Research Article

Numerical Computation and Stability Analysis for the Fractional Subdiffusions with Spatial Variable Coefficients

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In this paper, we propose an efficient compact finite difference method for a class of time-fractional subdiffusion equations with spatially variable coefficients. Based on the $L_2-1_\sigma$ approximation formula of the time-fractional derivative and a fourth-order compact finite difference approximation to the spatial derivative, an efficient compact finite difference method is developed. The local truncation error and the solvability of the developed method are discussed in detail. The unconditional stability of the resulting scheme and also its convergence of second-order in time and fourth-order in space are rigorously proved using a discrete energy analysis method. Numerical examples are provided to demonstrate the accuracy and the theoretical results.

1. Introduction

The fractional differential equations are widely used to describe plenty of nature phenomena in physics, biology, and chemistry [1–10]. Fractional subdiffusion is the evolution equation for the probability density function that describes particles diffusing with mean square displacement $\langle x^2(t) \rangle \sim t^\alpha (0 < \alpha < 1)$ [8].

In this paper, we consider a class of time-fractional subdiffusion equations with variable coefficients as follows:

$$
\frac{\partial u}{\partial t}(x, t) + C_0 D_t^\alpha u(x, t) = \frac{\partial}{\partial x} \left( \varphi(x) \frac{\partial u}{\partial x} \right) + f(x, t), \quad (x, t) \in (0, L) \times (0, T),
$$

with its initial boundary conditions

$$
\begin{align*}
& u(0, t) = \phi_0(t), \\
& u(L, t) = \phi_L(t), \\
& t \in (0, T),
\end{align*}
$$

$$
\begin{align*}
& u(x, 0) = \gamma(x), \quad x \in [0, L],
\end{align*}
$$

where $C_0 D_t^\alpha u(x, t)$ is the Caputo fractional derivative

$$
C_0 D_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} (t-s)^{-\alpha} ds, \quad 0 < \alpha < 1.
$$

In practice, the motion time and the status of particles are described by the time drift term $\partial u/\partial t$. Equation (1) has many important applications in various fields [11–13]. Generally, various numerical methods are used to obtain the numerical solutions of fractional differential, such as the finite difference method [14–18], the spectral method [20, 21], and so on.

In the construction of high-order numerical methods for equation (1), it is crucial to obtain a stable high-order approximation to the Caputo time-fractional derivative $C_0 D_t^\alpha v(x, t)(0 < \alpha < 1)$. A common technique to design a second-order or more higher-order approximation to the fractional derivative is to use the weighted and shifted Grünwald difference operator. The operator is often used to handle the Riemann–Liouville-type fractional derivative (see [22–25]). Some of its applications to the Caputo-type fractional derivative $C_0 D_t^\alpha v(x, t)(0 < \alpha < 1)$ can be found in [24, 26, 27]. Another alternative method was given in [28], where a modified $L_1$ approximation formula, called $L_1-2$ formula, was used to discretize the Caputo time-fractional
derivative $\frac{\partial^\alpha}{\partial t^\alpha}v(x,t) (0 < \alpha < 1)$, but the rigorous convergence analysis for the corresponding difference scheme has not been available. In [29], a new approximation to the Caputo time-fractional derivative $\frac{\partial^\alpha}{\partial t^\alpha}v(x,t) (0 < \alpha < 1)$ with the convergence order $3 - \alpha$, called $L_2-1_\alpha$ formula, was constructed.

Until now, there is a little discussion for time-fractional subdiffusion equations with spatial variable coefficients [30–32]. Zhao and Xu [33] proposed an efficient compact difference scheme for solving the fractional subdiffusion equation with the spatial variable coefficient. The stability and the convergence of the scheme are proved. In [34], Vong proposed a compact finite difference scheme for spatially variable coefficient fractional subdiffusion with Neumann boundary conditions. The analysis of stability and convergence is studied by the energy method. Unfortunately, the temporal accuracy of the aforementioned methods was still less than two order. In general, the weighted and shifted Grünwald difference operator [22–27] is a usual approximation method to the Caputo time-fractional derivative. In this paper, we propose some new methods to construct a high-order compact finite difference method for equation (1).

## 2. Compact Finite Difference Method

For positive integers $M$ and $N$, we let $\tau = T/N$ be the time step and $h = L/M$ be the spatial step. Let $t_n = nh (0 \leq n \leq N)$ and $t_{n-(a/2)} = (n - (a/2))h (1 \leq n \leq N)$. We partition $[0, L]$ into a mesh by the mesh points $x_i = ih (0 \leq i \leq M)$. Denote $x_{i-(1/2)} = (i - (1/2))h (1 \leq i \leq M)$. Define spatial difference operators as follows:

$$
\delta_x u_{i-(1/2)} = \frac{1}{h} (u_{i+1} - u_i),
$$

$$
\delta_x^2 u_i = \frac{1}{h^2} (u_{i+1} - 2u_i + u_{i-1}),
$$

$$
\mathcal{H}_x u_i = u_i + \frac{h^2}{12} \delta_x^2 u_i + \frac{h^2}{12} \left( \varphi' \right)_{u_i}.
$$

Let

$$
c_{k,n}^{(\alpha)} = \begin{cases} 
  a_0^{(\alpha)}, & k = 0, n = 1, \\
  c_k^{(\alpha)} + b_{k+1}^{(\alpha)} - b_k^{(\alpha)}, & 0 \leq k \leq n - 2, n \geq 2, \\
  a_{n-1}^{(\alpha)} - b_{n-1}^{(\alpha)}, & k = n - 1,
\end{cases}
$$

where

$$
\sigma = 1 - \frac{\alpha}{2} h = k + \sigma (k \geq 0),
$$

$$
a_0^{(\alpha)} = \eta_0^{1-\alpha},
$$

$$
0 < b_k^{(\alpha)} = \frac{\alpha}{2(2 - \alpha)} ((k + \sigma)^{1-\alpha} - (k - 1 + \sigma)^{1-\alpha}) (k \geq 1),
$$

$$
1 - \frac{\alpha}{2} (k + \sigma)^{1-\alpha} < a_k^{(\alpha)} - b_k^{(\alpha)} < (k + \sigma)^{1-\alpha}
$$

$$
- (k - 1 + \sigma)^{1-\alpha} (k \geq 1),
$$

$$
c_{k-1,n}^{(\alpha)} > c_{k,n}^{(\alpha)}, \quad (n \geq 2, 1 \leq k \leq n - 1),
$$

$$
(1 - \alpha) a_0^{(\alpha)} - a_{1,n}^{(\alpha)} > 0, \quad (n \geq 2).
$$

### Lemma 2 (assume $u(t) \in \mathbb{C}^3 [0, T]$). When $n \geq 1$, we have

$$
C^\alpha_0 D^\alpha_{(a/2)} u(t_{n-(a/2)}) = \frac{1}{\mu} \sum_{k=1}^n c_k^{(\alpha)} (u(t_k) - u(t_{k-1})) + \mathcal{O}(r^3),
$$

where $\mu = \tau^a (2 - \alpha)$.

