion system restricted to the center manifold is given by

$$\begin{aligned} \frac{dz}{dt} &= i\omega_0 z + \langle q^*, F_0 \rangle \\ &= i\omega_0 z + \sum_{2 \leq i+j \leq 3} \frac{g_{ij}}{i!j!} z^i \bar{z}^j + \mathcal{O}(|z|^4), \end{aligned} \quad (51)$$

where $g_{20} = \langle q^*, Q_{qq} \rangle$, $g_{11} = \langle q^*, Q_{q\bar{q}} \rangle$, $g_{02} = \langle q^*, Q_{\bar{q}\bar{q}} \rangle$, and

$$g_{21} = 2 \langle q^*, Q_{w_{11}, q} \rangle + \langle q^*, Q_{w_{20}, \bar{q}} \rangle + \langle q^*, C_{q, q, \bar{q}} \rangle. \quad (52)$$

The dynamics of (48) can be determined by the dynamics of (51).

We write the Poincaré normal form of (43) (for μ in a neighborhood of μ_*) in the following form:

$$\dot{z} = (\alpha(\mu) + i\omega(\mu))z + z \sum_{j=1}^M c_j(\mu) (z\bar{z})^j, \quad (53)$$

where z is a complex variable, $M \geq 1$, and $c_j(\mu)$ are complex-valued coefficients.

Then, we have

$$\begin{aligned} c_1(\mu) &= \frac{g_{20}g_{11}(3\alpha(\mu) + i\omega(\mu))}{2(\alpha^2(\mu) + \omega^2(\mu))} + \frac{|g_{11}|^2}{\alpha(\mu) + i\omega(\mu)} \\ &\quad + \frac{|g_{02}|^2}{2(\alpha(\mu) + 3i\omega(\mu))} + \frac{g_{21}}{2}. \end{aligned} \quad (54)$$

Thus,

$$\begin{aligned} c_1(\mu_*) &= \frac{i}{2\omega_0} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2} \\ &= \frac{i}{2\omega_0} \langle q^*, Q_{qq} \rangle \cdot \langle q^*, Q_{q\bar{q}} \rangle + \langle q^*, Q_{w_{11},q} \rangle \\ &\quad + \frac{1}{2} \langle q^*, Q_{w_{20},\bar{q}} \rangle + \frac{1}{2} \langle q^*, C_{q,q,\bar{q}} \rangle. \end{aligned} \quad (55)$$

Then, it remains to calculate $\langle q^*, Q_{qq} \rangle$, $\langle q^*, Q_{q\bar{q}} \rangle$, $\langle \bar{q}^*, Q_{qq} \rangle$, $\langle q^*, C_{qq\bar{q}} \rangle$, and $\langle q^*, C_{qq\bar{q}} \rangle$.

After direct calculations, we observe that all of the $f_{vv}(\mu_*, 0, 0)$, $f_{uvv}(\mu_*, 0, 0)$, $f_{vvv}(\mu_*, 0, 0)$, $g_{vv}(\mu_*, 0, 0)$, $g_{uvv}(\mu_*, 0, 0)$, and $g_{vvv}(\mu_*, 0, 0)$ are all zeros. And in particular,

$$\begin{aligned} f_{uu}(\mu_*, 0, 0) &= -2 + \frac{(1 - \mu_*^2)(3\alpha\mu_* + 1)}{4\mu_*^2(\alpha\mu_* + 1)^2}, \\ f_{uv}(\mu_*, 0, 0) &= -\frac{1}{2\mu_*(\alpha\mu_* + 1)^2}, \end{aligned}$$

$$f_{uuu}(\mu_*, 0, 0) = \frac{3(\mu_*^2 - 1)(5\alpha^2\mu_*^2 + 4\alpha\mu_* + 1)}{8\mu_*^4(\alpha\mu_* + 1)^3},$$

$$f_{uuv}(\mu_*, 0, 0) = \frac{3\alpha\mu_* + 1}{4(\mu_*)^3(\alpha\mu_* + 1)^3},$$

$$g_{uu}(\mu_*, 0, 0) = \frac{\beta(\mu_*^2 - 1)(3\alpha\mu_* + 1)}{4\mu_*^2(\alpha\mu_* + 1)^2},$$

