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Research Article

Social Optimization and Pricing Strategies in Unobservable Queues with Delayed Multiple Vacations

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The paper considers the single-server Markovian queues with delayed multiple vacations. Once the system becomes empty, the server will experience a changeover time to begin the vacation. By using the matrix-geometric solution method, we obtain the stationary probability distribution and the mean queue length under almost unobservable and fully unobservable cases. Based on the system status information and a linear reward-cost structure reflecting the desire of customers for service and their unwillingness for waiting, the social welfare function is determined. We also investigate optimal pricing strategies of the server under the ex-postpayment scheme and the ex-antepayment scheme, and we get the customer's equilibrium strategy under optimal pricing strategy. Finally, we illustrate the effect of several parameters and information levels on socially optimal strategies and optimal social benefit by numerical examples.

1. Introduction

Over the past few decades, more and more people have become keen to study queueing systems from a game-theoretic perspective. This study of queueing models was initiated by Naor [1] and Edelson and Hildebrand [2] who studied the $M/M/1$ model with a simple linear reward-cost structure. They obtained customers' equilibrium and socially optimal strategies for observable and unobservable cases. From then on, many authors have investigated the same problem for various queueing systems incorporating diverse characteristics. The monograph written by Hassin and Haviv [3] introduced the main results and solution methodology about these models with extensive bibliographical references.

Queueing models with vacations have been developed as useful performance analysis tools for computer systems, communication networks, and flexible manufacturing systems. As to strategic behavior in the vacation queues, Burnetas and Economou [4] first studied a Markovian single-server queueing system with setup times. They derived equilibrium strategies for the customers under different levels of information and analyzed social benefit under these strategies. Then, Economou and Kanta [5] considered equilibrium threshold strategies in the fully observable and almost observable Markovian single-server queue with breakdowns and repairs. Guo and Hassin [6] studied the strategic behavior and social optimization in Markovian queues with N policy. This work was extended by Guo and Hassin [7] to heterogeneous customers. Sun et al. [8, 9] analyzed customers' equilibrium and socially optimal balking strategies in the observable and unobservable queues with three types of setup/closedown policies, respectively. Liu et al. [10] investigated equilibrium thresholds of customers in observable M/M/1 queueing systems with a single vacation. Zhang et al. [11] and Sun and Li [12] studied almost simultaneously equilibrium balking strategies in M/M/1 queues with multiple working vacations for different cases with respect to the system information. But Sun and Li [12] also observed socially optimal joining strategies and optimal social benefit. Subsequently, Sun et al. [13, 14] considered the customers' optimal balking behavior in some single-server Markovian queues with two-stage working vacations and double adaptive working vacations, respectively. Wang and Zhang [15] discussed the equilibrium strategies in the Markovian queues with a single working vacation. Then, Tian et al. [16] studied equilibrium and optimal strategies in M/M/1 queues with multiple working vacations and vacation interruptions under three different levels of system information. Doo Ho Lee [17] observed customer's equilibrium joining/balking behaviors in M/M/1 queues with a single working vacation and vacation interruptions. About comprehensive and excellent study on strategic queueing systems with vacations, readers are referred to chapter 10 in the book of Hassin [18].

As for the work studying pricing strategy and server profits in queueing systems, Sun et al. [19] studied price decisions of the two servers in equilibrium under competition in batch-arrival queues with complementary services. They introduced two pricing scheme: ex-postpayment (EPP) scheme and ex-antepayment (EAP) scheme. Under EPP scheme, the server imposes a price proportional to sojourn time after completing service, while under EAP scheme the server charges a fee before offering service. Ma and Liu [20] observed customers' equilibrium behavior and optimal pricing strategies in a discrete-time queue under EPP and EAP schemes. Doo ho Lee [21] dealt with optimal pricing strategies in almost and fully unobservable queues with negative customers. Moreover, Sun et al. [9] analyzed social optimization and profit maximization and they got optimal price for motivating customers to adopt socially optimal strategies. Wang and Zhang [22] investigated the optimal price that maximized server profits in a retrial queue with delayed vacations.

In this paper, we consider almost and fully unobservable queueing models with delayed multiple vacations. As to the almost unobservable case, arriving customer can only observe the state of the server and cannot observe the number of present customers before making decision, while for the fully unobservable case, a customer cannot observe the states of the system. The aim of this paper is to study the socially optimal strategies and optimal pricing strategies in the context of an unobservable M/M/1 queue with delayed multiple vacations.

This paper is organized as follows. Descriptions of the model are given in Section 2. In Sections 3 and 4, we determine social benefit and optimal pricing strategies in the almost unobservable and fully unobservable cases, respectively. In Section 5, some numerical examples are presented to illustrate the effect of several parameters on socially optimal strategies and optimal social benefit (server profits) in two cases. Finally, conclusions are given in Section 6.

2. Model Description

Consider a single-server M/M/1 queue with delayed multiple vacations where customers arrive according to a Poisson process with a rate λ. In a busy period, service times are supposed to be exponentially distributed with a rate μ. Upon the completion of service, if there is no customer in the system, the server will stay in the system for a period of time, called the delayed period or the changeover time. The delayed period follows an exponential distribution with parameter ξ. If a customer arrives during the delayed period, the server starts to serve the customer immediately. Otherwise, after the delayed period, the server will take a vacation immediately at the end of delayed period. If the system is still empty after a vacation, the server will take another vacation; otherwise he begins to server the customers. The vacation times are exponentially distributed with a rate θ.

We assume that the interarrival times, the service times, and the vacation times are mutually independent. In addition, the service discipline is first in first out (FIFO).

Denote by $N(t)$ the number of customers in the system at time t and let

$$J(t) = \begin{cases} 0, & \text{the server is on vacation period at time } t, \\ 1, & \text{the server is busy or in a delayed time at time } t. \end{cases} \quad (1)$$

Then, the process $\{(N(t), J(t)), t \geq 0\}$ is a two-dimensional continuous-time Markov chain with the state space $\Omega = \{(n, i) \mid n \geq 0, i = 0, 1\}$.

Arriving customers are assumed to be identical and they can decide whether to join or balk upon arrival. We assume that every customer receives a reward of R units after completing service and suffers a waiting cost of C units per time unit that the customer remains in the system. Then, a customer's utility consists of a reward for receiving service minus a waiting cost based on a linear cost structure. Customers are risk neutral and maximize their expected net benefit. Finally, we assume that there are no retrials of balking customers nor reneging of waiting customers.

In the following sections, we consider social benefit and server profit under two different cases.

3. Almost Unobservable Queues

In this section, we focus on the almost unobservable case, where arriving customers can only observe the state of the server at their arrival instant and do not know the number of customers present.

3.1. Social Optimization. There are four pure strategies available for a customer, i.e., to join the queue or not to join the queue at the state i $(i = 0, 1)$. A pure or mixed strategy can be described by a fraction q_i $(0 \leq q \leq 1)$, which is the probability of joining when the server is at the state i $(i = 0, 1)$, and the effective arrival rate of the arriving customers is λq_i. The transition diagram is illustrated in Figure 1.

Using the lexicographical sequence for the states, the transition rate matrix $\mathbf{Q}$ can be written as the tri-diagonal block matrix:

$$\mathbf{Q} = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{C} & & & \\ \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{C} & & \\ & \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{C} & \\ & & \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{C} \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (2)$$

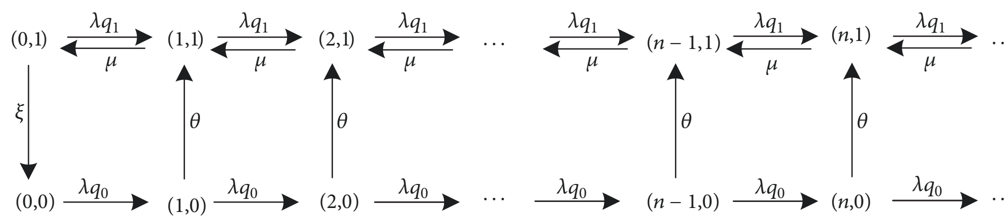

Figure 1: Transition rate diagram for the almost unobservable queue.

where

$$A_0=\begin{pmatrix}-\lambda q_0 & 0\\ \xi & -(\lambda q_1+\xi)\end{pmatrix},\quad B=\begin{pmatrix}0&0\\0&\mu\end{pmatrix},\quad A=\begin{pmatrix}-(\lambda q_0+\theta) & \theta\\ 0 & -(\lambda q_1+\mu)\end{pmatrix},\quad C=\begin{pmatrix}\lambda q_0 & 0\\ 0 & \lambda q_1\end{pmatrix}. \tag{3}$$

The structure of $\mathbf{Q}$ indicates that $\{(N(t),J(t)),t\geq 0\}$ is a quasibirth and death process (see Neuts [23]). To analyze this QBD process, we need to solve the matrix quadratic equation

$$\boldsymbol{R}^2\boldsymbol{B}+\boldsymbol{R}\boldsymbol{A}+\boldsymbol{C}=0, \tag{4}$$

for the minimal nonnegative solution and this solution is called the rate matrix and denoted by $\boldsymbol{R}$.

