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Research Article

A Unified Approach of Free Vibration Analysis for Stiffened Cylindrical Shell with General Boundary Conditions

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A unified approach of free vibration analysis for stiffened cylindrical shell with general boundary conditions is presented in this paper. The vibration of stiffened cylindrical shell is modeled mathematically involving the first-order shear deformation shell theory. The improved Fourier series is selected as the admissible displacement function while the arbitrary boundary conditions are simulated by adjusting the equivalent spring stiffness. The natural frequencies and modal shapes of the stiffened shell are obtained by solving the dynamic model with the Rayleigh-Ritz procedure. Various numerical results of free vibration analysis for stiffened cylindrical shell are obtained, including natural frequencies and modes under simply supported, free, and clamped boundary conditions. Moreover, the effects of stiffener on natural frequencies are discussed. Compared with several state-of-the-art methods, the feasibility and validity of the proposed method are verified.

1. Introduction

As one critical load-bearing component in structural engineering, stiffened cylindrical shells are widely used in various industrial equipment such as aerospace, rocket, and infrastructure [1–3]. On the premise of guaranteeing the durability and service life of whole structures, the stiffened structure is conducive to save the consumption of materials used, reduce weight, and improve the mechanical properties of the whole structures. However, compared with simple cylindrical shells, the application of stiffeners also brings in high-complexity in describing its dynamic behaviors and performing its performance analysis. Therefore, it is urgently desired to develop an efficient approach to analyze the dynamic characteristics of stiffened cylindrical shells. As an important analysis technique, free vibration analysis is frequently required and has always been one study focus since it can provide valuable insight into dynamic behaviors and vibration control of stiffened cylindrical shells [4–7]. In past several decades, the increasing application of stiffened cylindrical shells has motivated great efforts in developing more accurate mathematical models and corresponding approaches for analyzing their dynamic behaviors. In this case, a large variety of modern theories and advanced approaches for vibration analysis have been achieved and the dynamic response characteristics of stiffened cylindrical shells have been well studied [8–10]. For instance, Al-Najafi [8] chose each ring rib as a discrete element and utilized the finite element theory to study the free vibration of stiffened cylindrical shells with simply supported boundary at both ends. Jafari [9] selected power series as Ritz functions and investigated the effects of initial circumferential stress, centrifugal force, and rotation on the natural frequencies of ring-stiffened cylindrical shells. Chen [10] solved theoretical model and achieves vibration analysis of the ring-stiffened cylindrical shell by using wave method. The free vibration characteristics of ring-stiffened cylindrical shells under several classical boundary conditions were obtained, in which ring-stiffened and frame-stiffened plates were used. However, it appears that most of the previous studies on stiffened cylindrical shells are confined to classic boundary conditions. In fact, as we know, a variety of possible boundary conditions encountered in engineering may not always be classical in nature [11]. Besides, the existing solution procedures are often customized for a specific set of different boundary conditions; the constant model modifications would lead to massive solution procedure operations,

which result in very tedious calculations because even only the classic cases still need a total of hundreds of different combinations [12, 13]. Therefore, to simplify the calculation tasks and obtain the accurate vibration characteristics, a unified method is urgently desired to tackle with the general boundary conditions problems in free vibration analysis of stiffened cylindrical shells. In this paper, to address general boundary conditions issues and build unified free vibration analysis method, two obstacles should be overcome: (1) establishment of a high-accuracy dynamic model for stiffened cylindrical shells; (2) development of an efficient solution procedure to resolve the aforementioned dynamic model.

To achieve unified vibration analysis and obtain accurate solution results, as we know, it should be the first to construct high-accuracy mathematical models. The current shell deformation theories on model formulation are mainly classified into three categories: Classical Shell Deformation Theory (CSDT), First-order Shear Deformation Theory (FSDT), and Higher-order Shear Deformation Theory (HSDT). In view of the Kirchhoff-Love assumptions, CSDT neglects neglect of transverse shear strains and possesses the advantages of simple calculation tasks [14]. To date, various subcategory CSDTs were derived and employed in vibration analysis of thin cylindrical shells [15, 16]. Although CSDT can achieve accurate vibration results for thin shells, it is unsuitable to describe the vibration characteristics of the stiffened cylindrical shell with larger thickness value. As one important alternative, the managerial theory-based FSDT were developed, which abandons the hypothesis in CSDT that the transverse normal is still perpendicular to the mid-surface after structural deformation, thus overcoming the defect that the neglect of transverse shear deformation in CSDT [17]. In light of the increasing options of admissible displacement function, the FSDT have been widely used in modeling and analysis of medium thick plates, laminated plates, and moderately thick shells [18–20]. Moreover, to further enhance the analysis precision, HSDT were developed, in which the transverse normal is no longer perpendicular to the middle after deformation [21, 22]. Although the HSDTs are capable of solving the global dynamic problem of shells more accurately, it also brings sophisticated formulations and boundary terms, which extremely increases the computational complexity and tasks [23, 24]. Therefore, HSDT is deemed to be unpractical in vibration analysis of complex shell structures like stiffened cylindrical shells. In the present work, the FSDT with high computing accuracy and little computing burden was chosen to formulate the theoretical model of stiffened cylindrical shells.

Apart from the aforementioned models with shear deformation theories, to accomplish high-accuracy and high-efficiency unified vibration analysis under general boundary conditions, the corresponding efficient and accurate solution approaches also should be developed to determine the vibration characteristics of stiffened cylindrical shells. A number of computational methods are available for vibration analysis of cylindrical shells, such as polynomial Ritz method (PRM) [25, 26], wave propagation method (WPM) [27–29], transfer matrix method (TMM) [30, 31], Galletly Ritz method (GRM) [32, 33], Galerkin method [34–36], finite element method (FEM) [37–41], the continuous element method (CEM) [42], and Rayleigh-Ritz method [43–45]. By less computational effort to achieve more accurate results and smooth derivative terms, Rayleigh-Ritz method possesses the potential of alleviating computational burdens and improving solution accuracy in the unified vibration analysis [46]. Nevertheless, although lots of virtues the Rayleigh-Ritz procedure held, it is still insufficient to deal with general boundary conditions owing to the used polynomial displacement function, which can hardly form a complete solution set or obtain stable results due to computer rounding errors [47, 48]. Unfortunately, to the best knowledge of the authors, the corresponding topic on displacement functions is very limited. Recently, Li [49] derived a complementary function and applied it to Fourier cosine series to express the dynamic behavior of beams, which solved the transverse vibration problem with elastic constraints. Zhang [50] proposed an improved Fourier cosine method in which the displacement function was assumed as the superposition of Fourier cosine series and supplementary functions. Dai [51] combined displacement functions of plate's transverse vibration and in-plane vibration to adapt to the vibration analysis of thin cylindrical shells and solved the problems of cylindrical shell's vibration by treating the both kinematic equations and boundary condition equations simultaneously. In this case, in view of the major benefits of only needs the kinetic energy and potential energy calculations rather than the specific stress variation calculations, the improved Fourier cosine series and its supplementary functions are deemed suitable as admissible displacement function in Rayleigh-Ritz procedure [52–54].