**Proof.** The proof follows from Lemma 2 of [29].

### Lemma 3 (see [35]). Assume $u(t) \in \mathbb{C}^3 [0, T]$. We obtain

$$
\frac{du}{dt} (t_{n-(a/2)}) = \frac{2 - \alpha}{\tau} (u(t_i) - u(t_0)) - (1 - \alpha) \frac{du}{dt} (t_0) + \mathcal{O}(r^2),
$$

and for $n \geq 2$,

$$
\frac{du}{dt} (t_{n-(a/2)}) = \frac{1}{2\tau} ((3 - \alpha) u(t_n) - (4 - 2\alpha) u(t_{n-1})
$$

$$
+ (1 - \alpha) u(t_{n-2})) + \mathcal{O}(r^2).
$$

### Lemma 4 (see [35]). Suppose $u(t) \in \mathbb{C}^2 [0, T]$. When $n \geq 1$, we have

$$
u(t_{n-(a/2)}) = \frac{\alpha}{2} u(t_{n-1}) + (1 - \frac{\alpha}{2}) u(t_n) + \mathcal{O}(r^2).
$$

### Lemma 5. Suppose $u(x) \in \mathbb{C}^6 [0, L] \times [0, T]$. It holds that

$$
\mathcal{H}_x \left( \frac{\partial}{\partial x} \left( \varphi(x) \frac{\partial u}{\partial x} \right) \right) (x_i, t_n) = \delta_x (\psi \delta_x u)(x_i, t_n) + R(x_i, t_n),
$$

$$
1 \leq i \leq M - 1, 1 \leq n \leq N,
$$

where

$$
\psi = \varphi - \frac{h^2}{12} \left( \left( \varphi' \right)^2 \cdot \frac{1}{2} \varphi''' \right),
$$

$$
R(x_i, t_n) = \mathcal{O}(h^4).
$$

**Proof.** The proof follows from (31)–(33) in [32].
Next, we discretize (1) into a compact finite difference scheme. We define
\[
\delta_t u^{n^{(1/2)}} = \frac{1}{\tau} (u^n - u^{n-1}) (1 \leq n \leq N),
\]
\[
D_t u^n = \frac{1}{\tau} ((3 - \alpha)u^n - (4 - 2\alpha)u^{n-1} + (1 - \alpha)u^{n-2})
\cdot (2 \leq n \leq N),
\]
\[
u_n^{(n/2)} = \alpha u^{n-1} + \left(1 - \frac{\alpha}{2}\right) u^n (1 \leq n \leq N).
\]

Suppose \(u(x, t)\) be the solution of (1), we define the grid functions as follows:
\[
U_i^n = u(x_i, t_n),
\]
\[
W_i^n = \frac{\partial u}{\partial t} (x_i, t_n),
\]
\[
Z_i^n = \frac{\partial}{\partial x} \left( \varphi (x) \frac{\partial u}{\partial x} \right) (x_i, t_n),
\]
\[
\varphi_i = \varphi (x_i),
\]
\[
f_i^n = f(x_i, t_n),
\]
\[
f_i^{n-(a/2)} = f(x_i, t_{n-(a/2)}),
\]
\[
\phi_i^0 = \phi_0 (t_n),
\]
\[
\phi_i^1 = \phi_1 (t_n),
\]
\[
\gamma_i = \gamma (x_i).
\]

We consider equation (1) at the point \((x_i, t_{n-(a/2)})\), and
\[
W_i^{n-(a/2)} + \frac{c_{n-k,n}}{\mu} u(x_i, t_{n-(a/2)}) = dZ_i^{n-(a/2)} + f_i^{n-(a/2)}.
\]

From Lemmas 2–4, we have
\[
D_t U_i^n = \frac{1}{\mu} \sum_{k=1}^{n} c_{n-k,n} (U_k^i - U_{k-1}^i) = Z_i^{n-(a/2)} + f_i^{n-(a/2)}
\]
\[
+ (R_i^{(a)}), \quad 1 \leq i \leq M - 1, 2 \leq n \leq N,
\]

\[
(2 - \alpha) \delta_t U_i^{1/2} + \frac{1}{\mu} c_{n,i} (U_i^1 - U_i^0) = Z_i^{1/2} + (1 - \alpha)W_i^0 + f_i^{1/2} + (R_i^{1/2}), \quad 1 \leq i \leq M - 1,
\]

where
\[
(R_i^{(a)})_n = O\left(r^2\right) + O\left(r^{1-a}\right), \quad 1 \leq i \leq M - 1, 1 \leq n \leq N.
\]

Using Lemma 5, we obtain
\[
\mathcal{H}_x D_t U_i^n + \frac{1}{\mu} \sum_{k=1}^{n} c_{n-k,n} \mathcal{H}_x (U_k^i - U_{k-1}^i) = \delta_x \left( \psi \delta_x U_i^n \right) + \mathcal{H}_x f_i^{n-(a/2)} + (R_i^{n})_n, \quad 1 \leq i \leq M - 1, 1 \leq n \leq N.
\]

Since \(c_{n,k,n} u(x_i, 0) = 0 [4]\), we have \(W_i^0 = f_i^{n,0}\), where
\[
f_i^{n,0} = \frac{\partial^2 \gamma}{\partial x^2} (x_i) + f (x_i, 0), \quad 1 \leq i \leq M - 1.
\]

We obtain the following compact finite difference scheme:
\[
\begin{cases}
\mathcal{H}_x D_t U_i^n + \frac{1}{\mu} \sum_{k=1}^{n} c_{n-k,n} \mathcal{H}_x (U_k^i - U_{k-1}^i) = \delta_x \left( \psi \delta_x U_i^n \right) + \mathcal{H}_x f_i^{n-(a/2)}, & 1 \leq i \leq M - 1, 2 \leq n \leq N, \\
(2 - \alpha) \delta_t U_i^{1/2} + \frac{1}{\mu} c_{n,i} (U_i^1 - U_i^0) = \delta_x \left( \psi \delta_x U_i^{1/2} \right) + (1 - \alpha)\mathcal{H}_x f_i^{+0} + \mathcal{H}_x f_i^{1-(a/2)}, & 1 \leq i \leq M - 1, \\
u_0^n = \phi_0^n, u_M^n = \phi_M^n, & 1 \leq n \leq N, \\
u_0^0 = \gamma_0, & 0 \leq i \leq M.
\end{cases}
\]
3. Truncation Error and Solvability

Assume \( u(x,t) \in C^{6,3}([0,L] \times [0,T]) \), and there exists a positive constant \( C \) such that
\[
\begin{align*}
| (R^n_{x,1})_{i} | & \leq C \tau^2, \\
| (R^n_{x})_{i} | & \leq C \tau^3.
\end{align*}
\] (29)

Applying (29) to (26), we have the following theorem:

**Theorem 1.** Suppose \( u(x,t) \in C^{6,3}([0,L] \times [0,T]) \). The truncation error \( (R^n_{x,1})_{i} \) satisfies
\[
| (R^n_{x,1})_{i} | \leq C \tau^2 + h^4, \quad 1 \leq i \leq M - 1, 1 \leq n \leq N,
\] (30)

where \( C \) is a positive constant.