Because the coefficients of (4) are all upper triangular matrices, we can assume that $\boldsymbol{R}$ has the same structure as

$$\boldsymbol{R}=\begin{pmatrix}r_{11} & r_{12}\\ 0 & r_{22}\end{pmatrix}. \tag{5}$$

Substituting $\boldsymbol{R}^2$ and $\boldsymbol{R}$ into (4), we get the following results:

$$\boldsymbol{R}=\begin{pmatrix}\sigma_0 & \rho_0\\ 0 & \rho_1\end{pmatrix}, \tag{6}$$

and

$$\boldsymbol{R}^n=\begin{pmatrix}\sigma_0^n & \rho_0\sum_{j=0}^{n-1}\sigma_0^j\rho_1^{n-j-1}\\ 0 & \rho_1^n\end{pmatrix}, \tag{7}$$

where

$$\rho_0=\frac{\lambda q_0}{\mu},\quad \rho_1=\frac{\lambda q_1}{\mu},\quad \sigma_0=\frac{\lambda q_0}{\lambda q_0+\theta}. \tag{8}$$

Denote the stationary distribution as

$$\pi_{ni}=\lim_{t\to\infty}P\{N(t)=n,J(t)=i\},\quad (n,i)\in\Omega.$$
$$\boldsymbol{\pi}_n=(\pi_{n0},\pi_{n1}),\quad n\geq 0, \tag{9}$$

Theorem 1. *Assuming that $\rho_1<1$ and $\rho_1\neq\sigma_0$, the stationary probabilities $\{\pi_{ni}:(n,i)\in\Omega\}$ are as follows:*

$$\pi_{n0}=K\sigma_0^n,\quad n\geq 0, \tag{10}$$

$$\pi_{n1}=K\left(\rho_0\frac{\sigma_0^n-\rho_1^n}{\sigma_0-\rho_1}+\frac{\lambda q_0}{\xi}\rho_1^n\right),\quad n\geq 0, \tag{11}$$

where

$$K=(1-\sigma_0)(1-\rho_1)\left(1-\rho_1+\rho_0+\frac{\lambda q_0}{\xi}(1-\sigma_0)\right)^{-1}. \tag{12}$$

Proof. Using the matrix-geometric solution method, we have

$$\boldsymbol{\pi}_n=(\pi_{n0},\pi_{n1})=(\pi_{00},\pi_{01})\boldsymbol{R}^n,\quad n\geq 1, \tag{13}$$

and (π_{00},π_{01}) satisfies the set of equations

$$(\pi_{00},\pi_{01})B[\boldsymbol{R}]=0, \tag{14}$$

where $B[\boldsymbol{R}]=A_0+\boldsymbol{R}B$. Solving (14) and letting $\pi_{00}=K$, we get

$$(\pi_{00},\pi_{01})=K\left(1,\frac{\lambda q_0}{\xi}\right), \tag{15}$$

By (13) and (15), we obtain

$$\pi_{n0}=K\sigma_0^n,\quad n\geq 0,$$
$$\pi_{n1}=K\left(\rho_0\frac{\sigma_0^n-\rho_1^n}{\sigma_0-\rho_1}+\frac{\lambda q_0}{\xi}\rho_1^n\right),\quad n\geq 0, \tag{16}$$

Finally, $\pi_{00}=K$ can be determined by the normalization condition. □

From (10) and (11), we get the steady-state probability that the server is in state i $(i=0,1)$, denoted by p_i, as follows:

$$p_0=\sum_{n=0}^{\infty}\pi_{n0}=\frac{K}{1-\sigma_0}, \tag{17}$$

$$p_1=\sum_{n=0}^{\infty}\pi_{n1}=K\left(\frac{\rho_0}{(1-\sigma_0)(1-\rho_1)}+\frac{\lambda q_0}{\xi}\frac{1}{1-\rho_1}\right). \tag{18}$$

So the effective arrival rate is

$$\begin{aligned}\overline{\lambda} &= \lambda\left(p_0 q_0 + p_1 q_1\right) \\ &= \frac{\lambda q_0\left(\xi + \lambda q_1\left(1-\sigma_0\right)\right)}{\xi\left(1-\rho_1+\rho_0\right) + \lambda q_0\left(1-\sigma_0\right)} \\ &= \frac{\lambda q_0 \mu\left(\xi\left(\lambda q_0+\theta\right) + \lambda q_1 \theta\right)}{\xi\left(\lambda q_0+\theta\right)\left(\mu-\lambda q_1+\lambda q_0\right) + \lambda q_0 \mu\theta}.\end{aligned} \tag{19}$$

The probability generating function of the queue length, denoted by $L(z)$, is as follows:

$$\begin{aligned}L(z) &= \sum_{n=0}^{\infty} z^n\left(\pi_{n0}+\pi_{n1}\right) \\ &= K\left(\frac{1}{1-\sigma_0 z} + \frac{\rho_0 z}{\left(1-\sigma_0 z\right)\left(1-\rho_1 z\right)} + \frac{\lambda q_0}{\xi}\frac{1}{1-\rho_1 z}\right) \\ &= \frac{1-\rho_1}{1-\rho_1 z}\frac{K}{1-\rho_1}\left(\frac{1-\rho_1 z}{1-\sigma_0 z} + \frac{\rho_0 z}{1-\sigma_0 z} + \frac{\lambda q_0}{\xi}\right) \\ &= \frac{1-\rho_1}{1-\rho_1 z} \cdot \frac{K}{\left(1-\rho_1\right)\left(1-\sigma_0\right)}\left(\left(1-\rho_1 z+\rho_0 z\right)\frac{1-\sigma_0}{1-\sigma_0 z} + \frac{\lambda q_0}{\xi}\left(1-\sigma_0\right)\right).\end{aligned} \tag{20}$$

By the stochastic decomposition property, we get the mean number of the customers in the system, denoted by $L(q_0, q_1)$,

$$\begin{aligned}L\left(q_0,q_1\right) &= \frac{\rho_1}{1-\rho_1} + \frac{\xi\left(\rho_0-\rho_1+\sigma_0\right)}{\left(1-\sigma_0\right)\left(\xi\left(1-\rho_1+\rho_0\right)+\lambda q_0\left(1-\sigma_0\right)\right)} \\ &= \frac{\lambda q_1}{\mu-\lambda q_1} + \frac{\xi\left(\lambda q_0+\theta\right)\left(\left(\lambda q_0-\lambda q_1\right)\left(\lambda q_0+\theta\right)+\lambda q_0\mu\right)}{\theta\left(\xi\left(\lambda q_0+\theta\right)\left(\mu-\lambda q_1+\lambda q_0\right)+\lambda q_0\mu\theta\right)}.\end{aligned} \tag{21}$$

Hence, the mean sojourn time of a customer who decides to enter upon his arrival can be obtained by using Little's law

$$\begin{aligned}W\left(q_0,q_1\right) &= \frac{L\left(q_0,q_1\right)}{\overline{\lambda}} \\ &= \frac{\xi\sigma_0\left(1-\rho_1\right)^2 + \xi\rho_0\left(1-\sigma_0\rho_1\right) + \lambda q_0\rho_1\left(1-\sigma_0\right)^2}{\lambda q_0\left(1-\sigma_0\right)\left(1-\rho_1\right)\left(\xi+\lambda q_1\left(1-\sigma_0\right)\right)}.\end{aligned} \tag{22}$$

And the social benefit when all customers follow a mixed policy (q_0, q_1) can now be easily computed as follows:

$$\begin{aligned}S_{au}\left(q_0,q_1\right) &= \overline{\lambda}R - CL\left(q_0,q_1\right) = \overline{\lambda}\left(R - CW\left(q_0,q_1\right)\right) \\ &= \overline{\lambda}R - C\left(\frac{\lambda q_1}{\mu-\lambda q_1} + \frac{\xi\left(\lambda q_0+\theta\right)\left(\left(\lambda q_0-\lambda q_1\right)\left(\lambda q_0+\theta\right)+\lambda q_0\mu\right)}{\theta\left(\xi\left(\lambda q_0+\theta\right)\left(\mu-\lambda q_1+\lambda q_0\right)+\lambda q_0\mu\theta\right)}\right).\end{aligned} \tag{23}$$

The goal of a social planner is to maximize overall social welfare. Solving the nonlinear programming (NLP) problem max $S_{au}(q_0, q_1)$ (of course, $0 \leq q_i \leq 1$, $i = 0, 1$), we can obtain the socially optimal strategy (q_0^*, q_1^*) to maximize social welfare.