The objective of this paper is to develop a unified vibration analysis method for stiffened cylindrical shells with general boundary conditions. The first shear deformation theory is employed to formulate the theoretical model, the improved Fourier cosine series and two supplementary functions are adopted as admissible displacement function, and the corresponding Rayleigh-Ritz procedure is performed to obtain the exact vibration behaviors of stiffened cylindrical shells. Without remodeling and corresponding procedures, the unified vibration analysis can be efficiently performed and the corresponding accurately vibration behaviors are obtained by only assigning the spring stiffness values of general boundary conditions. The convergence and accuracy of the developed method are validated by numerous examples in respect of frequency comparisons and modal comparisons.

The remainder of this paper is structured as follows. Based on the first-order shear deformation theory and the Rayleigh-Ritz procedure, the essential methodology of the unified vibration analysis method for stiffened cylindrical shells with general boundary conditions is investigated in Section 2. In Section 3, the proposed method is validated by comparing its free vibration analysis results with that of finite element method and several state-of-the-art methods. Some conclusions on this study are discussed and summarized in Section 4.

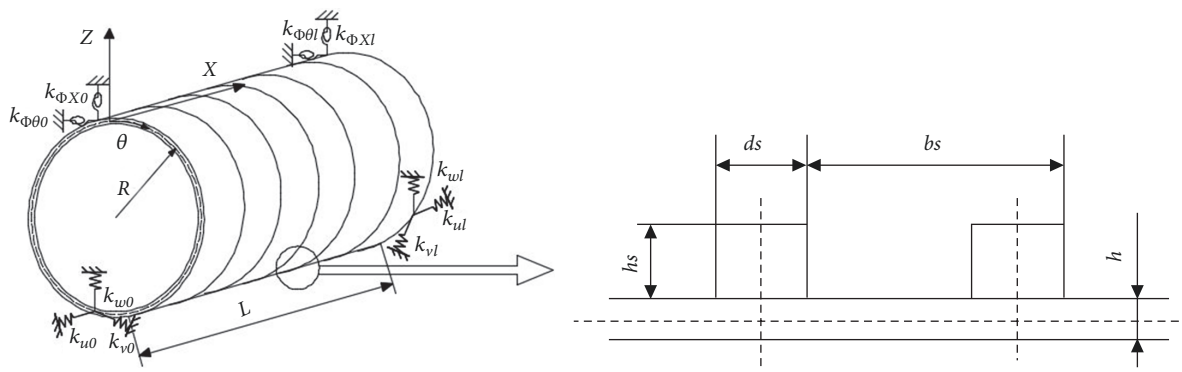

Figure 1: Schematic diagram of stiffened cylindrical shell.

2. Theoretical Formulations

2.1. Description of Stiffened Cylindrical Shell. The schematic diagram of stiffened cylindrical shell is shown in Figure 1. The basic parameters of stiffened cylindrical shell involve length L, radius R, thickness h, width of two adjacent stiffeners bs, and thickness of stiffeners hs. To analyze the vibration characteristics of stiffened cylindrical shell, the cylindrical coordinate system (x, θ, z) is built on the left-edge middle surface; herein, x represents the axis direction of the cylindrical shell; θ and z indicate the circumferential and radial directions of cylindrical shell. Moreover, to simulate various boundary conditions, the equivalent linear springs k_u, k_v, and k_w and torsion springs $k_{\phi x}$ and $k_{\phi\theta}$ are set at both edges of stiffened cylindrical shell.

2.2. Displacement Field and Geometric Equation. To describe and quantify the vibration behaviors of moderately thick shell, we construct the FSDT dynamic model by considering transverse shear deformation. Assuming that the transverse normal would not remain vertical and mid-plane after shell deformation, the displacement field is established as

$$\begin{aligned} u(x,\theta,z,t) &= u_0(x,\theta,t) + z\phi_x(x,\theta,t) \\ v(x,\theta,z,t) &= v_0(x,\theta,t) + z\phi_\theta(x,\theta,t) \\ w(x,\theta,z,t) &= w_0(x,\theta,t) \end{aligned} \tag{1}$$

where u_0, v_0, and w_0 indicate middle surface displacements along the x-axis, θ-axis, and z-axis, respectively; ϕ_x and ϕ_θ indicate the rotation angle along the θ-axis and x-axis, respectively; t indicates the time variable.

In view of the von Karman geometric nonlinearity rule, the geometric equation is built as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \gamma_{x\theta} \\ \gamma_{xz} \\ \gamma_{\theta z} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{\theta\theta}^{(0)} \\ \gamma_{x\theta}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{\theta z}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{\theta\theta}^{(1)} \\ \gamma_{x\theta}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{\theta z}^{(1)} \end{Bmatrix} \tag{2}$$

where

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{\theta\theta}^{(0)} \\ \gamma_{x\theta}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{\theta z}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{R\partial\theta} + \dfrac{w_0}{R} \\ \dfrac{\partial u_0}{R\partial\theta} + \dfrac{\partial v_0}{\partial x} \\ \phi_x + \dfrac{\partial w_0}{\partial x} \\ \phi_\theta + \dfrac{\partial w_0}{R\partial\theta} \end{Bmatrix},$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{\theta\theta}^{(1)} \\ \gamma_{x\theta}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{\theta z}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_\theta}{R\partial\theta} \\ \dfrac{\partial \phi_x}{R\partial\theta} + \dfrac{\partial \phi_\theta}{\partial x} \\ 0 \\ 0 \end{Bmatrix} \tag{3}$$

2.3. Constitutive Relation and Internal Force Relation. In light of the general Hook's law, to further reveal the stress behavior in terms of displacement, the constitutive relation of stiffened cylindrical shell is indicated as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \tau_{x\theta} \\ \tau_{xz} \\ \tau_{\theta z} \end{Bmatrix} = \begin{Bmatrix} Q_{11} & Q_{12} & & & \\ Q_{12} & Q_{22} & & & \\ & & Q_{44} & & \\ & & & Q_{55} & \\ & & & & Q_{66} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \gamma_{x\theta} \\ \gamma_{xz} \\ \gamma_{\theta z} \end{Bmatrix} \tag{4}$$

where elastic constants Q_{ij} is denotes as

$$\begin{aligned} Q_{11} &= Q_{22} = \frac{E}{1-\nu^2}, \\ Q_{12} &= \nu\frac{E}{1-\nu^2}, \\ Q_{44} &= Q_{55} = Q_{66} = \frac{E}{2(1+\nu)} \end{aligned} \tag{5}$$

Assuming that the stiffeners are of the same material with shell structures, the stress-strain relationship of stiffeners can be expressed as

$$\sigma_{\theta\theta}^{s} = E^{s}\varepsilon_{\theta\theta} \tag{6}$$

where E^s is Young's modulus of stiffener.