For convenience, we construct the matrix form of the compact difference scheme (28). Let
\[
u^{(a)}_{k,n} = \begin{cases}
\varepsilon^{(a)}_{0,1}, & k = 0, n = 1, \\
-\varepsilon^{(a)}_{0,1}, & k = 1, n = 1, \\
\varepsilon^{(a)}_{k,n}, & k = 0, n \geq 2, \\
-\varepsilon^{(a)}_{k-1,n}, & 1 \leq k \leq n - 1, n \geq 2, \\
-\varepsilon^{(a)}_{n-1,n}, & k = n, n \geq 2.
\end{cases}
\] (31)

Define the column vectors as follows:
\[
\begin{align*}
u^n & = (u^n_1, u^n_2, \ldots, u^n_{M-1})^T, \\
u^{n-2,*} & = (u^{n-2,*}_1, u^{n-2,*}_2, \ldots, u^{n-2,*}_{M-1})^T, \\
f^{(n/2)} & = (f^{(n/2)}_1, f^{(n/2)}_2, \ldots, f^{(n/2)}_{M-1})^T, \\
f^{*,0} & = (f^{*,0}_1, f^{*,0}_2, \ldots, f^{*,0}_{M-1})^T, \\
\Phi^n & = (\psi^n_1, \psi^n_2, \ldots, \psi^n_{M-1})^T,
\end{align*}
\] (32)

where
\[
u^{n-2,*}_i = \sum_{k=2}^{n} \omega^{(a)}_{k,n} \nu^{n-k}_i, \quad 1 \leq i \leq M - 1, 2 \leq n \leq N.
\] (33)

Define the following \((M-1)\)-order tridiagonal or diagonal matrices:
\[
A = \frac{1}{h^2} \begin{pmatrix}
\psi(x_{1/2}) + \psi(x_{3/2}) & -\psi(x_{3/2}) & & \\
-\psi(x_{1/2}) & \psi(x_{3/2}) + \psi(x_{5/2}) & -\psi(x_{5/2}) & & \\
& \ddots & \ddots & \ddots & \\
& & -\psi(x_{M-(3/2)}) & \psi(x_{M-(3/2)}) + \psi(x_{M-(1/2)}) & -\psi(x_{M-(1/2)}) \\
& & & \psi(x_{M-(3/2)}) + \psi(x_{M-(1/2)}) & \ddots \\
\end{pmatrix},
\] (34)

\[
B_1 = \frac{1}{12} \text{tridiag}(1, 10, 1),
\]

\[
B_2 = \frac{h}{24} \text{tridiag}(1, 0, -1),
\]

\[
Q = \text{diag}(\xi_1, \xi_2, \ldots, \xi_{M-1}),
\]

\[
Q_1 = \left( \frac{2 - \alpha}{\tau} - \frac{w^{(a)}_{0,1}}{\mu} \right) (B_1 + B_2 Q) + \left( \frac{2 - \alpha}{2h^2} - A, \right.
\]

\[
Q_n = \left( \frac{3 - \alpha}{2\tau} - \frac{w^{(a)}_{0,n}}{\mu} \right) (B_1 + B_2 Q) + \left( \frac{2 - \alpha}{2h^2} - A, \right. \quad 2 \leq n \leq N,
\]

\[
\widetilde{Q}_n = \left( \frac{2 - \alpha}{\tau} - \frac{w^{(a)}_{0,n}}{\mu} \right) (B_1 + B_2 Q) - \frac{\alpha}{2h^2} A, \quad 1 \leq n \leq N.
\]
According to some simple calculations, the scheme (28) can be expressed in the matrix form as

\[
\begin{align*}
Q_nu^n &= \tilde{Q}_nu^{n-1} - (B_1 + B_2)Q_n\left[\frac{(1 - \alpha)}{2\tau}u^{n-2} + \frac{1}{\mu}x^{n-2} - \Gamma^{-(a/2)}\right] \\
&\quad + r^n, \quad 2 \leq n \leq N, \\
Q_nu^1 &= \tilde{Q}_nu^0 + (B_1 + B_2)\left((1 - \alpha)\Gamma^0 + \Gamma^{-(a/2)}\right) + r^1,
\end{align*}
\]

where \(r^n\) absorbs the boundary values of the solution vector and the source term.

**Theorem 2.** The compact difference scheme (28) is uniquely solvable if and only if the matrices \(Q_n\) (1 \(\leq n \leq N\) are all nonsingular.

Assume

\[
c_1 \leq \psi \leq c_2, \quad \frac{|\psi|}{\psi} = |\xi| \leq c.
\]

A sufficient condition for the matrices \(Q_n\) (1 \(\leq n \leq N\) to be all nonsingular is given by

\[
hc < 8.
\]

**Corollary 1.** The compact difference scheme (28) is uniquely solvable if condition (37) holds true.

**Proof.** At first, we prove the matrix \(Q_n\) (2 \(\leq n \leq N\) is nonsingular. Since \(u_{h/n}^{(n)} = a_0^{(n)} + b_1^{(n)}\) (n \(\geq 2\)), we have \(Q_n = \text{tridiag}(p_{i-1}^n, q_i^n, p_i^n)\), where

\[
\begin{align*}
p_{i-1}^n &= \left(\frac{1}{12} + \frac{h}{24\tau}\right)\left((3 - \alpha)\frac{a_0^{(a)} + b_1^{(a)}}{2\tau} + \frac{2 - \alpha}{2h^2}\right) \\
\quad - (2 - \alpha)\left(\psi(x_{i-1/2})\right), \quad i \neq 1, \\
p_{i+1}^n &= \left(\frac{1}{12} + \frac{h}{24\tau}\right)\left((3 - \alpha)\frac{a_0^{(a)} + b_1^{(a)}}{2\tau} + \frac{2 - \alpha}{2h^2}\right) \\
\quad - (2 - \alpha)\left(\psi(x_{i+1/2})\right), \\
q_i^n &= \frac{5}{6}\left(\frac{3 - \alpha}{2\tau} + \frac{a_0^{(a)} + b_1^{(a)}}{\mu}\right) \\
\quad + (2 - \alpha)\left(\psi(x_{i-1/2}) + \psi(x_{i+1/2})\right).
\end{align*}
\]