Let

$$\begin{aligned}f &= f\left(q_0,q_1\right) \\ &= \left(\lambda q_0+\theta\right)\left(\left(\lambda q_0-\lambda q_1\right)\left(\lambda q_0+\theta\right)+\lambda q_0\mu\right),\end{aligned} \tag{24}$$

$$\begin{aligned}g &= g\left(q_0,q_1\right) \\ &= \xi\left(\lambda q_0+\theta\right)\left(\mu-\lambda q_1+\lambda q_0\right)+\lambda q_0\mu\theta.\end{aligned} \tag{25}$$

Then, we have

$$\overline{\lambda} = \frac{\lambda q_0\mu\left(\xi\left(\lambda q_0+\theta\right)+\lambda q_1\theta\right)}{g\left(q_0,q_1\right)}, \tag{26}$$

$$L\left(q_0,q_1\right) = \frac{\lambda q_1}{\mu-\lambda q_1} + \frac{\xi f\left(q_0,q_1\right)}{\theta g\left(q_0,q_1\right)}. \tag{27}$$

So the social benefit $S_{au}(q_0, q_1)$ can be expressed as follows:

$$\begin{aligned}S_{au}\left(q_0,q_1\right) = {} & \frac{\lambda q_0\mu\left(\xi\left(\lambda q_0+\theta\right)+\lambda q_1\theta\right)}{g\left(q_0,q_1\right)}R \\ & - C\left(\frac{\lambda q_1}{\mu-\lambda q_1} + \frac{\xi f\left(q_0,q_1\right)}{\theta g\left(q_0,q_1\right)}\right).\end{aligned} \tag{28}$$

The first partial derivative of the function in (26) and (27) can be expressed as

$$\frac{\partial\overline{\lambda}}{\partial q_0} = \frac{\lambda\mu\xi\left(\mu-\lambda q_1\right)\left(\xi\left(\lambda q_0+\theta\right)^2+\lambda^2 q_0^2\theta+\lambda q_1\theta^2\right)}{g^2}, \tag{29}$$

$$\frac{\partial\overline{\lambda}}{\partial q_1} = \frac{\lambda^2 q_0\mu\left(\xi^2\left(\lambda q_0+\theta\right)^2+\theta\xi\left(\lambda q_0+\theta\right)\left(\lambda q_0+\mu\right)+\lambda q_0\mu\xi^2\right)}{g^2}, \tag{30}$$

$$\frac{\partial L}{\partial q_0} = \frac{\lambda\xi\left(g\left(\left(\lambda q_0+\theta\right)\left(3\lambda q_0+\theta+\mu-2\lambda q_1\right)+\lambda q_0\mu\right)-f\left(\xi\left(2\lambda q_0+\theta+\mu-\lambda q_1\right)+\mu\theta\right)\right)}{\theta g^2}, \tag{31}$$

$$\frac{\partial L}{\partial q_1} = \frac{\lambda\xi\left(\lambda q_0+\theta\right)\left(\xi f-g\left(\lambda q_0+\theta\right)\right)}{\theta g^2}. \tag{32}$$

Therefore, the first and two partial derivatives of $S_{au}(q_0, q_1)$ are given as

$$\begin{aligned}\frac{\partial S_{au}}{\partial q_0} &= R\frac{\partial\overline{\lambda}}{\partial q_0} - C\frac{\partial L}{\partial q_0},\\ \frac{\partial S_{au}}{\partial q_1} &= R\frac{\partial\overline{\lambda}}{\partial q_1} - C\frac{\partial L}{\partial q_1},\end{aligned} \tag{33}$$

$$D_1 = \frac{\partial^2 S_{au}}{\partial q_0^2} = R\frac{\partial^2\overline{\lambda}}{\partial q_0^2} - C\frac{\partial^2 L}{\partial q_0^2}, \tag{34}$$

$$D_2 = \frac{\partial^2 S_{au}}{\partial q_0\partial q_1} = \frac{\partial^2 S_{au}}{\partial q_1\partial q_0} = R\frac{\partial^2\overline{\lambda}}{\partial q_0\partial q_1} - C\frac{\partial^2 L}{\partial q_0\partial q_1}. \tag{35}$$

$$D_3 = \frac{\partial^2 S_{au}}{\partial q_1^2} = R\frac{\partial^2\overline{\lambda}}{\partial q_1^2} - C\frac{\partial^2 L}{\partial q_1^2}. \tag{36}$$

Since the objective function expression is more complex, we only provide a specific and sufficient condition to ensure convex programming, and in this case we get the best joining probability.

Theorem 2. *If $D_1 < 0$ and $D_1D_3 - D_2^2 > 0$, the optimization problem of $S_{au}(q_0, q_1)$ is a convex maximization problem and there exists a unique optimal joining strategy $q_i^* \in [0,1]$, $i = 0, 1$.*

Proof. The optimization problem of $S_{au}(q_0, q_1)$ has three constraints: $0 \le q_i \le 1$ $(i = 0, 1)$ and $\lambda q_1 < \mu$, and they are all real-valued linear functions. Therefore, the set of constraints is convex.

Let $\boldsymbol{H}(q_0, q_1)$ be the Hessian matrix of the objective function, and it has the form of

$$\boldsymbol{H}(q_0, q_1) = \begin{pmatrix} \dfrac{\partial^2 S_{au}}{\partial q_0^2} & \dfrac{\partial^2 S_{au}}{\partial q_0\partial q_1} \\ \dfrac{\partial^2 S_{au}}{\partial q_1\partial q_0} & \dfrac{\partial^2 S_{au}}{\partial q_1^2} \end{pmatrix} = \begin{pmatrix} D_1 & D_2 \\ D_2 & D_3 \end{pmatrix}. \tag{37}$$

According to the definition of the convex maximization in Boyd and Vandenberghe [24], if $D_1 < 0$ and $D_1D_3 - D_2^2 > 0$, $\boldsymbol{H}(q_0, q_1)$ is negative definite and the objective function $S_{au}(q_0, q_1)$ should be strictly concave in the feasible region, which leads to the optimization problem of $S_{au}(q_0, q_1)$ being a convex maximization problem. One property of convex programming is that the local optimal solution is the global optimal solution.

According to the discussion above, we can use Lagrange multiplier method, Karush-Kuhn-Tucker conditions, or Newton method to find the optimal solutions. However, the explicit form of the optimal $q_i^* \in [0,1]$ $(i = 0, 1)$ is too long and complicated; we only introduce the method and give numerical results later. Details about two above methods can be found in Boyd and Vandenberghe [24]. This completes the proof. □

Figure 2 shows that social benefit $S_{au}(q_0, q_1)$ is strictly concave and takes the maximum. We also obtain $(q_0^*, q_1^*) = (0.8, 0.75)$.

3.2. Profit Maximization. Now, we consider a monopolistic server that sets a profit-maximizing price under EPP and EAP schemes.

(1) EPP scheme model. We first consider the EPP scheme model in which the server sets a fee proportional to the customer's sojourn time. Let T be the fee charged by the server and P_t be the expected server profit per time unit. In this case, we have the following results regarding the customer's utility.

Lemma 3. *In the almost unobservable queue, the expected utility of an ordinary customer who decides to join the system is $U_t = R - (C + T)W(q_0, q_1)$.*

Proof. Let W_i be the conditional expected sojourn time of an ordinary customer who decides to join at the state i $(i = 0, 1)$. Then, we have $W(q_0, q_1) = p_0W_0 + p_1W_1$. And the conditional expected benefit of such a customer who decides to enter the system at the state i is $U_i = U_i(q_0, q_1) = R - (C + T)W_i(q_0, q_1)$ $(i = 0, 1)$.