By integrating the in-plane stress in thickness and the stress in cross section, the relationship between force and moment, strain, and curvature on the middle plane is indicated as

$$\begin{Bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ Q_x \\ Q_\theta \end{Bmatrix} = \begin{Bmatrix} A_{11} & A_{12} & & & \\ A_{12} & A_{22} & & & \\ & & A_{33} & & \\ & & & kA_{44} & \\ & & & & kA_{55} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{0} \\ \varepsilon_{\theta\theta}^{0} \\ \varepsilon_{x\theta}^{0} \\ \varepsilon_{xz}^{0} \\ \varepsilon_{\theta z}^{0} \end{Bmatrix} + \begin{Bmatrix} B_{11} & B_{12} & & & \\ B_{12} & B_{22} & & & \\ & & B_{33} & & \\ & & & B_{44} & \\ & & & & B_{55} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{1} \\ \varepsilon_{\theta\theta}^{1} \\ \varepsilon_{x\theta}^{1} \\ \varepsilon_{xz}^{1} \\ \varepsilon_{\theta z}^{1} \end{Bmatrix} \tag{7}$$

$$\begin{Bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \end{Bmatrix} = \begin{Bmatrix} B_{11} & B_{12} & \\ B_{12} & B_{22} & \\ & & B_{33} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{0} \\ \varepsilon_{\theta\theta}^{0} \\ \varepsilon_{x\theta}^{0} \end{Bmatrix} + \begin{Bmatrix} D_{11} & D_{12} & \\ D_{12} & D_{22} & \\ & & D_{33} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{1} \\ \varepsilon_{\theta\theta}^{1} \\ \varepsilon_{x\theta}^{1} \end{Bmatrix}$$

$$N_{\theta\theta}^{s} = A_s\varepsilon_{\theta\theta}^{0} + B_s\varepsilon_{\theta\theta}^{1},$$

$$M_{\theta\theta}^{s} = B_s\varepsilon_{\theta\theta}^{0} + D_s\varepsilon_{\theta\theta}^{1}$$

where N_{xx}, $N_{\theta\theta}$, and $N_{x\theta}$ represent the normal force in x-direction, θ-direction, and shear force resultants, respectively; M_{xx}, $M_{\theta\theta}$, and $M_{x\theta}$ represent the bending moment in x-direction, θ-direction, and twisting moment resultants, respectively; $N_{\theta\theta}^{s}$ and $M_{\theta\theta}^{s}$ represent the membrane stress and moment respect to stiffens, respectively; k represents the shear correction factor used to correct the discrepancy between the constant stress state and real stress state, which is supposed as 5/6 in this paper. Moreover, the coefficients A_{ij}, B_{ij}, D_{ij}, A_s, B_s, and D_s can be further expressed as

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{-h/2}^{h/2} Q_{ij}\left(1, z, z^2\right) dz \tag{8}$$

$$\left(A_s, B_s, D_s\right) = \int_{h/2}^{h/2+h_s} E^s \frac{d_s}{b_s}\left(1, z, z^2\right) dz \tag{9}$$

It should be noted that, for a symmetric cylindrical shell, B_{ij}=0, the bending and stretching vibration of the shell are uncoupled. Moreover, for the stiffeners distributed in shell, $B_s \neq 0$, the calculation burdens have increased dramatically. Thus, the general boundary conditions of the elastically restrained shell can be described as follows:

At left edge (x=0):

$$\begin{aligned} k_{u_0}u_0 &= N_{xx}, \\ k_{v_0}v_0 &= N_{x\theta} + \frac{M_{x\theta}}{R}, \\ k_{w_0}w_0 &= Q_x + \frac{\partial M_{x\theta}}{R\partial\theta} \\ k_{\phi x_0}\phi_x &= -M_{xx}, \\ k_{\phi\theta_0}\phi_\theta &= -M_{x\theta} \end{aligned} \tag{10}$$

At right edge (x=L):

$$\begin{aligned} k_{u_l}u_0 &= -N_{xx}, \\ k_{v_l}v_0 &= -N_{x\theta} - \frac{M_{x\theta}}{R}, \\ k_{wl}w_0 &= -Q_x - \frac{\partial M_{x\theta}}{R\partial\theta} \\ k_{\phi x_l}\phi_x &= M_{xx}, \\ k_{\phi\theta_l}\phi_\theta &= M_{x\theta} \end{aligned} \tag{11}$$

To this extent, all the needed parts of the first-order shear deformation theory (FSDT) are present, and they may be combined to obtain the desired form of energy expressions.

2.4. Energy Expressions. In view of the efficiency and reliability of the results in modeling and solution procedure, the energy-oriented Rayleigh-Ritz method is employed in the present work. To define the energy expressions of the stiffened cylindrical shell, we assume that the spring retains a certain stiffness without considering mass; the total kinetic energy T of the stiffened cylindrical shell is consisting of the kinetic energy of cylindrical shell T_1 and the kinetic energy of stiffener T_2. In light of kinetic energy theory, the total kinetic energy of stiffened cylindrical shell can be described as

$$\begin{aligned} T &= T_1 + T_2 \\ &= \frac{1}{2}\left[\int_A \int_{-h/2}^{h/2} \rho_0 \left(\left(\dot{u}_0 + z\dot{\phi}_x\right)^2 + \left(\dot{v}_0 + z\dot{\phi}_\theta\right)^2 + \dot{w}_0{}^2\right) dzRd\theta dx + \int_A \int_{h/2}^{h/2+h_s} \frac{d_s}{b_s}\rho_s \left(\left(\dot{u}_0 + z\dot{\phi}_x\right)^2 + \left(\dot{v}_0 + z\dot{\phi}_\theta\right)^2 + \dot{w}_0{}^2\right) dzRd\theta dx\right] \\ &= \frac{1}{2}\int_A \left[I_0\left(\dot{u}_0{}^2 + \dot{v}_0{}^2 + \dot{w}_0{}^2\right) + 2I_1\left(\dot{\phi}_x\dot{u}_0 + \dot{\phi}_\theta\dot{v}_0\right) + I_2\left(\dot{\phi}_x{}^2 + \dot{\phi}_\theta{}^2\right)\right] Rd\theta dx \end{aligned} \tag{12}$$

where the ρ_0 and ρ_s represent the density of cylindrical shell and stiffener, respectively. The inertia moment I_i (i= 0, 1, 2) can be further represented as

$$[I_0, I_1, I_2] = \left(\int_{-h/2}^{h/2} \rho_0 \left[1, z, z^2\right] dz + \int_{h/2}^{h/2+hs} \frac{ds}{bs} \rho_s \left[1, z, z^2\right] dz \right) \tag{13}$$

Similarly, by integrating the strain potential energy P_1 induced by deformation and the elastic potential energy P_2 induced by springs, the potential energy P of stiffened cylindrical shell is expressed as