$$g_{uv}(\mu_*, 0, 0) = \frac{\beta}{2\mu_*(\alpha\mu_* + 1)^2},$$

$$g_{uuu}(\mu_*, 0, 0) = -\frac{3\beta(\mu_*^2 - 1)(5\alpha^2\mu_*^2 + 4\alpha\mu_* + 1)}{8\mu_*^4(\alpha\mu_* + 1)^3},$$

$$g_{uuv}(\mu_*, 0, 0) = -\frac{\beta(3\alpha\mu_* + 1)}{4\mu_*^3(\alpha\mu_* + 1)^3}.$$

(56)

Then, from [26], we have

$$\begin{aligned} c_0 &:= f_{uu}a_0^2 + 2f_{uv}a_0b_0 \\ &= \frac{8\alpha^2\mu_*^4 + 19\alpha\mu_*^3 - 3\alpha\mu_* + 9\mu_*^2 - 1 + 4i\omega_0(\alpha\mu_* + 1)}{4\omega_0^2(\alpha\mu_* + 1)^4}, \end{aligned}$$

$$\begin{aligned} d_0 &:= g_{uu}a_0^2 + 2g_{uv}a_0b_0 \\ &= \beta \frac{-3\alpha\mu_*^3 - \mu_*^2 + 3\alpha\mu_* + 1 + 4i\omega_0(\alpha\mu_* + 1)}{4\omega_0^2(\alpha\mu_* + 1)^4}, \end{aligned} \quad (57)$$

$$\begin{aligned} e_0 &:= f_{uu}|a_0|^2 + f_{uv}(a_0\bar{b}_0 + \bar{a}_0b_0) \\ &= \frac{-8\alpha^2\mu_*^4 - 19\alpha\mu_*^3 + 3\alpha\mu_* - 9\mu_*^2 + 1}{4\omega_0^2(\alpha\mu_* + 1)^4}, \end{aligned}$$

and

$$f_0 := g_{uu}|a_0|^2 + g_{uv}(a_0\bar{b}_0 + \bar{a}_0b_0) = \frac{\beta(\mu_*^2 - 1)(3\alpha\mu_* + 1)}{4\omega_0^2(\alpha\mu_* + 1)^4},$$

$$g_0 := f_{uuu}|a_0|^2 a_0 + f_{uuv}(2|a_0|^2 \bar{b}_0 + a_0^2 \bar{b}_0) = \frac{2\omega_0(3\alpha^2\mu_*^2 + 4\alpha\mu_* + 1) - 3i(\mu_*^2 - 1)(5\alpha^2\mu_*^2 + 4\alpha\mu_* + 1)}{8\omega_0^3\mu_*(\alpha\mu_* + 1)^6}, \quad (58)$$

$$h_0 := g_{uuu}|a_0|^2 a_0 + g_{uuv}(2|a_0|^2 \bar{b}_0 + a_0^2 \bar{b}_0) = \frac{-2\beta\omega_0(\alpha\mu_* + 1)(3\alpha\mu_* + 1) + 3i\beta(\mu_*^2 - 1)[\alpha\mu_*(5\alpha\mu_* + 4) + 1]}{8\omega_0^3\mu_*(\alpha\mu_* + 1)^6}.$$

Again, by [26], we have

$$\begin{aligned}\langle q^*, Q_{qq} \rangle &= \int_0^{\ell\pi} \overline{q}^{*T} (c_0, d_0)^T dx \\ &= -\frac{\beta\mu_* (\mu_*^2 - 1) (3\alpha\mu_* + 1) + 4\omega_0^2 (\alpha\mu_* + 1)^2}{8\omega_0^2\mu_* (\alpha\mu_* + 1)^4} \\ &\quad + i\frac{\mu_* (8\alpha^2\mu_*^3 + 19\alpha\mu_*^2 + 9\mu_* - 3\alpha - 4\beta) - 1}{8\omega_0\mu_* (\alpha\mu_* + 1)^3}, \\ \langle q^*, Q_{q\overline{q}} \rangle &= \int_0^{\ell\pi} \overline{q}^{*T} (e_0, f_0)^T dx \\ &= \frac{\beta(\mu_*^2 - 1)(3\alpha\mu_* + 1)}{8\omega_0^2(\alpha\mu_* + 1)^4}\end{aligned}$$