So for any customer who is willing to enter the system, his expected utility is

$$\begin{aligned}U_t &= p_0U_0 + p_1U_1\\ &= R - (C+T)\,W\left(q_0, q_1\right).\end{aligned} \tag{38}$$

At customer's equilibrium, if $U_i = 0$ $(i = 0, 1)$, we also have $U_t = 0$ and obtain $R = (C + T)W(q_0, q_1)$. Hence, at

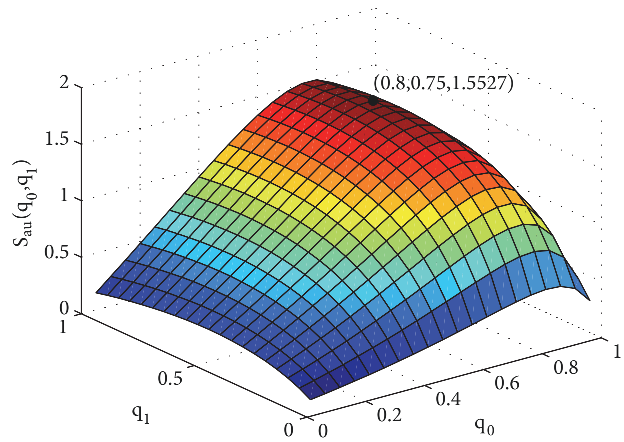

FIGURE 2: Social benefit of almost unobservable queue for $R = 3, C = 1, \lambda = 1.2, \mu = 1.5, \theta = 0.8, \xi = 1$.

customer's equilibrium, we find that the toll is a function of the customer's joining probabilities, denoted by q_{0t} and q_{1t}

$$T = \frac{R}{W(q_{0t}, q_{1t})} - C. \tag{39}$$

For the system with delayed multiple vacations, the monopoly's problem is a nonlinear programming problem to maximize P_t with q_{0t} and q_{1t}:

$$P_t = \overline{\lambda} T W(q_{0t}, q_{1t}) = \overline{\lambda} R - C L(q_{0t}, q_{1t}). \tag{40}$$

Here, q_0 and q_1 in $\overline{\lambda}$ are replaced by q_{0t} and q_{1t}. Then the objectives of a monopolistic server and the society are consistent with each other, so we can induce the socially optimal joining probabilities (q_0^*, q_1^*) by an appropriate price, which also maximizes profit of the server.

Moreover, as mentioned above, if the server chooses the optimal pricing strategy, the expected benefit of the customer equals

$$\begin{aligned} U_t &= R - (C + T^*) W(q_0, q_1) \\ &= R - \frac{R}{W(q_0^*, q_1^*)} W(q_0, q_1). \end{aligned} \tag{41}$$

□

Based on the above analysis, we can give the following theorem.

Theorem 4. *In the EPP scheme model, if U_0 and U_1 have the same sign, there exists a unique equilibrium strategy where customers join the queue with probabilities q_0^* and q_1^* and the optimal price $T^* = R/W(q_0^*, q_1^*) - C$, and the server's maximal profit is $P_t^* = S_{au}(q_0^*, q_1^*)$.*

Proof. Using the standard methodology of equilibrium analysis, we derive the following equilibria.

If $R \le (C + T^*)W(0,0)$, that is, $U_t \le 0$: Because U_0 and U_1 have the same sign, so we have $U_0 \le 0$ and $U_1 \le 0$. In this case, even if no other customer joins, the residual utility of a customer who joins is less than or equal to 0. Therefore, $q_0^e = q_1^e = 0$ is an equilibrium strategy. Moreover, in this case, balking is a dominant strategy.

If $R \ge (C + T^*)W(1,1)$, that is, $U_t \ge 0$, $U_0 \ge 0$ and $U_1 \ge 0$: In this case, for every strategy of the other customers, the tagged customer has a positive expected net benefit if he decides to enter. Hence, the equilibrium strategy $q_0^e = q_1^e = 1$.

If $(C + T^*)W(0,0) < R < (C + T^*)W(1,1)$: In this case, if all customers who find the system empty enter the system with probability $q_0^e = q_1^e = 1$, then the tagged customer suffers a negative expected benefit if he decides to enter. Hence, $q_0^e = q_1^e = 1$ does not lead to an equilibrium. Similarly, if all customers use $q_0^e = q_1^e = 0$ then the tagged customer receives a positive benefit from entering; thus $q_0^e = q_1^e = 0$ also cannot be part of an equilibrium mixed strategy. Therefore, there exists a unique $q_0^e = q_1^e$ satisfying $U_t = 0$, $U_0 = 0$, and $U_1 = 0$. From (41), it can be seen that (q_0^*, q_1^*) is the zero point of the function U_t. In other words, they are also customers' equilibrium strategy. □

(2) EAP scheme model. Under the EAP scheme model, the server imposes a flat entry fee F for all those who decide to join. Let P_f be the server's expected profit per time unit. The expected benefit for a customer is equal to $U_f = R - F - CW(q_0, q_1)$. We can obtain customers' equilibrium when $U_f = 0$, and we obtain $R = F + CW(q_0, q_1)$. At customer's equilibrium, F is a function of q_{0f} and q_{1f} and has the form of $F = R - CW(q_{0f}, q_{1f})$.

Similar to the EPP scheme, we establish the following NLP problem to maximize P_f with respect to q_{0f} and q_{1f}:

$$P_f = \overline{\lambda} F = \overline{\lambda} R - C L(q_{0f}, q_{1f}). \tag{42}$$

And we also have the following theorem.

Theorem 5. *In the EAP scheme model, if U_0 and U_1 have the same sign, there exists a unique equilibrium strategy where customers join the queue with probabilities q_0^* and q_1^* and the optimal price $F^* = R - W(q_0^*, q_1^*)$ and the server's maximal profit is $P_f^* = S_{au}(q_0^*, q_1^*)$.*

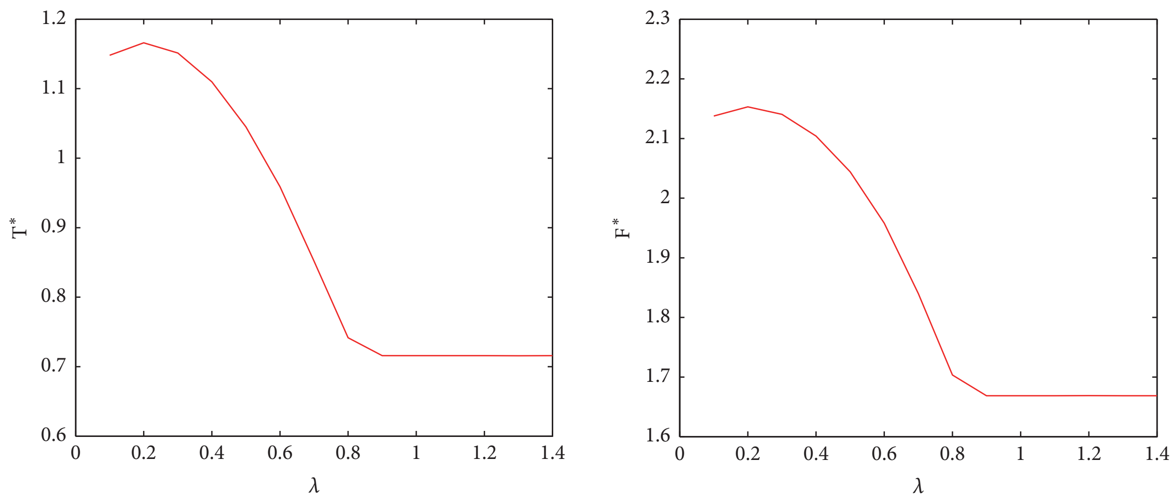

FIGURE 3: Monopoly price T^* and F^* for $R = 4, C = 1, \mu = 1.5, \theta = 0.8, \xi = 1$.

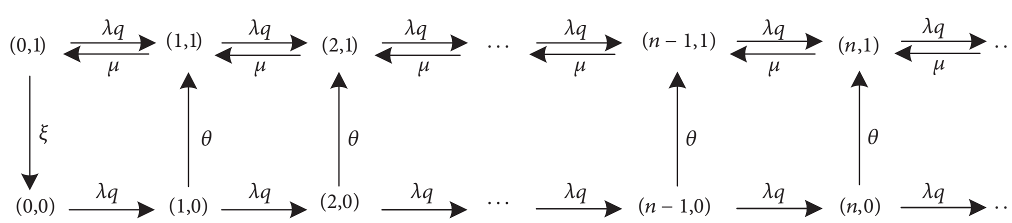

FIGURE 4: Transition rate diagram for the fully unobservable queues.