$$P = P_1 + P_2 \tag{14}$$

where the strain potential energy P_1 is

$$\begin{aligned} P_1 &= \frac{1}{2} \int_V \{\sigma_{xx}\varepsilon_{xx} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{x\theta}\varepsilon_{x\theta} + \sigma_{xz}\varepsilon_{xz} \\ &\quad + \sigma_{\theta z}\varepsilon_{\theta z}\} dV + \frac{1}{2} \int_{V_s} \sigma_{xx}^s \varepsilon_{xx} dV_s \\ &= \int_A \left(N_{xx}\varepsilon_{xx}^0 + M_{xx}\varepsilon_{xx}^1 + \left(N_{\theta\theta} + N_{\theta\theta}^s\right)\varepsilon_{\theta\theta}^0 \right. \\ &\quad + \left(M_{\theta\theta} + M_{\theta\theta}^s\right)\varepsilon_{\theta\theta}^1 + N_{x\theta}\gamma_{x\theta}^0 + M_{x\theta}\gamma_{x\theta}^1 + N_{xz}\gamma_{xz}^0 \\ &\quad \left. + M_{xz}\gamma_{xz}^1 + N_{\theta z}\gamma_{\theta z}^0 + M_{\theta z}\gamma_{\theta z}^1 \right) Rd\theta dx = \frac{1}{2} \\ &\quad \cdot \int_A \left[N_{xx}\left(\frac{\partial u_0}{\partial x}\right) + M_{xx}\frac{\partial \phi_x}{\partial x} \right. \\ &\quad + \left(N_{\theta\theta} + N_{\theta\theta}^s\right)\left(\frac{\partial v_0}{R\partial\theta} + \frac{w_0}{R}\right) + \left(M_{\theta\theta} + M_{\theta\theta}^s\right)\frac{\partial \phi_\theta}{R\partial\theta} \\ &\quad + N_{x\theta}\left(\frac{\partial u_0}{R\partial\theta} + \frac{\partial v_0}{\partial x}\right) + M_{x\theta}\left(\frac{\partial \phi_x}{R\partial\theta} + \frac{\partial \phi_\theta}{\partial x}\right) \\ &\quad \left. + Q_x\left(\phi_x + \frac{\partial w_0}{\partial x}\right) + Q_\theta\left(\phi_\theta + \frac{\partial w_0}{R\partial\theta}\right) \right] Rd\theta dx \\ &= \frac{1}{2} \int_A \left[N_{xx}\left(\frac{\partial u_0}{\partial x}\right) + M_{xx}\frac{\partial \phi_x}{\partial x} \right. \\ &\quad + N_\theta\left(\frac{\partial v_0}{R\partial\theta} + \frac{w_0}{R}\right) + M_\theta\frac{\partial \phi_\theta}{R\partial\theta} \\ &\quad + N_{x\theta}\left(\frac{\partial u_0}{R\partial\theta} + \frac{\partial v_0}{\partial x}\right) + M_{x\theta}\left(\frac{\partial \phi_x}{R\partial\theta} + \frac{\partial \phi_\theta}{\partial x}\right) \\ &\quad \left. + Q_x\left(\phi_x + \frac{\partial w_0}{\partial x}\right) + Q_\theta\left(\phi_\theta + \frac{\partial w_0}{R\partial\theta}\right) \right] Rd\theta dx \end{aligned} \tag{15}$$

in which

$$\begin{aligned} N_\theta &= N_{\theta\theta} + N_{\theta\theta}^s, \\ M_\theta &= M_{\theta\theta} + M_{\theta\theta}^s \end{aligned} \tag{16}$$

Besides, the elastic potential energy P_2 at the boundary is obtained as

$$\begin{aligned} P_2 &= \left(\frac{1}{2} \int \left\{ k_{u_0} {u_0}^2 + k_{v_0} {v_0}^2 + k_{w_0} {w_0}^2 + k_{\phi x_0} {\phi_x}^2 \right. \right. \\ &\quad \left. \left. + k_{\phi\theta_0} {\phi_\theta}^2 \right\} Rd\theta \right)_{x=0} + \left(\frac{1}{2} \int \left\{ k_{u_l} {u_0}^2 + k_{v_l} {v_0}^2 \right. \right. \\ &\quad \left. \left. + k_{w_l} {w_0}^2 + k_{\phi x_l} {\phi_x}^2 + k_{\phi\theta_l} {\phi_\theta}^2 \right\} Rd\theta \right)_{x=L} \end{aligned} \tag{17}$$

2.5. Admissible Displacement Functions and Deriving Solutions. To derive the discrete motion equation in the Rayleigh-Ritz procedure, the displacement and rotation components should be expanded with one admissible displacement function. For shell problem, the most commonly used form of admissible function is orthogonal polynomial and Fourier series. However, the low-order polynomials cannot form a complete solution set while the high-order polynomials tend to be numerically unstable due to computer rounding errors, which may lead to very tedious calculations and may be inundated with various boundary conditions. Although the Fourier series constitutes a complete solution set and holds good numerical stability by the Fourier series expansion, the convergence difficulties and low solution accuracy problems are still emerged in vibration analysis with general boundary condition. In this paper, to overcome the difficulty and satisfy the general boundary conditions, by using two supplementary functions, the displacement and rotation components of the middle surface of stiffened cylindrical shells are expanded into an improved form of Fourier cosine series

$$\begin{aligned} u(x,\theta,t) &= \left[\sum_{m=0}^{M} \sum_{n=0}^{N} U_{mn} \cos\lambda_m x \cos(n\theta) \right. \\ &\quad \left. + \sum_{l=1}^{2} \sum_{n=0}^{N} a_{\ln} \zeta_l(x) \cos(n\theta) \right] e^{i\omega t} \\ v(x,\theta,t) &= \left[\sum_{m=0}^{M} \sum_{n=0}^{N} V_{mn} \cos\lambda_m x \sin(n\theta) \right. \\ &\quad \left. + \sum_{l=1}^{2} \sum_{n=0}^{N} b_{\ln} \zeta_l(x) \sin(n\theta) \right] e^{i\omega t} \\ w(x,\theta,t) &= \left[\sum_{m=0}^{M} \sum_{n=0}^{N} W_{mn} \cos\lambda_m x \cos(n\theta) \right. \\ &\quad \left. + \sum_{l=1}^{2} \sum_{n=0}^{N} c_{\ln} \zeta_l(x) \cos(n\theta) \right] e^{i\omega t} \\ \phi_x(x,\theta,t) &= \left[\sum_{m=0}^{M} \sum_{n=0}^{N} \Psi_{xmn} \cos\lambda_m x \cos(n\theta) \right. \\ &\quad \left. + \sum_{l=1}^{2} \sum_{n=0}^{N} d_{\ln} \zeta_l(x) \cos(n\theta) \right] e^{i\omega t} \end{aligned}$$

$$\phi_\theta(x,\theta,t) = \left[\sum_{m=0}^{M}\sum_{n=0}^{N}\Psi_{\theta mn}\cos\lambda_m x\sin(n\theta) + \sum_{l=1}^{2}\sum_{n=0}^{N}e_{\ln}\zeta_l(x)\sin(n\theta)\right]e^{i\omega t} \tag{18}$$

where m and n indicate the axial wave number and circumferential wave number; λ_m indicates the nondimensional parameter related to the axial wavenumber whose value is $m\pi/L$; U_{mn}, V_{mn}, W_{mn}, Ψ_{xmn}, and $\Psi_{\theta mn}$ indicate the Fourier expansion coefficients; a_{ln}, b_{ln}, c_{ln}, d_{ln}, and e_{ln} the auxiliary function coefficients. The supplementary function $\zeta_l(x)$ can be indicated as

$$\zeta_1(x) = x\left(\frac{x}{L}-1\right)^2, \qquad \zeta_2(x) = \frac{x^2}{L}\left(\frac{x}{L}-1\right) \tag{19}$$

The uniformly convergent series expansions can be obtained by differentiating operation of series expansion. In fact, the solution holds arbitrary precision since the series expression has to be truncated numerically. In general, to obtain acceptably accurate solution, the infinite series expression is truncated to M and N in actual calculations.