Since \(q_i^n > 0\) (1 \(\leq i \leq M - 1\)), we have

\[
|p_{i-1}^n| + |p_{i+1}^n| \leq \left(\frac{1}{6} + \frac{h}{24}(\tau - \zeta_{i-1} + \zeta_{i+1})\right)\left((3 - \alpha)\frac{a_0^{(a)} + b_1^{(a)}}{2\tau} + \frac{2 - \alpha}{2h^2}\right) \\
\quad + \frac{(2 - \alpha)}{2h^2}\left(\psi(x_{i-1/2}) + \psi(x_{i+1/2})\right)
\]

\[
\leq \frac{1}{6} + \frac{h}{12\tau}\left((3 - \alpha)\frac{a_0^{(a)} + b_1^{(a)}}{2\tau} + \frac{2 - \alpha}{2h^2}\right) \\
\quad + \frac{(2 - \alpha)}{2h^2}\left(\psi(x_{i-1/2}) + \psi(x_{i+1/2})\right)
\]

\[
< \frac{5}{6}\left((3 - \alpha)\frac{a_0^{(a)} + b_1^{(a)}}{2\tau} + \frac{2 - \alpha}{2h^2}\right) + (2 - \alpha)\left(\psi(x_{i-1/2}) + \psi(x_{i+1/2})\right) = |q_i^n|.
\]

Similarly,

\[
|p_{i-1}^n| \leq \left(\frac{1}{12} + \frac{h}{24\tau}\right)\left((3 - \alpha)\frac{a_0^{(a)} + b_1^{(a)}}{2\tau} + \frac{2 - \alpha}{2h^2}\right) \\
\quad + \frac{(2 - \alpha)}{2h^2}\psi(x_{3/2})< \frac{5}{6}\left((3 - \alpha)\frac{a_0^{(a)} + b_1^{(a)}}{2\tau} + \frac{2 - \alpha}{2h^2}\right) + (2 - \alpha)\left(\psi(x_{3/2})\right) + \psi(x_{1/2}) = |q_i^n|,
\]

\[
|p_{M-1}^n| \leq \frac{1}{12} + \frac{h}{24\tau}\left((3 - \alpha)\frac{a_0^{(a)} + b_1^{(a)}}{2\tau} + \frac{2 - \alpha}{2h^2}\right) \\
\quad + \frac{(2 - \alpha)}{2h^2}\psi(x_{M-3/2})< \frac{5}{6}\left((3 - \alpha)\frac{a_0^{(a)} + b_1^{(a)}}{2\tau} + \frac{2 - \alpha}{2h^2}\right) + (2 - \alpha)\left(\psi(x_{M-3/2})\right) + \psi(x_{1/2}) = |q_{M-1}^n|.
\]

Therefore, \(Q_n\) (1 \(\leq n \leq N\) are strictly diagonally dominant and thus nonsingular [36].

**4. Stability and Convergence of the Proposed Difference Scheme**

Let \(\delta_h = \{u | u = (u_0, u_1, \ldots, u_M), u_0 = u_M = 0\}\), and for any grid functions \(u, v \in \delta_h\), we define
\begin{align}
(\delta_x u, \delta_x v) &= \frac{h^M}{M} \sum_{i=0}^{M-1} \delta_x u_{i+1/2} \delta_x v_{i+1/2}, \\
|u_1| &= (\delta_x u, \delta_x u)^{1/2}, \\
\|u_1\| &= \left(\|u_1\|^2 + |u_1|^2\right)^{1/2},
\end{align}

\begin{equation}
(\delta_x u, \delta_x v)_\nu = h \sum_{i=0}^{M-1} \psi(x_{i+1/2}) \delta_x u_{i+1/2} \delta_x v_{i+1/2},
\end{equation}

\begin{equation}
\left\|\delta_x u \right\| = (\delta_x u, \delta_x u)^{1/2}.
\end{equation}

For any grid function \( u, v \in \delta_h \), we have
\begin{equation}
(\delta_x^2 u, v) = -(\delta_x u, \delta_x v),
\end{equation}

\begin{equation}
\begin{aligned}
&h\|\delta_x^2 u\| \leq 2|u_1|, \\
&h|u_1| \leq 2\|u_1\|.
\end{aligned}
\end{equation}

Using Lemma 2.6 and Theorem 2.1 in [33], we obtain \((H, u, \mu) \geq (1/4)\|u\|^2\). Next, we define the discrete inner product as follows:
\begin{equation}
\langle u, \nu \rangle = (H, u, \nu),
\end{equation}

\begin{equation}
\|u_\nu\| = \langle u, u \rangle^{1/2}.
\end{equation}

Lemma 6 (see [35]). For any grid function \( u = \{u^n_i| 0 \leq i \leq M, 0 \leq n \leq N\} \in \delta_h \), we have
\begin{equation}
(H, D u^n, u_n^{(a,2)}) \geq \frac{1}{4\varepsilon}(E - E^n), 2 \leq n \leq N,
\end{equation}

where for \( 1 \leq n \leq N \),
\begin{equation}
E^n = (3 - \alpha) \|u^n\|_2^2 - (1 - \alpha) \|u^{n-1}\|_2^2
+ \frac{(4 - \alpha)(1 - \alpha)}{2} \|u^n - u^{n-1}\|_2^2 \geq \frac{2 - 2\alpha}{2 \varepsilon} \|\phi\|_2^2.
\end{equation}

Lemma 7. For any grid function \( u^n_i \in \delta_h \), we obtain
\begin{equation}
\sum_{k=1}^{n} \epsilon_{n-k, k} (H, u^k - u^{k-1}, u_n^{(a,2)}) \geq \frac{1}{2} \sum_{k=1}^{n} \epsilon_{n-k, k} (\|u^k\|_2^2 - \|u^{k-1}\|_2^2), 1 \leq n \leq N.
\end{equation}

\begin{equation}
\left\langle \sum_{k=1}^{n} \epsilon_{n-k, k} (H, u^k - u^{k-1}, u_n^{(a,2)}) \right\rangle = \left\langle \sum_{k=1}^{n} \epsilon_{n-k, k} (u^k - u^{k-1}, u_n^{(a,2)}) \right\rangle.
\end{equation}

Applying Corollary 1 [29], we have
\begin{equation}
\left\langle \sum_{k=1}^{n} \epsilon_{n-k, k} (u^k - u^{k-1}, u_n^{(a,2)}) \right\rangle \leq \frac{1}{2} \sum_{k=1}^{n} \epsilon_{n-k, k} (\|u^k\|_2^2 - \|u^{k-1}\|_2^2).
\end{equation}

The desired result (48) follows immediately.