From the above analysis, we can see that the server can make the equilibrium strategy and the socially optimal strategy unified by setting the appropriate price.

Figure 3 shows that optimal prices T^* and F^* first increase when the arrival rate λ is very low and then decrease. Therefore, an increase in demand may result in a reduction in price.

4. Fully Unobservable Queues

In this section, arriving customers can observe neither the state of the server at their arrival instant nor the number of customers present.

4.1. Social Optimization. There are two pure strategies available for a customer, i.e., to join the queue or not to join the queue. A strategy can be described by a joining probability q $(0 \leq q \leq 1)$, and the joining rate is λq. The transition diagram is illustrated in Figure 4.

The stationary distribution of the system when all customers follow a given strategy q can be obtained by taking $q_0 = q_1 = q$ in the almost unobservable queue. So we have

$$\pi_{n0} = K\sigma^n, \quad n \geq 0,$$
$$\pi_{n1} = K\left(\rho\frac{\sigma^n - \rho^n}{\sigma - \rho} + \frac{\lambda q}{\xi}\rho^n\right), \quad n \geq 0, \tag{43}$$

where

$$\rho = \frac{\lambda q}{\mu},$$
$$\sigma = \frac{\lambda q}{\lambda q + \theta}, \tag{44}$$
$$K = \frac{\xi(1-\rho)(1-\sigma)}{\xi + \lambda q(1-\sigma)}.$$

From (43), we get the mean queue length, denoted by $L(q)$, as follows:

$$L(q) = \frac{\lambda q}{\mu - \lambda q} + \frac{\xi\lambda q(\lambda q + \theta)}{\theta(\xi(\lambda q + \theta) + \lambda q\theta)}. \tag{45}$$

By using Little's law, the mean sojourn time of a customer who decides to enter upon his arrival is as follows:

$$W(q) = \frac{1}{\lambda q}L(q) = \frac{1}{\mu - \lambda q} + \frac{\xi(\lambda q + \theta)}{\theta(\xi(\lambda q + \theta) + \lambda q\theta)}. \tag{46}$$

So the social benefit per time unit can now be easily computed as

$$S_{fu}(q) = \lambda q(R - CW(q))$$
$$= \lambda qR$$
$$- C\left(\frac{\lambda q}{\mu - \lambda q} + \frac{\xi\lambda q(\lambda q + \theta)}{\theta(\xi(\lambda q + \theta) + \lambda q\theta)}\right). \tag{47}$$

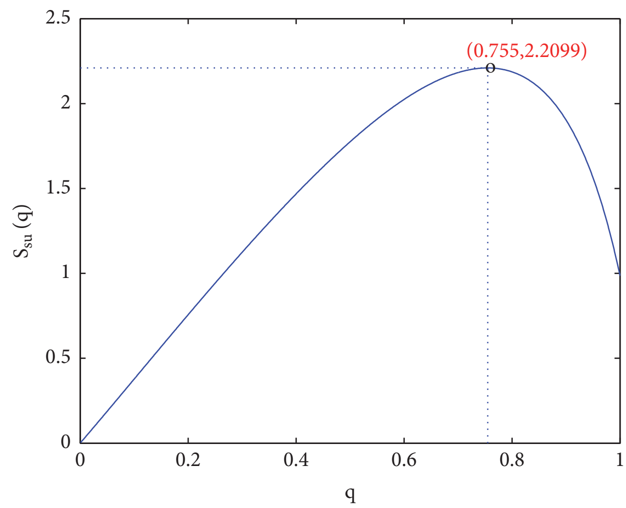

FIGURE 5: Social benefit of unobservable case when $R = 5, C = 1, \lambda = 1.2, \mu_1 = 1.5, \xi = 1, \theta = 0.8$.

Let q^* denote the socially optimal joining probability q^*, and let x^* be the root of the first-order optimal condition $S'_{fu}(q) = 0$. They have the following relation.

Theorem 6. *If $0 < x^* < 1$ and $S''_{fu}(q) \leq 0$, then $q^* = x^*$; if $0 < x^* < 1$ and $S''_{fu}(q) > 0$ or $x^* \geq 1$, then $q^* = 1$.*

In the analysis below, we will provide a specific and sufficient condition to ensure that the function in (47) is unimodal and strictly convex, and under this condition we obtain the optimal joining probability.

Theorem 7. *Suppose $\rho < 1$. If $0 \leq q \leq 1$ and $\lambda^3 q^3 - \mu^3 + 3\lambda q\mu(\lambda q + \mu) - \theta^3 \geq 0$, then*

(1) the social benefit $S_{fu}(q)$ is unimodal and strictly convex in $[q_1, q_2]$, where

$$[q_1, q_2] = [0, 1] \cap \{q \mid \lambda^3 q^3 - \mu^3 + 3\lambda q\mu(\lambda q + \mu) - \theta^3 \geq 0\}; \tag{48}$$

(2) if $0 < x^* < 1$, then $q^* = x^*$. Otherwise, $q^* = 1$.

Proof. Let $\lambda_q = \lambda q$. The first two derivatives of the mean sojourn time $W(\lambda_q)$ are given by

$$W'(\lambda_q) = \frac{1}{(\mu - \lambda_q)^2} - \frac{\theta\xi}{(\lambda_q(\theta + \xi) + \theta\xi)^2}, \tag{49}$$

and

$$W''(\lambda_q) = \frac{2}{(\mu - \lambda_q)^3} + \frac{2\theta\xi(\theta + \xi)}{(\lambda_q(\theta + \xi) + \theta\xi)^3}. \tag{50}$$

The second-order derivative of the social benefit is

$$S''_{fu}(\lambda_q) = -2CW'(\lambda_q) - C\lambda_q W''(\lambda_q) = -\frac{2C\mu}{(\mu - \lambda_q)^3} - \frac{2C\theta^2\xi^2}{(\theta^2 + \theta\lambda_q + \lambda_q^2)^3}. \tag{51}$$

Thus, if $\lambda^3 q^3 - \mu^3 + 3\lambda q\mu(\lambda q + \mu) - \theta^3 \geq 0$, then $S''_{fu}(\lambda_q) < 0$, and the social benefit $S_{fu}(q)$ is unimodal and strictly convex in $[q_1, q_2]$. □

In Figure 5, we observe that $S_{fu}(q)$ first increases and then decreases with q and have a maximum value at $q = 0.755 < 1$. Hence, we have $q^* = 0.755$.

4.2. Profit Maximization. Under the EPP scheme, the price proportional to the customer's waiting time is charged by the server. Thus, the expected residual utility of a customer is $U_t = R-(C+T)W(q)$. If $U_t = 0$, we have $R = (C+T)W(q)$. Hence, at customer's equilibrium, the price is expressed with the joining probability q_t,

$$T = \frac{R}{W(q_t)} - C. \tag{52}$$

The monopoly's problem is a NLP problem to maximize P_t with q_t:

$$P_t = \lambda q_t TW(q_t) = \lambda q_t R - CL(q_t). \tag{53}$$

This is also a problem of maximizing social benefit. The price that induces social optimization and profit maximization equals $T^* = R/W(q^*) - C$, and the server's maximal profit is $P_t^* = S_{fu}(q^*)$. What is more, under the optimal price, q^* is also an equilibrium strategy.