Once the admissible displacement functions and energy functions of the stiffened cylindrical shell are established, the following tasks are to determine the coefficients in the admissible functions. In light of (12) and (14), the Lagrange energy function L can be indicated in terms of kinetic energy and potential energy of the stiffened cylindrical shell as

$$L = T - P \tag{20}$$

In view of energy-oriented Rayleigh-Ritz procedure, the total expression of the Lagrangian energy function is minimized with respect to the undetermined coefficients

$$\frac{\partial L}{\partial \alpha} = 0, \qquad \alpha = U_{mn}, V_{mn}, W_{mn}, \Psi_{xmn}, \Psi_{\theta mn}, a_{ln}, b_{ln}, c_{ln}, d_{ln}, e_{ln} \tag{21}$$

Assuming that the vector of unknown coefficients are $X = [X^u, X^v, X^w, X^{\phi x}, X^{\phi\theta}]^{\mathrm{T}}$, the corresponding $(M+3)\times(5N+3)$ coefficient equations can be retrieved as

$$\left(\boldsymbol{K} - \omega^2 M\right)\boldsymbol{X} = 0 \tag{22}$$

where

$$\begin{aligned}
\boldsymbol{X}^u &= [U_{00},\cdots,U_{mn},a_{10},\cdots,a_{1n},a_{20},\cdots a_{2n}]\\
\boldsymbol{X}^v &= [V_{00},\cdots,V_{mn},b_{10},\cdots,b_{1n},b_{20},\cdots b_{2n}]\\
\boldsymbol{X}^w &= [W_{00},\cdots,W_{mn},c_{10},\cdots,c_{1n},c_{20},\cdots c_{2n}]\\
\boldsymbol{X}^{\phi x} &= [\Psi_{x00},\cdots,\Psi_{xmn},d_{10},\cdots,d_{1n},d_{20},\cdots d_{2n}]\\
\boldsymbol{X}^{\phi\theta} &= [\Psi_{\theta 00},\cdots,\Psi_{\theta mn},e_{10},\cdots,e_{1n},e_{20},\cdots e_{2n}]
\end{aligned} \tag{23}$$

The natural frequencies and modes can be determined readily by solving the characteristic equation (22). Each column of the eigenvector matrix is the set of coefficients for the corresponding modes. By substituting the solution X from (23) into (18), the corresponding mode shapes can be retrieved accordingly.

3. Numerical Study

To verify the feasibility and validity of the presented approach, we compare the analysis results with several state-of-the-art methods from multiple perspectives, including the convergence analysis performed in Section 3.1, the frequency comparisons illustrated in Section 3.2, and modal comparisons depicted in Section 3.3. Besides, we summarize the detailed discussions in Section 3.4.

3.1. Convergence Analysis. To verify the convergence of this method, the natural frequencies under elastic boundary conditions are calculated and compared with the finite element method. Axial direction is set as elastic and the stiffness coefficients are set as 10^5 while others are set as infinite large. The geometrical and material parameters of cylindrical shells are set as L=39.45×10^{-2} m, R=4.9759×10^{-2} m, h=0.1651×10^{-2} m, ρ_0=2762 kg/m^3, μ=0.3, E=68.95 GPa, ds=0.3175×10^{-2}m, hs=0.5334×10^{-2} m, bs=1.9725×10^{-2} m, and N=5. The calculation results are shown in Table 1; the data reveal good convergence and the results between present method and FEM fit well. It is clarified that all computations are performed on an Intel (R) Pentium (R) Desktop Computer (CPU G3260 3.30 GHz and 4.00 GB RAM). It is obvious that the present solution results hold excellent convergence and possesses high computing efficiency. For instance, the modes can be obtained within 0.5 s even the series are truncated as M=15.

3.2. Frequency Comparisons. In this subsection, to validate the feasibility and effectiveness of the presented method, the free vibration analysis of stiffened cylindrical shell with general boundary conditions is performed by the presented method, and the obtained natural frequencies are compared with several state-of-the-art methods firstly. By gained confidence in the present method, we further investigate the effects of stiffeners or not on the natural frequencies. After that we the contribution of first-order shear deformation theory (FSDT) on moderate-thickness stiffened cylindrical shells in contrast with classic shear deformation theory (CSDT).

Classic boundary support can be considered as special cases of elastic boundary conditions. Different types of classic boundary support can be readily generated by adjusting the stiffness values of the translational springs and rotational springs which are uniformly distributed along the ends of cylindrical shell. Although we can gain the exact solutions for stiffened cylindrical shell with general boundary conditions, to facilitate the method comparisons, in this paper we select three typical boundary conditions that are frequently encountered in engineering practices: free-free (F-F), simple-simple (S-S), and clamped-clamped (C-C). Taking edge x=0

Table 1: Comparison of natural frequencies of stiffened cylindrical shells with free boundaries.

Truncated	Modes								Time consuming, s
	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	
M=5	1290.34	1661.19	2298.11	3612.72	3693.16	4455.88	4472.75	4756.00	0.26357
M=7	1261.44	1645.61	2252.24	3578.23	3587.75	4454.88	4460.87	4715.33	0.34305
M=9	1244.95	1635.48	2226.08	3533.22	3560.78	4453.71	4453.94	4692.21	0.37890
M=11	1235.15	1628.70	2209.30	3500.31	3550.77	4448.73	4453.12	4676.70	0.39036
M=13	1225.68	1621.08	2189.77	3465.19	3540.46	4443.05	4451.44	4658.80	0.41890
M=15	1225.44	1620.93	2189.49	3465.14	3540.21	4442.22	4451.90	4657.96	0.45036

Table 2: Comparison of natural frequencies of stiffened cylindrical shells with simple support boundaries.

n	m=1				m=2				m=3			
	PRM	WPM	FEM	present	PRM	WPM	FEM	present	PRM	WPM	FEM	present
1	1199	1216	1244	1223.49	3493	3536	3634	3435	5839	5907	6080	5943
2	1564	1635	1642	1596.01	2113	2176	2202	21535	3378	3430	3190	3555
3	4387	4578	4015	4356.80	4400	4573	4109	4439	4595	4788	4428	4527
4	8377	8781	7409	8105.72	8392	8731	7221	8468	8449	8728	8229	8538
5	13490	14172	13143	12979	13508	14119	13356	1387	13555	14069	13589	14389

as an example, the corresponding spring stiffness of three classical boundary conditions are introduced as follows:

$$\begin{aligned} &\text{(F-F)} \quad k_{u0} = k_{v0} = k_{w0} = k_{\phi x0} = k_{\phi\theta 0} = 0 \\ &\text{(S-S)} \quad k_{u0} = k_{v0} = k_{w0} = 10^{15} \\ &\qquad\quad\; k_{\phi x0} = k_{\phi\theta 0} = 0 \\ &\text{(C-C)} \quad k_{u0} = k_{v0} = k_{w0} = k_{\phi x0} = k_{\phi\theta 0} = 10^{15} \end{aligned} \tag{24}$$