$$+ i\frac{1 - \mu_* (8\alpha^2\mu_*^3 + 19\alpha\mu_*^2 - 3\alpha + 9\mu_*)}{8\omega_0\mu_* (\alpha\mu_* + 1)^3},$$

$$\begin{aligned}\langle \overline{q}^*, Q_{qq} \rangle &= \int_0^{\ell\pi} q^{*T} (c_0, d_0)^T dx \\ &= \frac{4\omega_0^2 (\alpha\mu_* + 1)^2 - \beta\mu_* (\mu_*^2 - 1) (3\alpha\mu_* + 1)}{8\omega_0^2\mu_* (\alpha\mu_* + 1)^4} \\ &\quad + i\frac{1 - \mu_* (8\alpha^2\mu_*^3 + 19\alpha\mu_*^2 + 9\mu_* - 3\alpha + 4\beta)}{8\omega_0\mu_* (\alpha\mu_* + 1)^3},\end{aligned}\tag{59}$$

and

$$\begin{aligned}\langle \overline{q}^*, Q_{q\overline{q}} \rangle &= \int_0^{\ell\pi} q^{*T} (e_0, f_0)^T dx = \frac{\beta(\mu_*^2 - 1)(3\alpha\mu_* + 1)}{8\omega_0^2(\alpha\mu_* + 1)^4} + i\frac{\mu_* (8\alpha^2\mu_*^3 + 19\alpha\mu_*^2 - 3\alpha + 9\mu_*) - 1}{8\omega_0\mu_* (\alpha\mu_* + 1)^3}, \\ \langle q^*, C_{q\overline{q}} \rangle &= \int_0^{\ell\pi} \overline{q}^{*T} (g_0, h_0)^T dx \\ &= \frac{\mu_* [15\alpha^2\mu_*^3 + 12\alpha\mu_*^2 + \mu_* (-15\alpha^2 - 6\alpha\beta + 3) - 12\alpha - 2\beta] - 3}{16\omega_0^2\mu_*^2 (\alpha\mu_* + 1)^5} \\ &\quad + i\frac{3\beta\mu_* (\mu_*^2 - 1) [\alpha\mu_* (5\alpha\mu_* + 4) + 1] + 2\omega_0^2 (\alpha\mu_* + 1)^2 (3\alpha\mu_* + 1)}{16\omega_0^3\mu_*^2 (\alpha\mu_* + 1)^6}.\end{aligned}\tag{60}$$

Then, from [26], we have

$$H_{20} := \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} - \langle q^*, Q_{qq} \rangle \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \langle \overline{q}^*, Q_{qq} \rangle \begin{pmatrix} \overline{a}_0 \\ \overline{b}_0 \end{pmatrix}\tag{61}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies that $w_{20} = 0, w_{11} = 0$. Therefore,

$$\langle q^*, Q_{w_{11}q} \rangle = \langle q^*, Q_{w_{20}\overline{q}} \rangle = 0.\tag{62}$$

$$H_{11} := \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} - \langle q^*, Q_{q\overline{q}} \rangle \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \langle \overline{q}^*, Q_{q\overline{q}} \rangle \begin{pmatrix} \overline{a}_0 \\ \overline{b}_0 \end{pmatrix}$$

So far, based on the aforementioned computations, we have

$$\begin{aligned}\text{Re}(c_1(\mu_*)) &= \Gamma_0(\mu_*) = \Gamma_0(\mu_*) := \text{Re} \left\{ \frac{i}{2\omega_0} \langle q^*, Q_{qq} \rangle \cdot \langle q^*, Q_{q\overline{q}} \rangle + \frac{1}{2} \langle q^*, C_{q\overline{q}} \rangle \right\} \\ &= \frac{-17\alpha^2\mu_*^4 - 15\alpha^2\mu_*^2 - 6\alpha\beta\mu_*^2 - 64\alpha\mu_*^3 - 2\beta\mu_* - 33\mu_*^2 + 1}{32(\omega^H)^2\mu_*^2(\alpha\mu_* + 1)^5} \\ &\quad - \frac{\beta(\mu_*^2 - 1)(3\alpha\mu_* + 1)(8\alpha^2\mu_*^4 + 19\alpha\mu_*^3 - 3\alpha\mu_* - 2\beta\mu_* + 9\mu_*^2 - 1)}{16\omega_0^4\mu_* (\alpha\mu_* + 1)^7}.\end{aligned}\tag{63}$$