In the EAP scheme model where the server imposes a flat fee for services, the expected benefit for a customer is equal to $U_f = R - F - CW(q)$. When $U_f = 0$, we obtain customers' equilibrium strategies and $R = F + CW(q)$. At customers' equilibrium, F is a function of joining probability q_f and has

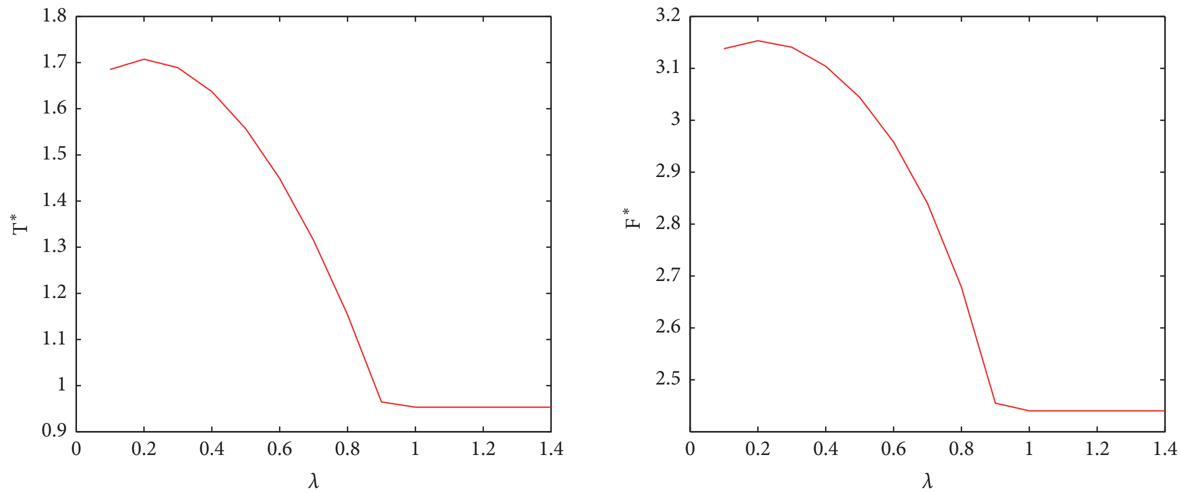

Figure 6: Monopoly price T^* and F^* for $R = 5, C = 1, \mu = 1.5, \theta = 0.8, \xi = 1$.

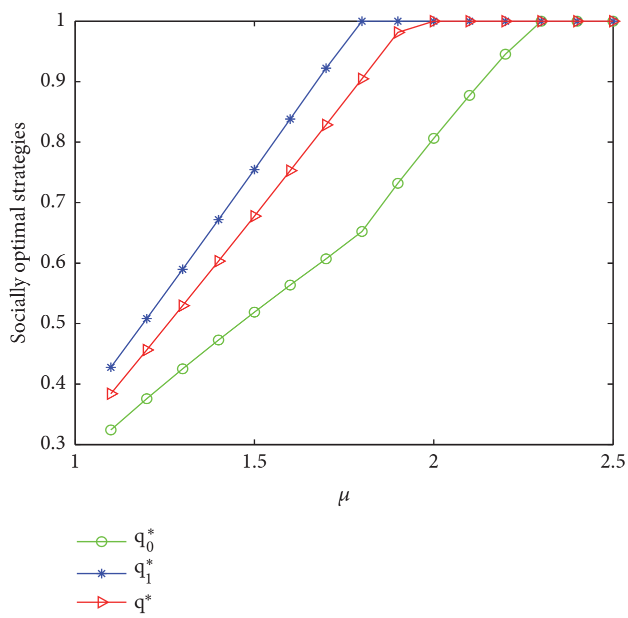

Figure 7: Socially optimal strategies when $R = 3, C = 1, \lambda = 1, \theta = 0.8, \xi = 1$.

the form of $F = R - CW(q_f)$. The monopoly's problem is to maximize $P_f = \lambda q_f F = \lambda q_f R - CL(q_f)$.

So the optimal price is $F^* = R - W(q^*)$, and the server's maximal profit is $P_f^* = S_{fu}(q^*)$. This price makes customers' equilibrium strategy consistent with the socially optimal strategy.

Figure 6 shows the socially optimal and monopolistic price for the unobservable queue.

From the above analysis, we can observe that the EPP and EAP mechanisms do not affect the customer equilibrium joining rate and the server's maximum profits.

5. Numerical Examples

In this section, we present some numerical analysis to investigate the sensitivity of socially optimal joining probabilities for customers and optimal social benefit (expected server profits) on system parameters.

We first explore the sensitivity of the socially optimal entrance probabilities. From Figures 7–10, we can make the following observations of socially optimal joining strategies for the almost and fully unobservable cases. The optimal probability in the fully unobservable queue is always between two optimal joining strategies in the almost unobservable case. However, the relative ordering of q_0^* and q_1^* cannot be determined. Therefore, when customers cannot observe the information about server's state, they enter the queue with an intermediate strategy between those that the customers use when they are given the server's state. As for the sensitivity about the system's parameters, we observe that socially optimal strategies are first increasing with respect to service rate μ in Figure 7, and when μ continues to increase all

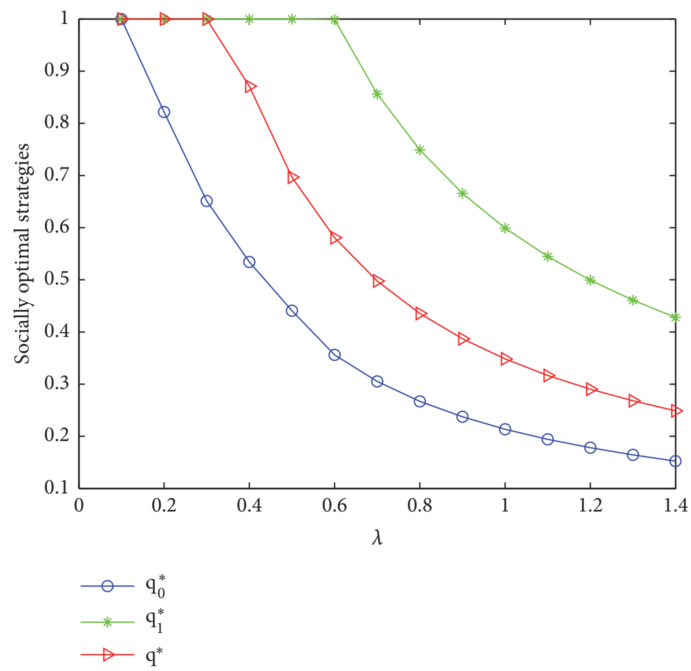

FIGURE 8: Socially optimal strategies when $R = 2, C = 1, \mu = 1.5, \theta = 0.8, \xi = 1$.

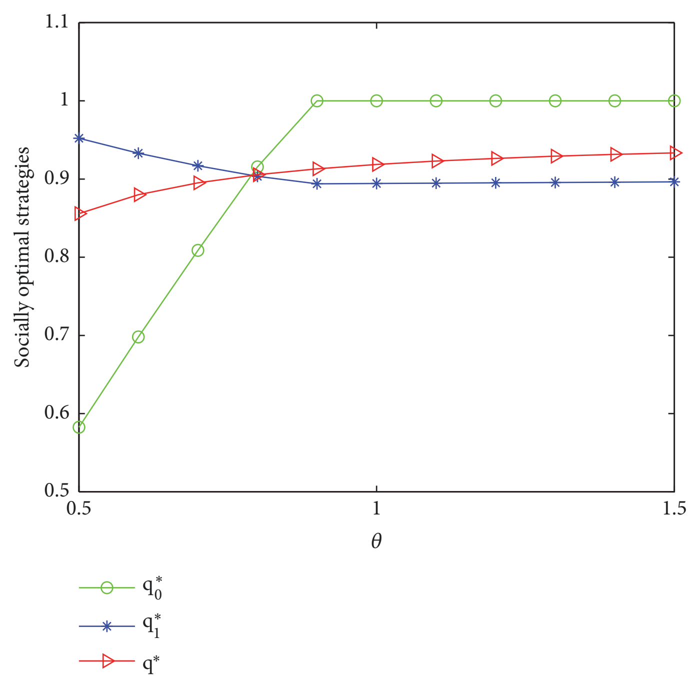

FIGURE 9: Socially optimal strategies when $R = 5, C = 1, \lambda = 1, \mu = 1.5, \xi = 1$.

probabilities are equal to 1. It is intuitive. That is, regardless of whether the customer has status information for the server, an increase in the service rate will reduce the expected waiting time of the customer. Figure 8 shows that optimal strategies decrease with arrival rate λ. This is because the system is more congested and an arriving customer is less willing to join the system when λ increases. In Figures 9 and 10, the behavior of optimal probabilities varies. While for the most part they are all increasing with θ, which is intuitive, there is a range of small values of θ in which q_1^* is decreasing. When the vacation time becomes shorter, the customer will be more willing to enter in vacation, which will lead to system congestion in the busy period. When ξ increases, the server can make the vacation faster. This may result in too many customers entering the system during the vacation period and the customers who arrive on busy period are less willing to enter.