Lemma 8 (discrete Gronwall lemma [37]). Suppose \( \{K_n\} \) and \( \{|S_n|\} \) are nonnegative sequences, and \( \{\lambda_n\} \) satisfies
\begin{equation}
\lambda_0 \leq g_0,
\end{equation}

\begin{equation}
\lambda_n \leq g_0 + \sum_{l=0}^{n-1} S_l + \sum_{l=0}^{n-1} K_l \lambda_l, n \geq 1,
\end{equation}

where \( g_0 \geq 0 \). Then, the sequence \( \{\lambda_n\} \) satisfies
\begin{equation}
\lambda_n \leq \left( \frac{g_0}{\sum_{l=0}^{n-1} S_l} \right) \exp \left( \frac{\sum_{l=0}^{n-1} K_l}{\sum_{l=0}^{n-1} S_l} \right), n \geq 1.
\end{equation}

Theorem 3. Suppose \( u_n = (u^n_0, u^n_1, \ldots, u^n_M) \) be the solution of the compact difference scheme (28) with \( u^n_0 = u^n_M = 0 \). Then, when \( \tau \leq (1/4 - 2\alpha)^3 \), it holds
\begin{equation}
\|u^n\|_2^2 \leq \exp(4(2 - \alpha)(2 - \alpha)^2 \tau) G_n^*, 1 \leq n \leq N,
\end{equation}

where
\begin{equation}
G_n^* = (2 - \alpha) \left( 1 + \frac{2(2 - \alpha)(2\tau)^{1-a}}{\Gamma(2 - \alpha)} \right) G_0 + 8rgG_n,
\end{equation}

with
\begin{equation}
G_0 = \left( 3 - \alpha + \frac{2(1 - a)}{\Gamma(2 - \alpha)} + 4\alpha^2 \right) \|\phi\|_2^2.
\end{equation}

\begin{equation}
G_n = \left( 1 + \frac{2(2 - \alpha)(2\tau)^{1-a}}{\Gamma(2 - \alpha)} \right) \|1 - \alpha\| + \|H_xf^{1-a}\|_2^2
+ \sum_{k=2}^{n} \|H_xf^{1-a}\|_2^2.
\end{equation}

Proof. Applying the inner product to equation (28) with \( u_n^{(a,2)} \), we obtain
\begin{equation}
(H, D u^n, u_n^{(a,2)}) + \frac{1}{\mu} \left( \sum_{k=1}^{n} \epsilon_{n-k, k} (H, u^k - u^{k-1}, u_n^{(a,2)}) \right)
= -\left\langle \delta_x u_n^{(a,2)} \right\rangle \left\langle (H, f^{a-2}, u_n^{(a,2)}) \right\rangle, 2 \leq n \leq N.
\end{equation}

By Lemmas 6 and 7,
\[
\frac{1}{4\tau}(E^n - E^{n-1}) + \frac{1}{2\mu} \sum_{k=0}^{n-1} c_{n-k,n}(|u^k|^2 - |u^{k-1}|^2) \\
\leq \left( \mathcal{H}_x f^{(a/2)}, u^{n,(a/2)} \right), \quad 2 \leq n \leq N,
\]

where
\[
E^n = (3 - \alpha)\|u^n\|_e^2 - (1 - \alpha)\|u^{n-1}\|_e^2 \\
+ \frac{(4 - \alpha)(1 - \alpha)}{2}\|u^n - u^{n-1}\|_e^2 \\
\geq \frac{2}{\alpha}\|u^n\|_e^2.
\]

Using the definition of \(c_{n-k,n}^{(a)}\), we have
\[
\sum_{k=0}^{n-1} c_{n-k,n}^{(a)} \left( |u^k|^2 - |u^{k-1}|^2 \right) \\
= \left( b_{n-1}^{(a)} - a_{n-1}^{(a)} \right) |u^n|^2 - \left( b_{n-1}^{(a)} - b_{n-k}^{(a)} \right) |u^{n-1}|^2 \\
+ \sum_{k=0}^{n-2} \left( a_k^{(a)} + b_{k+1}^{(a)} - b_k^{(a)} \right) \|u^{n-k}\|_e^2 - \sum_{k=0}^{n-2} \left( a_k^{(a)} + b_{k+1}^{(a)} - b_k^{(a)} \right) \|u^{n-1-k}\|_e^2.
\]

By the Cauchy–Schwarz inequality and Lemma 6, we obtain
\[
\left( \mathcal{H}_x f^{(a/2)}, u^{n,(a/2)} \right) \leq \frac{1}{4} \left( (2 - \alpha)^2\|u^n\|_e^2 + \alpha^2\|u^{n-1}\|_e^2 \right) \\
+ \frac{1}{2}\left( E^n - E^{n-1} \right)^2.
\]

Let
\[
F^n = \frac{1}{4\tau}(E^n - E^{n-1}) + \frac{1}{2\mu} \sum_{k=0}^{n-1} \left( a_k^{(a)} + b_{k+1}^{(a)} - b_k^{(a)} \right) \|u^{n-k}\|_e^2, \quad n \geq 1.
\]

Substituting (60) and (61) into (58), we have
\[
F^n \leq F^{n-1} + \frac{1}{4} \left( (2 - \alpha)^2\|u^n\|_e^2 + \alpha^2\|u^{n-1}\|_e^2 \right) \\
+ \frac{1}{2\mu} \left( b_n^{(a)}\|u^n\|_e^2 + (a_{n-1}^{(a)} - b_{n-1}^{(a)})\|u^{n-1}\|_e^2 \right) \\
+ \frac{1}{2}\left( \mathcal{H}_x f^{(a/2)} \right)^2, \quad 2 \leq n \leq N,
\]

or equivalently,
\[
F^n \leq F^{n-1} + \frac{1}{4} \left( (2 - \alpha)^2\|u^n\|_e^2 + \alpha^2\|u^{n-1}\|_e^2 \right) \\
+ \frac{1}{2\mu} \left( b_n^{(a)}\|u^n\|_e^2 + (a_{n-1}^{(a)} - b_{n-1}^{(a)})\|u^{n-1}\|_e^2 \right) \\
+ \frac{1}{2}\left( \mathcal{H}_x f^{(a/2)} \right)^2, \quad 2 \leq n \leq N.
\]

Taking the inner product of equation (28) with \(u^{1,(a/2)}\), we obtain
\[
(2 - \alpha)\left( \mathcal{H}_x \delta_{1/2} u^{1/2} - u^{1,(a/2)} \right) + \frac{1}{\mu c_{0,1}} \mathcal{H}_x (u^{1} - u^{0}) = -\left\| \delta_x u^{1,(a/2)} \right\|_e^2 + \left( (1 - \alpha)\mathcal{H}_x f^{*0} + \mathcal{H}_x f^{1,(a/2)} \right). \tag{65}
\]

It is easy to see that
\[
\left( \mathcal{H}_x \delta_{1/2} u^{1/2}, u^{1,(a/2)} \right) = \frac{1}{2\tau}\left( \|u^0\|_e^2 - \|u^0\|_e^2 + (1 - \alpha)\|u^{1} - u^{0}\|_e^2 \right) \tag{66}
\]

Applying Lemma 7, we have
\[
\left( c_{0,1} \mathcal{H}_x (u^{1} - u^{0}), u^{1,(a/2)} \right) \geq \frac{1}{2\tau c_0} \left( \|u^1\|_e^2 - \|u^0\|_e^2 \right). \tag{67}
\]