Then, from [26], we complete the proof. $\square$

FIGURE 2: Numerical simulations of a stable homogeneous equilibrium solution of system with parameters in (64). The left: component u ; the right: component v .

FIGURE 3: Numerical simulations of the Hopf bifurcating spatially homogeneous periodic solutions of system (1) with parameters in (65). The left: component u ; the right: component v .

3. Numerical Simulations

In this section, we include some numerical simulations to support our theoretical analysis. We use the symbolic mathematical software *Matlab* (Version 7.0) to plot numerical graphs.

The model (1) has five different system parameters, which are: α , β , γ , d_1 , and d_2 . On one hand, we choose the system parameters in the following way:

$$\begin{aligned}\alpha &= 1.7, \\ \beta &= 0.9, \\ \gamma &= 0.3, \\ d_1 &= 1, \\ d_2 &= 3.\end{aligned}\tag{64}$$

Under (64) and after direct calculations, we have $(u_*, v_*) = (0.3748, 0.7811)$. In this case, the positive equilibrium solution (u_*, v_*) is locally asymptotically stable, and no Hopf bifurcations can be observed. This is shown in Figure 2, where the initial condition is taken by $u(0) = 0.3748 + 0.01 \cos(2\pi x/3)$, $v(0) = 0.7811 + 0.01 \cos(2\pi x/3)$ (Figure 2).

On the other hand, we choose the system parameters in the following way:

$$\begin{aligned}\alpha &= 2, \\ \beta &= 9,\end{aligned}$$

$$\gamma = 0.2851,$$

$$d_1 = 1,$$

$$d_2 = 3.$$

(65)

Under (65) and after direct calculations, we have $(u_*, v_*) = (0.6632, 0.8643)$. In this case, (u_*, v_*) is unstable and a family of periodic solutions bifurcating from (u_*, v_*) can be observed. This is shown in Figure 3, where the initial condition is taken at $u(0) = 0.6632 + 0.01 \cos(2\pi x/3)$, $v(0) = 0.8643 + 0.01 \cos(2\pi x/3)$ (Figure 3).

4. Conclusions

In this paper, we considered a new homogeneous diffusive predator-prey system with herd behavior in terms of the square root of the prey population. By using the classical Hopf bifurcation theorem applicable to the general reaction-diffusion equations due to [26], we are capable of showing the existence of both spatially homogeneous/inhomogeneous periodic solutions bifurcating from the positive constant steady state solutions. For the homogeneous periodic solutions, we derive the formulae to determine their stability and instability. In this paper, there are three highlights: (1) we consider the more general case when $\alpha > 0$, instead of $\alpha = 0$. (2) We use μ , or equiv. the first component u_* of the positive equilibrium (u_*, v_*) , as the bifurcation parameter, instead of

s and the diffusion coefficient; (3) we show that the Hopf bifurcation points of the system are much more complex than those in [26]. Our results allow for the clearer understanding of the mechanism of the spatiotemporal pattern formations of the predator-prey interactions in ecology.

Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Prof. Meng Lin, Prof. Yanyou Chai, and Prof. Xuguang Yang contributed to the Hopf bifurcation analysis. Dr. Yufeng Wang contributed to the numerical simulations in the revision stage of the manuscript. All authors read and approved the final manuscript.

Acknowledgments

The final version of the paper was completed when the corresponding author Yanyou Chai visited University of Central Florida (UCF), and he would like to thank the hospitality of Prof. Z. Shuai in UCF. This work is supported by the National Natural Science Foundation of China (No. 61801141) and the Fundamental Research Funds for the Central Universities.

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