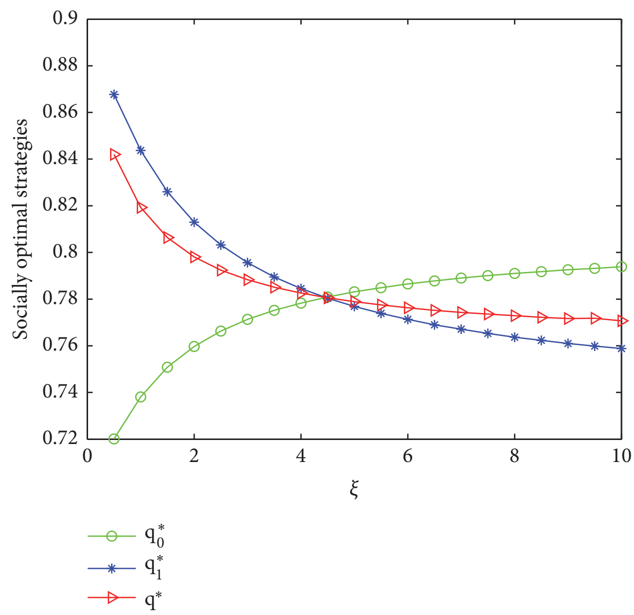

Figure 10: Socially optimal strategies when $R = 4, C = 1, \lambda = 1, \mu = 1.5, \theta = 0.8$.

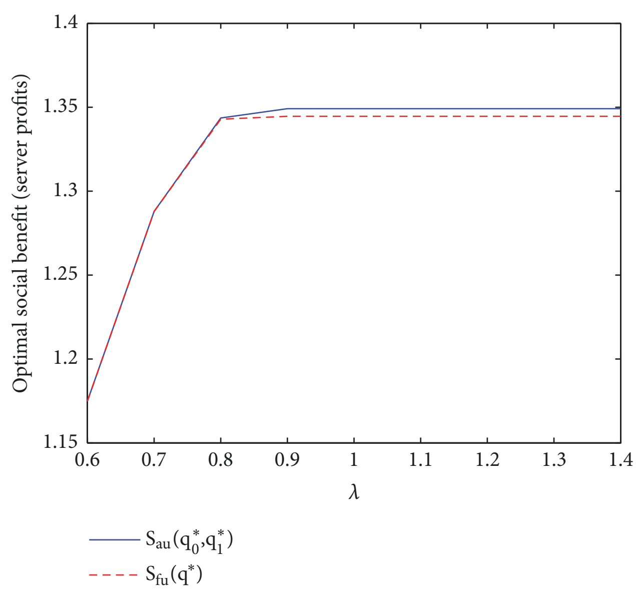

Figure 11: Optimal social benefit when $R = 4, C = 1, \mu = 1.5, \theta = 0.8, \xi = 1$.

Subsequently, we turn to the social benefit or server profit under optimal strategy for two information levels. In Figure 11, the social benefit or server profit first monotonously increases and then keeps stable when λ exceeds the certain value. The reason is that customers' waiting time is shorter when λ is small, which makes the social welfare improve. And when the arrival rate becomes larger, the customer's maximum arrival rate tends to be stable. Figures 12 and 13 show that the social benefit or server profit is always increasing with service rate μ and vacation rate θ, which is intuitive. This is because the utility of individual customer becomes larger and the social benefits are also improved. Figure 14 shows that the social benefit or server profit decreases with respect to ξ. This is because shorter delayed time results in too many customers entering the vacation, which leads to system congestion and reduces social benefits. From these figures we find that $S_{fu}(q^*) \leq S_{au}(q_0^*, q_1^*)$. In order to get more social benefits, the social planner can appropriately disclose the status information of the server to the customers.

Finally, we are concerned with the delayed period which is the special feature of this paper. Taking the fully unobservable queue as an example, comparison work is carried out to show which mechanism, multiple vacations or delayed multiple vacations, is better in controlling the mean queue length and improving the social welfare. Figure 15 shows that mean queue length $L(q)$ is increasing with respect to q, and the

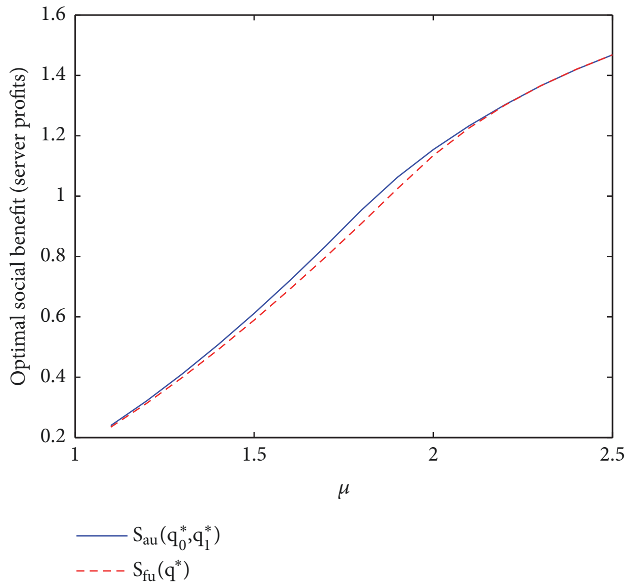

Figure 12: Optimal social benefit when $R = 3, C = 1, \lambda = 1, \theta = 0.8, \xi = 1$.

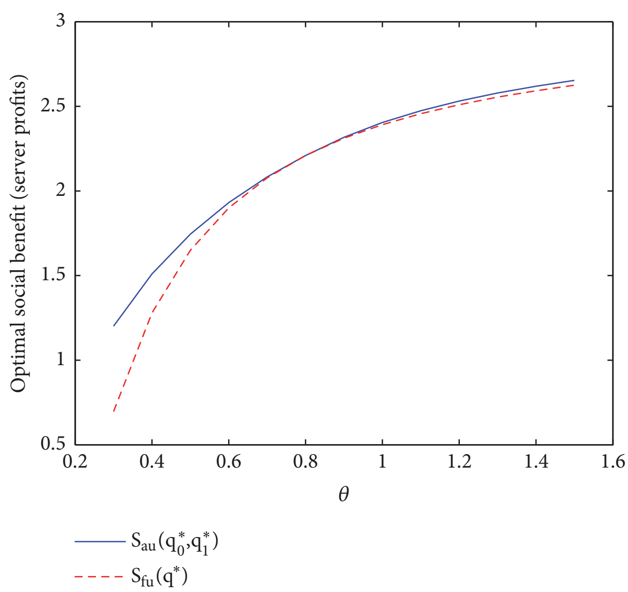

Figure 13: Optimal social benefit when $R = 5, C = 1, \lambda = 1, \mu = 1.5, \xi = 1$.

social benefit $S_{fu}(q)$ first increases and then decreases with q and has a maximum in Figure 16. On the other hand, Figures 15 and 16 show that the mean queue length for queues with vacations is always larger than that for queues with delayed vacations, and the social benefit for queues with delayed vacations is always greater than that for queues with vacations. Because in queues with vacations, when the system is empty, the server immediately begins to take a vacation and customers suffer a longer waiting time and a greater waiting cost, while in queues with delayed vacations. Some of the customers who arrived after the system was empty are served during the delayed time. All in all, delayed vacation somewhat relieves the system congestion problem and increases social benefits.

6. Conclusion

In this paper, we discuss the socially optimal joining probabilities and optimal pricing strategies of the server in the almost and fully unobservable M/M/1 queueing systems with delayed multiple vacations. We first obtain the steady-state performance measures and social benefit in two different cases. Meanwhile, we assume that the server can choose the optimal pricing strategies that maximize the monopoly profit under ex-postpayment (EPP) scheme and ex-antepayment (EAP) scheme. Moreover, the servers' maximum profits in the two pricing schemes are also identical and have equal optimal social benefit. Finally, we compared socially optimal strategy and optimal social

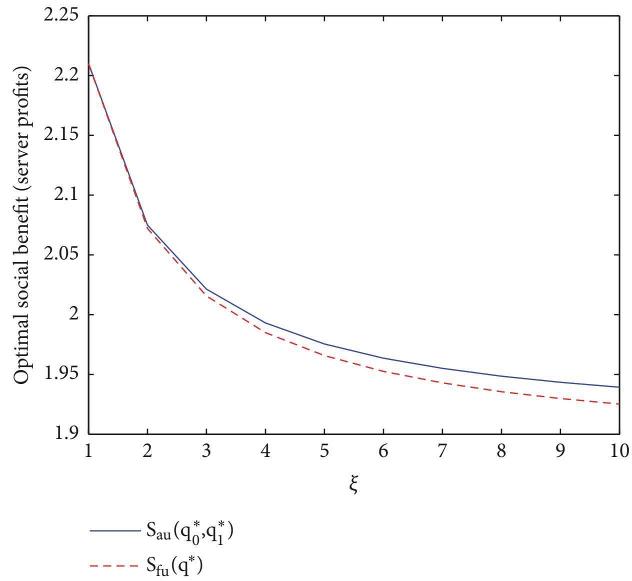

Figure 14: Optimal social benefit when $R = 5, C = 1, \lambda = 1, \mu = 1.5, \theta = 0.8$.