Substituting (66) and (67) into (65), we obtain
\[
\frac{1}{2\tau} \left( (2 - \alpha)\left( \|u^1\|_e^2 + (1 - \alpha)\|u^{1} - u^{0}\|_e^2 \right) + \frac{1}{2\mu c_{0,1}}\|u^1\|_e^2 \right) \\
\leq \frac{1}{2} \left( (2 - \alpha) + \frac{1}{\mu c_{0,1}} \right)\|u^1\|_e^2 + \frac{1}{4} \left( (2 - \alpha)^2\|u^1\|_e^2 + \alpha^2\|u^{0}\|_e^2 \right) \\
+ \frac{1}{2}\left( (1 - \alpha)\mathcal{H}_x f^{*0} + \mathcal{H}_x f^{1,(a/2)} \right)^2. \tag{68}
\]

Since \(3 - \alpha < 2(2 - \alpha)\) and \(4 - \alpha < 4(2 - \alpha)\), we have
\[
F^1 < \frac{1}{2\tau} \left( (2 - \alpha)\left( \|u^1\|_e^2 + (1 - \alpha)\|u^{1} - u^{0}\|_e^2 \right) + \frac{1}{2\mu c_{0,1}}\|u^1\|_e^2 \right) \\
- \frac{1}{4\tau} (1 - \alpha)\|u^0\|_e^2 + \frac{1}{2\mu c_{0,1}}\|u^1\|_e^2.
\]

Thus,
\[
F^1 < \frac{1}{2} \left( \frac{1}{2\tau} \left( (3 - \alpha) + \frac{1}{\mu c_{0,1}} \right)\|u^0\|_e^2 + \frac{1}{2\mu c_{0,1}}\|u^1\|_e^2 \right) \\
+ \frac{1}{4} \left( (2 - \alpha)^2\|u^1\|_e^2 + \alpha^2\|u^{0}\|_e^2 \right) \tag{70}
\]

Using (63), we have
\[
F^n \leq \frac{1}{4r} (3 - \alpha) \| u^0 \|^2_i + \frac{1}{4} \left( (2 - \alpha)^2 \sum_{k=1}^{n} \| u^k \|^2_i + \alpha^2 \sum_{k=0}^{n-1} \| u^k \|^2_i \right) \\
+ \frac{1}{2 \mu} \sum_{k=1}^{n} \left( b_k^{(a)} \| u^k \|^2_i + (a_k^{(a)} - b_k^{(a)}) \| u^0 \|^2_i \right) \\
+ \frac{1}{2} \left( \| (1 - \alpha) \mathcal{H}_x f^{*, \alpha} + \mathcal{H}_x f^{1-(a/2)} \|_i^2 \right) \\
+ \sum_{k=2}^{n} \| \mathcal{H}_x f^{k-(a/2)} \|_i^2 , \quad 2 \leq n \leq N. 
\]

(71)

By the definition of \( F^n \), equation (60), and \( a_k^{(a)} + b_k^{(a)} > 0 \) (see (10)), we have

\[
\frac{1}{2(2 - \alpha) \tau} \| u^n \|^2_i \leq \frac{1}{4} (3 - \alpha) \left( \alpha^2 \| u^0 \|^2_i + \frac{3}{4} \| u \|^2 \right) \\
\leq \frac{1}{4r} (3 - \alpha) + \frac{1}{2 \mu} \sum_{k=1}^{n} \left( a_k^{(a)} - b_k^{(a)} \right) \| u^k \|^2_i + \frac{1}{2} \left( \| (1 - \alpha) \mathcal{H}_x f^{*, \alpha} + \mathcal{H}_x f^{1-(a/2)} \|_i^2 \right) \\
+ \sum_{k=2}^{n} \| \mathcal{H}_x f^{k-(a/2)} \|_i^2 , \quad 2 \leq n \leq N. 
\]

(72)

Applying Lemma 6, we obtain

\[
\left( \frac{1}{2(2 - \alpha) \tau} - (2 - \alpha)^2 \right) \| u^n \|^2_i \\
\leq \frac{1}{4r} (3 - \alpha) + \frac{1}{2 \mu} \sum_{k=1}^{n} \left( a_k^{(a)} - b_k^{(a)} \right) \| u^k \|^2_i + \frac{1}{2} \left( \| (1 - \alpha) \mathcal{H}_x f^{*, \alpha} + \mathcal{H}_x f^{1-(a/2)} \|_i^2 \right) \\
+ \sum_{k=2}^{n} \| \mathcal{H}_x f^{k-(a/2)} \|_i^2 , \quad 2 \leq n \leq N. 
\]

(73)

When \( \tau \leq (1/4 \ (2 - \alpha)^3) \), we have

\[
\| u^n \|^2_i \leq (2 - \alpha) \left( \left( 3 - \alpha \right) + \frac{2 \tau}{\mu} \sum_{k=1}^{n} \left( a_k^{(a)} - b_k^{(a)} \right) \right) \| u^0 \|^2_i \\
+ 4 \alpha^2 \| u^n \|^2_i + \frac{2 \tau}{\mu} \left( \sum_{k=1}^{n} b_k^{(a)} \right) \| u^k \|^2_i \\
+ 8 \tau \left( \| (1 - \alpha) \mathcal{H}_x f^{*, \alpha} + \mathcal{H}_x f^{1-(a/2)} \|_i^2 \right) \\
+ 4 \tau (2 - \alpha) (2 - \alpha)^2 + \alpha^2 \sum_{k=1}^{n} \| u^k \|^2_i , \quad 2 \leq n \leq N. 
\]

(74)

By Lemma 1,

\[
\frac{\tau}{\mu} \sum_{k=1}^{n} \left( a_k^{(a)} - b_k^{(a)} \right) < \frac{T^{1-a}}{\Gamma (2 - \alpha)} n^{1-a} \leq \frac{T^{1-a}}{\Gamma (2 - \alpha)} 
\]

(75)

Therefore, we have

\[
\| u^n \|^2_i \leq (2 - \alpha) \left( G_0 + \frac{2 \tau (2T)^{1-a}}{\Gamma (2 - \alpha)} \| u^0 \|^2_i \right) \\
+ 8 \tau (2 - \alpha) \left( \| (1 - \alpha) \mathcal{H}_x f^{*, \alpha} + \mathcal{H}_x f^{1-(a/2)} \|_i^2 \right) \\
+ \frac{\tau}{\mu} \left( \sum_{k=1}^{n} \| \mathcal{H}_x f^{k-(a/2)} \|_i^2 \right) \\
+ 4 \tau (2 - \alpha) (2 - \alpha)^2 + \alpha^2 \sum_{k=1}^{n} \| u^k \|^2_i , \quad 2 \leq n \leq N, 
\]

(76)

where \( G_0 \) is defined by (55). Similarly, using (70), we have

\[
\| u^n \|^2_i \leq (2 - \alpha) \left( G_0 + 8 \tau (1 - \alpha) \| \mathcal{H}_x f^{*, \alpha} + \mathcal{H}_x f^{1-(a/2)} \|_i^2 \right). 
\]

(77)

Substituting (77) into (76), we get

\[
\| u^n \|^2_i \leq (2 - \alpha) \left( 1 + \frac{2 \tau (2T)^{1-a}}{\Gamma (2 - \alpha)} \right) \left( G_0 + 8 \tau G_0 \right) \\
+ 4 \tau (2 - \alpha) (2 - \alpha)^2 + \alpha^2 \sum_{k=1}^{n} \| u^k \|^2_i , \quad 2 \leq n \leq N, 
\]

(78)

where \( G_0 \) is defined by (56). From Lemma 3, this completes the proof.