At first, regarding the geometrical and material parameters as L=39.45×10^{-2} m, R=4.9759×10^{-2} m, h=0.1651×10^{-2} m, ρ_0=2762 kg/m^3, μ=0.3, E=68.95 GPa, M=15, N=5, ds=0.3175×10^{-2}m, hs=0.5334×10^{-2} m, bs=1.9725×10^{-2} m, axial wave number m=1, 2, 3, and circumference wave number n=1, 2, 3, 4, 5, the natural frequencies of cylindrical shell with S-S boundary conditions are studied. The frequency analysis results of polynomial Ritz method (PRM) [25], wave propagation method (WPM) [27], finite element method (FEM), and the present method are shown in Table 2. Then, with the F-F boundary condition and the geometrical and material parameters L=47.09×10^{-2} m, R=10.37×10^{-2} m, h=0.119×10^{-2} m, ρ_0=7700 kg/m^3, μ=0.3, E=2.06 GPa, M=15, N=5, ds=0.218×10^{-2} m, and bs=3.14×10^{-2} m, through introducing dimensionless parameters $\Omega = \omega \times R[\rho(1\text{-}\mu^2)/E]^{0.5}$, the corresponding analysis results are obtained by the present method are compared with that of Galletly Ritz method (GRM) [32], PRM [26], and FEM, which are illustrated in Table 3. Finally, to validate the generality of the presented method in elastic boundary conditions, we compare the presented method with a state-of-the-art Galerkin method (GM) [34]; the comparison results are shown in Tables 4–6. Herein, the elastic boundary conditions are simulated by F-F, S-S and C-C boundary conditions, and the geometrical and material parameters are chosen as L=0.502 m, R=0.0635 m, h=0.00163 m, ρ_0=7800 kg/m^3, μ=0.28, E=2.1e+11 N/m^2, M=15, N=5, ds=0.01 m, hs=0.003 m, and bs=L/7. From the above tables, it is obvious that the current solutions are consistent with that of referential methods, which validates the feasibility and effectiveness of the present method.

Table 3: Comparison of natural frequencies of stiffened cylindrical shells with free boundaries with m=1.

n	GRM	PRM	FEM	present
1	0.1997	0.1983	0.2065	0.2064
2	0.0851	0.0851	0.0895	0.0895
3	0.0689	0.0704	0.0742	0.0744
4	0.1092	0.1147	0.1123	0.1157
5	0.1728	0.1836	0.1569	0.1656

Table 4: Comparison of natural frequencies of cylindrical shells with clamped boundaries.

n	M=15, N=5, k=10^{15}		
	FEM	GM	present
1	983.34	988.32	980.25
2	1718.24	1717.66	1715.78
3	1736.55	1738.45	1734.99
4	1947.46	1949.45	1942.42

Having gained confidence in the present method, considering the geometrical and material parameters as L=0.502 m, R=0.0635 m, h=0.00163 m, ρ_0=7800 kg/m^3, μ=0.28, E=2.1e+11 N/m^2, M=15, N=5, ds=0.01 m, hs=0.003 m, and bs=L/7, we analyze the effects of stiffeners on the natural frequencies of cylindrical shells. Therein, by setting stiffener thickness hs as 0 in (13), the stiffened dynamics model is degraded as an unstiffened dynamics model, and the natural frequencies of unstiffened cylindrical shell are obtained. The frequency differences between cylinders with and without stiffening are

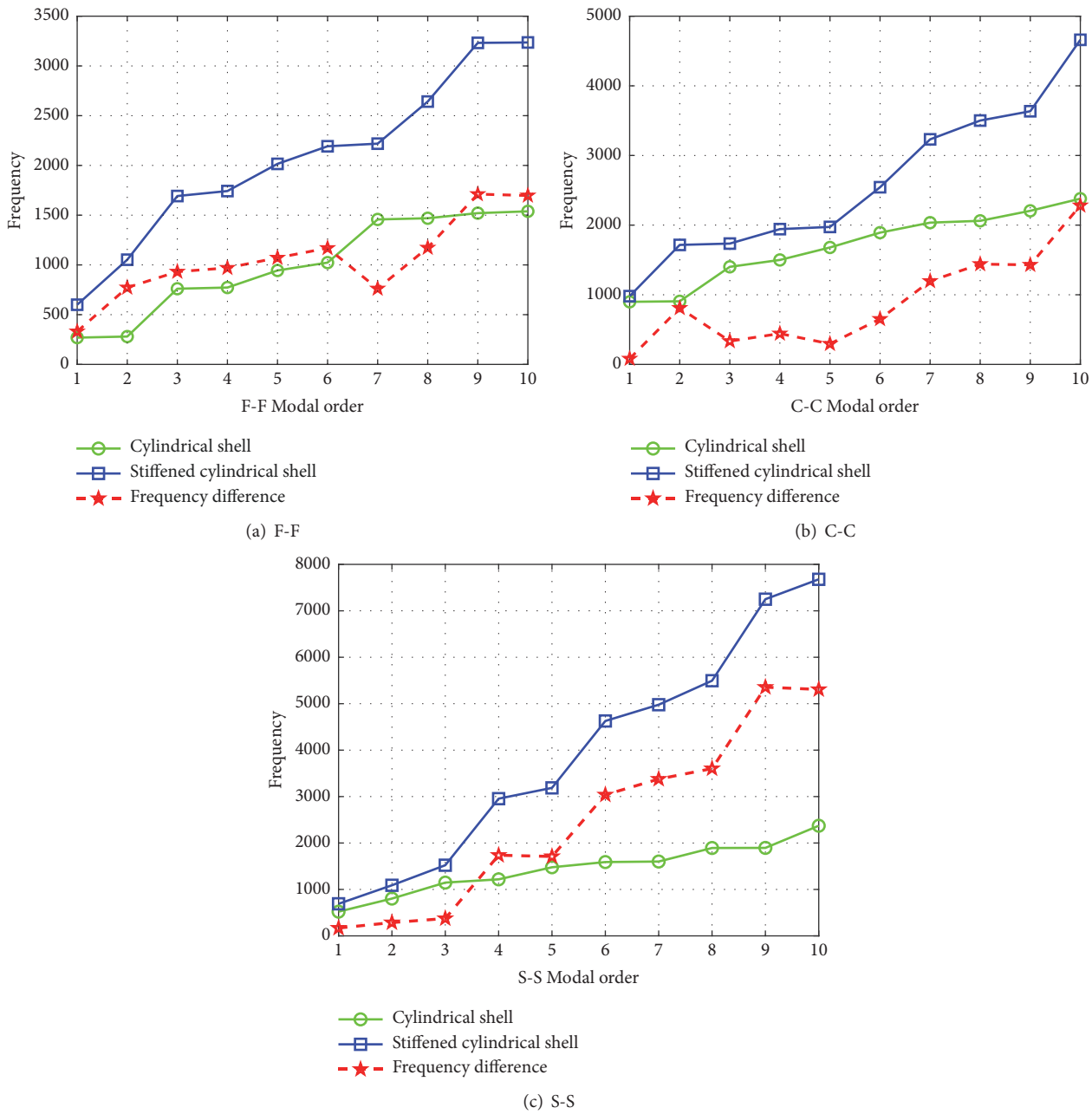

Figure 2: Frequency differences of cylinders with and without stiffening.