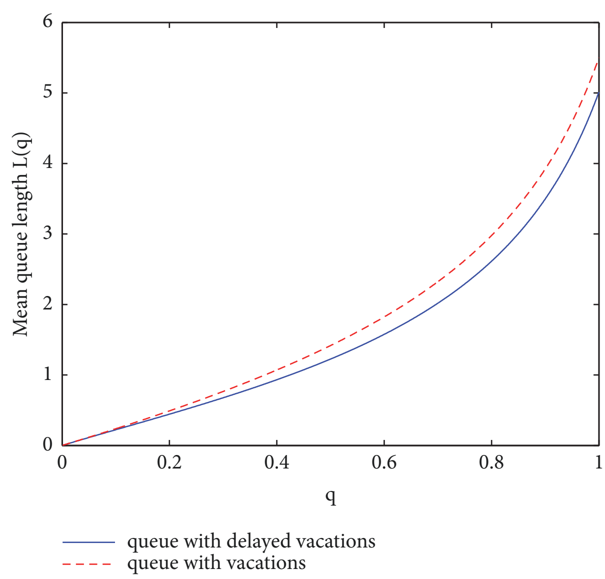

Figure 15: Mean queue length $L(q)$ of unobservable queues with delayed vacations and queues with vacations when $R = 5, C = 1, \lambda = 1.2, \mu = 1.5, \theta = 0.8, \xi = 1$.

benefit for almost and fully unobservable cases numerically, and we observed that the disclosure of system information had certain benefits for social planner. Furthermore, the direct generalization is the study of the corresponding non-Markovian queues. One can also study the pricing strategies for other queues with sojourn time-dependent reward.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The author declares that they have no conflicts of interest.

Acknowledgments

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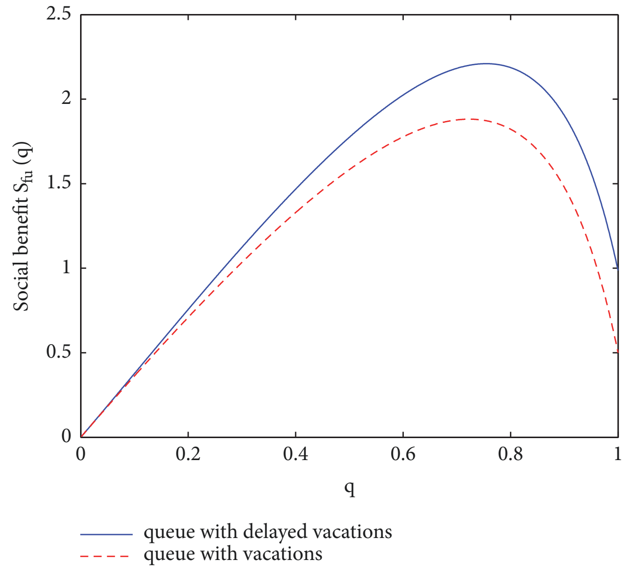

Figure 16: Social benefit of unobservable queues with delayed vacations and queues with vacations when $R = 5, C = 1, \lambda = 1.2, \mu = 1.5, \theta = 0.8, \xi = 1$.

References

[1] P. Naor, "The regulation of queue size by levying tolls," *Econometrica*, vol. 37, no. 1, pp. 15–24, 1969.

[2] N. M. Edelson and D. K. Hildebrand, "Congestion tolls for Poisson queueing processes," *Econometrica*, vol. 43, no. 1, pp. 81–92, 1975.

[3] R. Hassin and M. Haviv, *Equilibrium Behavior in Queueing Systems: To Queue or Not to Queue*, Kluwer Academic, Boston, Mass, USA, 2003.

[4] A. Burnetas and A. Economou, "Equilibrium customer strategies in a single server Markovian queue with setup times," *Queueing Systems*, vol. 56, no. 3-4, pp. 213–228, 2007.

[5] A. Economou and S. Kanta, "Equillibrium balking strategies in the observable single-server queue with breakdowns and repairs," *Operations Research Letters*, vol. 36, no. 6, pp. 696–699, 2008.

[6] P. Guo and R. Hassin, "Strategic behavior and social optimization in Markovian vacation queues," *Operations Research*, vol. 59, no. 4, pp. 986–997, 2011.

[7] P. Guo and R. Hassin, "Strategic behavior and social optimization in Markovian vacation queues: the case of heterogeneous customers," *European Journal of Operational Research*, vol. 222, no. 2, pp. 278–286, 2012.

[8] W. Sun, P. Guo, and N. Tian, "Equilibrium threshold strategies in observable queueing systems with setup/closedown times," *Central European Journal of Operations Research*, vol. 18, no. 3, pp. 241–268, 2010.

[9] W. Sun, Y. Wang, and N. Tian, "Pricing and setup/closedown policies in unobservable queues with strategic customers," *4OR*, vol. 10, no. 3, pp. 287–311, 2012.

[10] W. Liu, Y. Ma, and J. Li, "Equilibrium threshold strategies in observable queueing systems under single vacation policy," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 36, no. 12, pp. 6186–6202, 2012.

[11] F. Zhang, J. Wang, and B. Liu, "Equilibrium balking strategies in Markovian queues with working vacations," *Applied Mathematical Modelling*, vol. 37, no. 16-17, pp. 8264–8282, 2013.

[12] W. Sun and S. Li, "Equilibrium and optimal behavior of customers in Markovian queues with multiple working vacations," *TOP*, vol. 22, no. 2, pp. 694–715, 2014.

[13] W. Sun, S. Li, and Q.-L. Li, "Equilibrium balking strategies of customers in Markovian queues with two-stage working vacations," *Applied Mathematics and Computation*, vol. 248, pp. 195–214, 2014.

[14] W. Sun, S. Li, and N. Tian, "Equilibrium and optimal balking strategies of customers in unobservable queues with double adaptive working vacations," *Quality Technology and Quantitative Management*, vol. 14, no. 1, pp. 94–113, 2017.

[15] F. Wang, J. Wang, and F. Zhang, "Equilibrium customer strategies in the GEO/GEO/1 queue with single working vacation," *Discrete Dynamics in Nature and Society*, vol. 2014, Article ID 309489, 9 pages, 2014.

[16] R. Tian, L. Hu, and X. Wu, "Equilibrium and optimal strategies in M/M/1 queues with working vacations and vacation interruptions," *Mathematical Problems in Engineering*, vol. 2016, Article ID 9746962, 10 pages, 2016.

[17] D. H. Lee, "Equilibrium balking strategies in Markovian queues with a single working vacation and vacation interruption," *Quality Technology & Quantitative Management*, pp. 1–22, 2018.

[18] R. Hassin, *Rational Queueing*, Chapman and Hall/CRC, New York, NY, USA, 2016.

[19] W. Sun, S. Li, N. Tian, and H. Zhang, "Equilibrium analysis in batch-arrival queues with complementary services," *Applied Mathematical Modelling*, vol. 33, no. 1, pp. 224–241, 2009.

[20] Y. Ma and Z. Liu, "Pricing analysis in Geo/Geo/1 queueing system," *Mathematical Problems in Engineering*, vol. 2015, Article ID 181653, 5 pages, 2015.

[21] D. H. Lee, "Optimal pricing strategies and customers' equilibrium behavior in an unobservable M/M/1 Queueing system with negative customers and repair," *Mathematical Problems in Engineering*, vol. 2017, Article ID 8910819, 11 pages, 2017.

[22] J. Wang and F. Zhang, "Monopoly pricing in a retrial queue with delayed vacations for local area network applications," *IMA Journal of Management Mathematics*, vol. 27, no. 2, pp. 315–334, 2016.

[23] M. Neuts, *Matrix-Geometric Solution in Stochastic Models*, The Johns Hopkins University Press, Baltimore, Md, USA, 1981.

[24] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.