Let \( \varepsilon^n = U^n - u^n \). From (24), (25), and (28), we get the following error equation:
Applying (79) and Theorem 3, we have
\[
\begin{align*}
H D \varepsilon_i^n + \frac{1}{\mu} \sum_{k=1}^{n} \gamma^{(a)}_{\pm \pm} H D \varepsilon_i^{(i-1)} = \delta^{2} \varepsilon_i^{(n+2)} + (R_{\varphi_{\pm}}^{n})^{i}, & \quad 1 \leq i \leq M - 1, 2 \leq n \leq N, \\
(2 - \alpha) H D \varepsilon_i^{(i/2)} + \frac{1}{\mu^{(a)}} H D \varepsilon_i^{(i)} = \delta^{2} \varepsilon_i^{(1+n/2)} + (R_{\varphi_{\pm}}^{n})^{i}, & \quad 1 \leq i \leq M - 1, \\
e_0^{i} = e_0^{i}, & \quad 1 \leq n \leq N, \\
e_0^{i} = 0, & \quad 0 \leq i \leq M.
\end{align*}
\]  

Theorem 4. Suppose \( \varphi_i^{n} \) denotes the value of the solution \( u(x, t) \) of (1) at the mesh point \( (x_i, t_n) \) and let \( \varphi_i^{n} = (\varphi_i^{n}, \varphi_i^{n}, \ldots, \varphi_i^{n}) \) be the solution of the compact difference scheme (28), assume that the condition in Theorem 1 is satisfied. Then, when \( \tau \leq (1/4(2 - \alpha)^3) \), it holds
\[
\| \varphi^n - \varphi^n \| \leq C_2 (\tau^2 + h^4), \quad 1 \leq n \leq N,
\]
where
\[
C_2 = (\varepsilon_2^{*} \exp(4(2 - \alpha)(2 - \alpha)^2 + \alpha^2) \tau)^{1/2},
\]

\[
\varepsilon_2^{*} = 8(2 - \alpha)LT \varepsilon_R^{2} \left( 1 + \frac{2(2 - \alpha)(2T)^{1/2}}{\Gamma(2 - \alpha)} \right).
\]

Proof. Applying (79) and Theorem 3, we have
\[
\| e_i^{n} \|^2 \leq (2 - \alpha) G_R \exp(4(2 - \alpha)(2 - \alpha)^2 + \alpha^2) \tau, \quad 1 \leq n \leq N,
\]
where
\[
G_R = 8 \left( 1 + \frac{2(2 - \alpha)(2T)^{1/2}}{\Gamma(2 - \alpha)} \right) h^2 \| R_{\varphi_{\pm}}^{n} \|^{2} + 8 \tau \sum_{k=1}^{n} \| R_{\varphi_{\pm}}^{n} \|^{k}.
\]

Using Theorem 1, we obtain
\[
\| e_i^{n} \|^2 \leq C_2 \tau^2 (h^2 + h^4)^2.
\]

The proof is completed.

5. Applications and Numerical Results

In this section, we apply the proposed compact finite difference method to two model problems in the form (1)–(3). Let \( V_i^{n} = v(x_i, t_n) \) be the value of the solution \( v(x, t) \) of problem (1)–(3) at the mesh point \( (x_i, t_n) \). We have from (30) that
\[
\| V_i^{n} - v \| \leq C_3 (r^2 + h^4), \quad 1 \leq n \leq N, \quad (v = 1, 2, \infty),
\]
where \( C_3 \) is a positive constant. In this paper, we compute norm errors as follows:
\[
E_2 (\tau, h) = \max_{\theta \leq n \leq N} \| V_i^{n} - v \|, \\
E_3 (\tau, h) = \max_{\theta \leq n \leq N} \| V_i^{n} - v \|, \quad (v = 1, \infty).
\]

The temporal and spatial convergence orders are computed, respectively, by
\[
O_3' (\tau, h) = \log_2 \left( \frac{E_3 (2\tau, h)}{E_3 (\tau, h)} \right),
\]
\[
O_3' (\tau, h) = \log_2 \left( \frac{E_3 (2 h, 2 \tau)}{E_3 (\tau, h)} \right), \quad (v = 1, 2, \infty).
\]

Example 1. We consider a problem which is governed by equation (1) in \([0, \pi/4] \times [0, 1]\), and
\[
f (x, t) = 3\tau^2 \cos x \left( 1 + \frac{2(2 - \alpha)^{1/2}}{\Gamma(4 - \alpha)} \right) + t^3 \tau^2 \cos x + \cos x).
\]

The boundary and initial conditions are given by (2) and (3) with
\[
\phi_0 (t) = 3\tau^2 \left( 1 + \frac{2(2 - \alpha)^{1/2}}{\Gamma(4 - \alpha)} \right) + t^3 \tau^2 \cos x, \\
\phi_1 (t) = -3\tau^2 \left( 1 + \frac{2(2 - \alpha)^{1/2}}{\Gamma(4 - \alpha)} \right) + t^3 \tau^2 \cos x, \\
\varphi (x) = \tau^2 \cos x.
\]

It is easy to check that \( v(x, t) = t^3 \cos x \) is the solution of this problem.

At first, we let the spatial step \( h = \pi/400 \). Table 1 gives the errors \( E_3 (\tau, h) \) \((v = 1, 2, \infty)\) and the temporal convergence orders \( O_3' (\tau, h) \) \((v = 1, 2, \infty)\). Obviously, the computed solution \( v_i^{n} \) has the second-order temporal accuracy, and this coincides well with the analysis. For comparison, in Table 2,
we list the corresponding errors $E_\nu(\tau, h)(\nu = 1, 2, \infty)$ and temporal convergence order $O_k(\tau, h)(\nu = 1, 2, \infty)$ of the compact scheme (6) in [33]. This demonstrates that the scheme (6) in [33] has only the temporal accuracy of order $2 - \alpha$.