Table 5: Comparison of natural frequencies of cylindrical shells with free boundaries.

n	M=15, N=5, k=0		
	FEM	GM	present
0	598.17	595.34	599.84
1	625.00	623.45	624.24
2	662.26	661.42	663.69
3	1098.20	1096.37	1099.48

shown in Figure 2. From the table, we can see that the natural frequencies have a distinct increase by adding stiffeners. To further investigate the stiffener thickness variation effects on natural frequencies, we further investigate the vibration characteristics of stiffened cylindrical shell under three stiffener thickness values hs= 0.291 m, 0.582 m, and 0.873 m. The solution results of WPM [27], modified variational method (MVM) [55], FEM, and the presented method have been presented in Table 7. In this case, the geometrical and material parameters of cylindrical shells are L=47.09×10^{-2} m, R=10.37×10^{-2} m, h=0.119×10^{-2} m, ρ_0=7700 kg/m^3, μ=0.3, E=2.06 GPa, M=15, N=5, ds=0.218×10^{-2} m, and bs=3.14×10^{-2} m. It is obvious that the present solutions have excellent agreements with the results of several literatures. Besides, the natural frequencies of stiffened cylindrical shell are

Table 6: Comparison of natural frequencies (HZ) of cylindrical shells with simply supported boundaries.

n	m=1			m=2		
	FEM	GM	present	FEM	GM	present
1	1147.0	1143.5	1145.4	1257.7	1253.3	1255.5
2	1021.0	1019.5	1020.2	1681.8	1678.6	1679.2
3	1960.5	1958.3	1959.3	2214.5	2213.6	2212.1
4	3440.5	3435.6	3438.9	3795.2	3792.9	3794.3

Table 7: Comparison of natural frequencies by different methods.

n	hs=0.291				hs=0.582				hs=0.873			
	MVM	WPM	FEM	present	MVM	WPM	FEM	present	MVM	WPM	FEM	present
2	4550	4470	4559	4558.44	4580	4450	4660	4649.62	5040	4885	5432	5098.44
3	3870	3655	3897	3790.55	6710	6235	6778	6370.77	10330	9500	9564	9842.43
4	6550	5950	6435	6443.51	12830	11790	10098	11345.66	20200	18010	19797	18902.30
5	10000	9510	9667	9559.39	20120	19020	18890	19554.33	31800	25570	29905	30222.30

Table 8: Comparison of natural frequencies of stiffened cylindrical shells with clamped boundaries h= 0.00326 m.

n	CSDT	FEM	present
1	1095.4	1099.4	1098.8
2	1104.1	1105.9	1106.4
3	1792.3	1793.8	1794.3
4	2076.3	2079.9	2078.4
5	2434.7	2436.3	2437.1

Table 9: Comparison of natural frequencies of stiffened cylindrical shells with clamped boundaries h=0.005 m.

n	CSDT	FEM	present
1	1299.6	1301.6	1302.3
2	1314.2	1316.4	1317.5
3	1820.1	1822.5	1823.1
4	2254.9	2256.2	2257.0
5	2263.2	2266.3	2267.4

increasing along with the heightening of thickness value and vice versa.

It is noted that all the numerical cases above are based on FSDT, the analysis results which can hardly show the significant difference with CSDT, since thin stiffened shells are considered in those examples. In this case, to show the contribution of FSDT, we present the examples of moderately thick stiffened cylindrical shells. The natural frequencies with larger h parameters are presented in Tables 8 and 9. The geometrical and material parameters of cylindrical shells are L=0.502 m, R=0.0635 m, ρ_0=7800 kg/m^3, μ=0.28, E=2.1e+11 N/m^2, M=15, N=5, ds=0.01 m, hs=0.003 m, and bs=L/7. When the thickness increases, the solution based on FSDT is more accurate than CSDT. It is obvious to see that the obtained natural frequency based on FSDT, especially the higher-order frequency, is more accurate than that of CSDT.

It is noted that the partial data in Tables 2–6, 8, and 9 are reproduced from the reference [42] (under the Creative Commons Attribution License/public domain).

3.3. Modal Comparisons. To verify the validity of the developed method on mode characteristics in free vibration analysis, we compare the mode shapes obtained by the presented method with that of FEM. Fourth-order mode shapes with three types of boundary conditions are presented. The comparisons of two approaches with the F-F, S-S, and C-C boundary conditions are depicted in Figures 3–5, respectively. It is noted that the right columns are the results obtained by our methods and the left columns are by FEM method in all subfigures. From these figures, the vibration behaviors of the stiffened cylindrical shell can be inspected straightforwardly and an excellent consistency of FEM and the presented method can be concluded.

3.4. Discussions. From the above convergence analysis, frequency comparisons, and modal comparisons, the feasibility and effectiveness of the proposed unified approach have been validated in the vibration analysis of stiffened cylindrical shell with general boundary conditions. Some discussions and findings from the vibration analysis are summarized as follows.

(1) As shown in Table 1, it is obvious that the presented approach solution holds a high calculation efficiency and shows a great agreement with FEM. The excellent convergence of the presented method is attributed to the following key factors: (i) unlike the high-order shear deformation theory (HSDT) in theoretical formulation, the first-order shear deformation theory (FSDT) we adopted in this paper avoids redundant boundary condition terms and complex high-order integral calculations, which is conducive to simplify the computational tasks in theoretical modeling process; (ii) a unified approach combining the modified Fourier series and the Rayleigh-Ritz method only calculates the kinetic energy and potential energy in the motion process rather than the specific stress variation calculations; the infinite degree-of-freedom system is transformed into a generalized

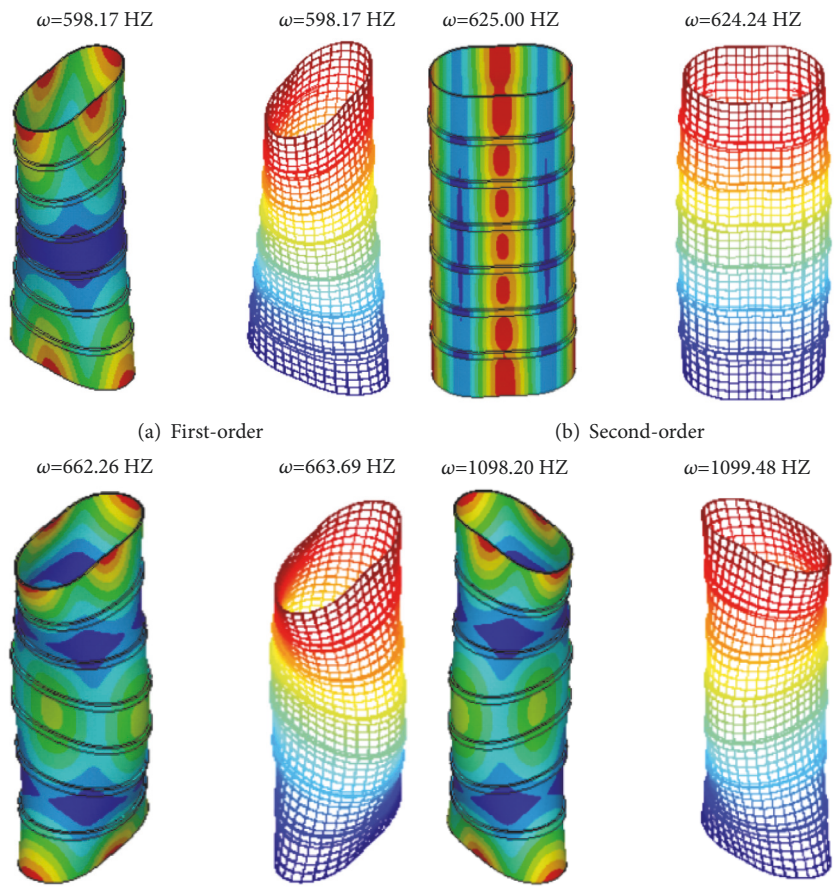

Figure 3: First four modes with free boundary.

multidegree-of-freedom problem by the basis function, thus reducing the computational workloads. Therefore, the proposed method combining the FSDT with Rayleigh-Ritz procedure can efficiently improve the computational efficiency and saves calculation time in free vibration analysis of stiffened cylindrical shells.

(2) As shown in Tables 2–9 and Figures 2–5, the solutions obtained by proposed method are consistent with that of several state-of-the-art methods. The accuracy virtues of the presented method might be attributed to the following: (i) by considering the effect of transverse shear deformation with the Reissner-Mindlin displacement hypothesis, the FSDT can build highly accurate theoretical model for stiffened cylindrical shell; (ii) the superposition of Fourier cosine series and supplementary function are chosen as the admissible displacement function in Rayleigh-Ritz procedure, which ensures the convergence of the second derivative and smooths the admissible displacement function adequately throughout the entire solution domain. Therefore, the proposed method can perform highly accurate vibration analysis for stiffened cylindrical shells.

In summary, the presented method greatly improves computational efficiency while maintaining computational accuracy and hereby is feasible and effective approach in the free vibration analysis of stiffened cylindrical shells.

4. Conclusions

The objective of this paper is to propose a unified approach for free vibration analysis of stiffened cylindrical shell with general boundary condition. In view of the first-order shear deformation shell theory, the vibration analysis of stiffened cylindrical shell is modeled mathematically. Without requiring any special remodeling or procedures, the arbitrary boundary conditions including free, simple, and clamped supports are readily simulated by simply adjusting the equivalent spring stiffness. By choosing the superposition function of Fourier series and complementary function as admissible displacement function, the natural frequencies and modes are obtained by Rayleigh-Ritz procedure based on the energy expression of stiffened cylindrical shell. In contrast to several state-of-the-art methods, the feasibility and effectiveness of the presented method are examined and the corresponding vibration characteristics of stiffened cylindrical shells are required. Some conclusions have been summarized as follows:

(1) In view of the fast computing time and high calculation efficiency, the presented approach solution shows an excellent convergence in free vibration analysis.

(2) For stiffened cylindrical shells subject to general boundary conditions, the presented method possesses steady

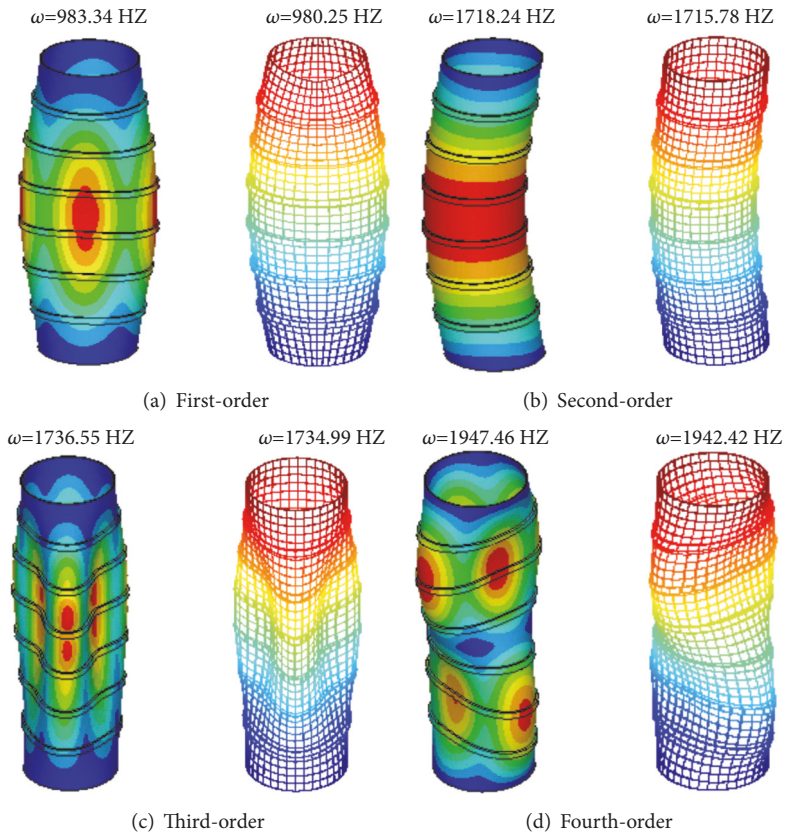

Figure 4: First four modes with clamped boundary.

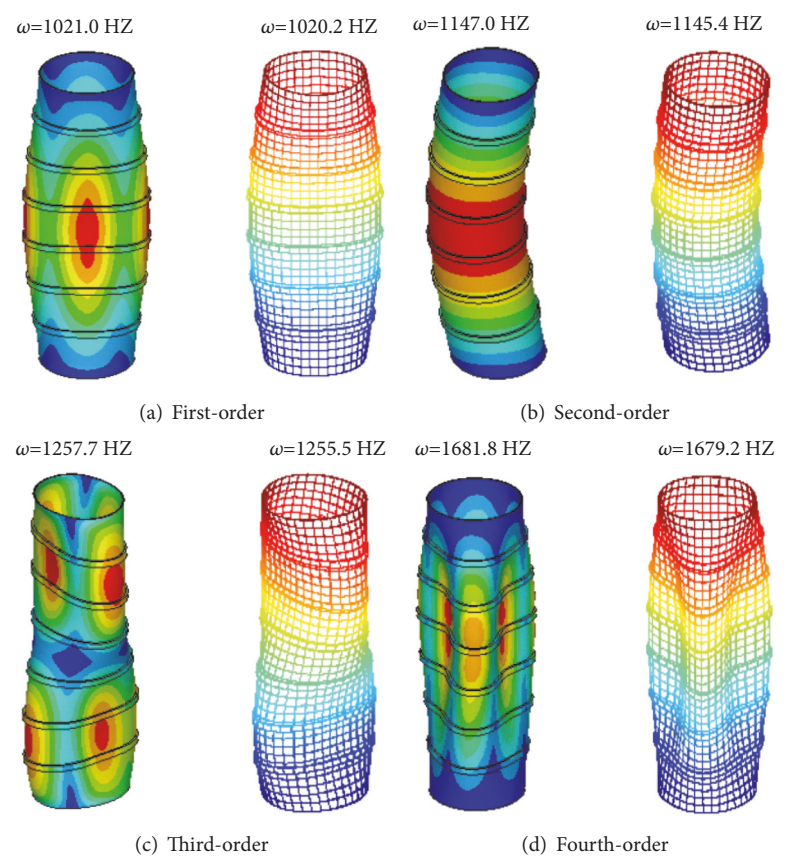

Figure 5: First four modes with simply supported boundary.

calculation results and high solution accuracy in free vibration analysis.

(3) It is found that the natural frequencies have a distinct increase with adding several stiffeners. Besides, with the higher stiffener thickness, the natural frequencies show an increasing tendency.

(4) For stiffened cylindrical shells with moderately thick, the first-order shear deformation theory which considering the transverse shear deformation holds higher computing accuracy than classic shear deformation theory.

(5) The modal shapes acquired by the presented method show good consistent with that of FEM, which further validate the effectiveness of the method presented from modal shapes perspective.

(6) The current efforts provide a unified accurate alternative to other analytical techniques and shed lights on the subsequent nonlinear vibration analysis.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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