Next, we compute the spatial convergence order of the compact difference scheme (28). Table 3 presents the errors $E_\nu(\tau, h)(\nu = 1, 2, \infty)$ and the spatial convergence orders $O_k(\tau, h)(\nu = 1, 2, \infty)$ with the time step $\tau = 1/10000$. As expected, the compact difference scheme (28) has the fourth-order spatial accuracy. Figure 1 shows that the difference scheme (28) is more accurate with small $\alpha$.

**Example 2.** Consider equation (1) in the domain $[0, \pi] \times [0, 1]$, and

\[ f(x, t) = 3t^2(1 + x)^2 \left(1 + \frac{2t^{1-\alpha}}{\Gamma(4 - \alpha)}\right) - t^3(6x^2 + 4x + 2). \]

\[ (90) \]
Table 3: The errors and the spatial convergence orders of the compact difference scheme (28) for Example 1 (\(\tau = 1/10000\)).

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(h)</th>
<th>(E_1 (\tau, h))</th>
<th>(O_1 (\tau, h))</th>
<th>(E_2 (\tau, h))</th>
<th>(O_2 (\tau, h))</th>
<th>(E_{\infty} (\tau, h))</th>
<th>(O_{\infty} (\tau, h))</th>
</tr>
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<tr>
<td>(\pi/4)</td>
<td>4.604088e-03</td>
<td>4.604088e-03</td>
<td>3.680647e-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\pi/8)</td>
<td>2.841961e-04</td>
<td>2.841961e-04</td>
<td>2.401639e-04</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\pi/6)</td>
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<td>1.769697e-05</td>
<td>1.496024e-05</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\pi/32)</td>
<td>1.103983e-06</td>
<td>1.103983e-06</td>
<td>9.380629e-07</td>
<td></td>
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</tr>
<tr>
<td>(\pi/64)</td>
<td>6.794511e-07</td>
<td>6.794511e-08</td>
<td>5.769242e-08</td>
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<td></td>
</tr>
<tr>
<td>(\pi/4)</td>
<td>6.370119e-03</td>
<td>6.370119e-03</td>
<td>3.511266e-03</td>
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<tr>
<td>(\pi/8)</td>
<td>4.015891e-04</td>
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<td>2.319704e-04</td>
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<td>(\pi/16)</td>
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<tr>
<td>(\pi/32)</td>
<td>1.569810e-06</td>
<td>1.569810e-06</td>
<td>9.029468e-07</td>
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<tr>
<td>(\pi/64)</td>
<td>9.567678e-08</td>
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<td>5.493287e-08</td>
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</table>

Figure 1: Error planes for \(\tau = 1/20\) and \(h = \pi/100\) for different \(\alpha\). (a) \(\alpha = 0.3\). (b) \(\alpha = 0.5\). (c) \(\alpha = 0.7\). (d) \(\alpha = 0.9\).

The boundary and initial conditions are given by (2) and (3) with

\[
\phi_0 (t) = 3 t^2 \left( 1 + \frac{2 t^{1-\alpha}}{\Gamma (4 - \alpha)} \right) - 2 t^3,
\]

\[
\phi_1 (t) = 12 t^2 \left( 1 + \frac{2 t^{1-\alpha}}{\Gamma (4 - \alpha)} \right) - 12 t^3, \quad (91)
\]

\[
\varphi (x) = 1 + x^2.
\]

It is clear that \(v (x, t) = t^3 (1 + x^2)\) is the exact analytical solution of this problem.

In Table 4, we give the errors \(E_v (\tau, h)\) \((v = 1, 2, \infty)\) and the temporal convergence orders \(O_v (\tau, h)\) \((v = 1, 2, \infty)\) with the spatial step \(h = 1/200\). Obviously, the computed solution \(v_0^h\) has the second-order temporal accuracy.

Table 5 presents the errors \(E_v (\tau, h)\) \((v = 1, 2, \infty)\) and the spatial convergence orders \(O_v (\tau, h)\) \((v = 1, 2, \infty)\) with the time step \(\tau = 1/15000\). These results show that the compact difference scheme (28) has the fourth-order spatial accuracy. Figure 2 presents the probability density function \(u (x, t)\) and describes the anomalous sub-diffusive particle versus the anomalous diffusion exponent \(\alpha\).
### Table 4: The errors and the temporal convergence orders of the compact difference scheme (28) for Example 2 ($h = \pi/400$).

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\tau$</th>
<th>$E_1(\tau, h)$</th>
<th>$O_1(\tau, h)$</th>
<th>$E_2(\tau, h)$</th>
<th>$O_2(\tau, h)$</th>
<th>$E_\infty(\tau, h)$</th>
<th>$O_\infty(\tau, h)$</th>
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<td>1.20449e-02</td>
<td>6.018456e-02</td>
<td>2.45709e-02</td>
<td>3.29575e-02</td>
<td>1.97209</td>
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<td>1/20</td>
<td>1.53431e-02</td>
<td>9.96643e-02</td>
<td>1.99345e-02</td>
<td>1.99352e-02</td>
<td>3.29575e-02</td>
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<tr>
<td>1/40</td>
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<td>1.99357e-02</td>
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<td>3.98830</td>
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<td>1/640</td>
<td>1/1600</td>
<td>1.56463e-02</td>
<td>1.99884e-02</td>
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<td>1.10094e-02</td>
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### Table 5: The errors and the spatial convergence orders of the compact difference scheme (28) for Example 2 ($\tau = 1/15000$).

<table>
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<tr>
<th>$\alpha$</th>
<th>$h$</th>
<th>$E_1(\tau, h)$</th>
<th>$O_1(\tau, h)$</th>
<th>$E_2(\tau, h)$</th>
<th>$O_2(\tau, h)$</th>
<th>$E_\infty(\tau, h)$</th>
<th>$O_\infty(\tau, h)$</th>
</tr>
</thead>
<tbody>
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<td>$\pi/4$</td>
<td>1.66127e-02</td>
<td>8.93363e-03</td>
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<td>$\pi/32$</td>
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<td>3.99421e-05</td>
<td>4.01713e-05</td>
<td>4.01713e-05</td>
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<td>4.14682e-06</td>
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</tbody>
</table>

### Figure 2: Continued.
6. Conclusions

In this paper, the $L^2$-$L^2$ approximation formula is applied for the Caputo time-fractional derivative and a fourth-order compact finite difference approximation is used for the spatial derivative. The proposed method extends the known method in [29] to a class of more general time-fractional subdiffusion equations with spatially variable coefficients. We have proved that the proposed scheme is uniquely solvable, unconditionally stable, and convergent. We have also provided the optimal error estimates in the discrete $H^1$, $L^2$, and $L^{\infty}$ norms. The error estimates show that the proposed method has the second-order temporal accuracy and the fourth-order spatial accuracy, and thus improves the temporal accuracy of the method given in [33]. Numerical results confirm our analysis and demonstrate the efficiency of our method. Applying the proposed method to the time multiterm fractional subdiffusion equations will be our further work.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References


