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Research Article

Robust Admissibilization for Discrete-Time Singular Systems with Time-Varying Delay

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The problems of the admissibility and state feedback stabilization for discrete-time singular systems with interval time-varying delay and norm-bounded uncertainty are studied. The system is equivalently transformed into a new comparison form by decomposition. By taking advantage of the Seuret summation inequality, the reciprocally convex inequality, and some relaxation techniques, a delay-dependent criterion that ensures the admissibility of the concerned systems is established. The result on robust stabilization is also obtained by fixing some parameters. It should be pointed out that the results are less dependent on the parameters so that some conservatism is reduced. A numerical example is included to illustrate the effectiveness and improvement of the proposed methods.

1. Introduction

Discrete-time systems, which are the analogues of the continuous-time systems, have more advantages in flexibility, anti-interference, precision, and economy. The discrete-time systems inherit the dynamic behavior of the corresponding continuous-time systems, while there is a lot of difference of the stability analysis and control synthesis between them. Intensive results have been given to study the problems for various discrete-time systems.

Most physical systems and processes in the real world can be represented as singular systems which contain not only differential equations but also nondynamic constraints and even improper parts of the systems. Considerable attention has been paid to singular systems [1–3]. It should be pointed out that the study of singular systems is much more complicated than that of state-space systems [4, 5]. In addition to stability, two more things called regularity and impulse-free (causality for the discrete-time cases) are introduced which are irrespective in state-space models [6, 7]. Time delay is common in the practical engineering and communication application [8–11]. It can be a source of instability of performance degradation which is needed to be eliminated [12]. Consequently, a lot of studies of time-delay systems have been carried out [13–20]. For discrete-time singular systems with time delay, several important results have been obtained. For example admissibility, new criteria are obtained by choosing the Lyapunov–Krasovskii functionals (LKFs) properly and using new inequality technology. In [21], delay-dependent stability criteria are proposed depending on the minimum and maximum delay bounds. By adopting an improved reciprocally convex combination approach, a less conservative criterion is built to ensure the system to be admissible [22]. Banu and Balasubramaniam [23] and Wu et al. [24] give sufficient condition via delay division approaches. The stability analysis for a class of discrete-time T-S fuzzy singular systems with time delays [25, 26] is studied. Furthermore, the filtering designs are studied in [27, 28]. Zheng and Bejarano [29] and Lin et al. [30] have considered the observer design problems. The admissibilization problem is studied in [31–39] by the LKF methods. The related stabilization conditions are usually nonlinear matrix inequalities. The static output feedback control problem is studied in [31, 32, 40, 41] by adopting new

summation inequalities and establishing iterative algorithms. Zhu et al. [39] and Feng and Lam [34] solve the dissipative control problems under actuator saturation. Ma et al. [36] and Feng et al. [42] give the H_∞ state feedback controller design. The delay-dependent stabilization for discrete-time interval TCS fuzzy systems is considered in [43]. Wu et al. [38] study the problem of robust state feedback control for discrete-time singular systems with time delays. By fixing some variables, the above criteria are linear matrix inequalities (LMIs) which are solvable by LMI toolbox [44], but they also bring conservatism. Cui et al. [33] give an iterative linear matrix inequality approach with initial condition optimisation which can reduce the conservatism. However, it has the shortcomings of the dependence on initial values and big calculation.

In this paper, the problem of a robust controller design for discrete-time singular systems with time delay is investigated. By utilizing the Seuret summation inequality, the reciprocally convex technique, and some transformations, new sufficient condition which contains auxiliary matrices for the admissibility is derived in terms of LMIs. Moreover, the problem of robust admissibilization of uncertain systems is also studied and a new condition is proposed. If the parameter ε is prescribed, the admissibilization condition is LMIs. For more auxiliary matrices added, the dependence on ε is reduced in the condition.

Notation: throughout the paper, standard notations are used. The superscript T stands for the matrix transpose, and -1 stands for the matrix inverse. "$*$" denotes the elements that are induced by symmetry. I and 0 represent the identity matrix and zero matrix. diag$\{\cdot\}$ denotes the block diagonal matrix.

2. Problem Formulation and Preliminaries

Consider the discrete-time singular system with time delay which can be represented by

$$\begin{aligned} Ex(k+1) &= (A+\Delta A)x(k) + \left(A_d + \Delta A_d\right)x(k-d(k)) \\ &\quad + Bu(k), \\ x(k) &= \varphi(k), k = -d_2, -d_2+1, \ldots, 0, \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector of the system, $u(k) \in \mathbb{R}^p$ is the control input, $\varphi(k)$ is a given initial condition, and $d(k)$ denotes the time delay satisfying $d_1 \le d(k) \le d_2$, where d_1 and d_2 are positive integers representing the lower and upper bounds of the delay, respectively.

$E \in \mathbb{R}^{n\times n}$ is a known matrix, and we assume that $\operatorname{rank}(E) = r < n$. A, A_d, and B are known real constant matrices. ΔA and ΔA_d are real-valued unknown matrices representing time-varying parameter uncertainties, respectively, and are assumed as

$$\begin{aligned} \Delta A &= MF(k)N, \\ \Delta A_d &= M_d F(k) N_d, \end{aligned} \tag{2}$$

where M, M_d, N, and N_d are known real constant matrices and $F(\cdot) : \mathbb{N} \longrightarrow \mathbb{R}^{l_1\times l_2}$ are known time-varying matrix function satisfying

$$F_i(k)^T F_i(k) \le I, \quad \forall k. \tag{3}$$

For the discrete time-delay system:

$$Ex(k+1) = Ax(k) + A_d x(k-d(k)), \tag{4}$$

we need to introduce some definitions.

Definition 1 (see [12]). (i) The pair (E, A) is said to be regular if $\det(zE - A)$ is not identically zero. (ii) The pair (E, A) is said to be causal if $\deg(\det(zE - A)) = \operatorname{rank}(E)$.

Definition 2 (see [12]). (i) For given integers d_1 and d_2, the discrete time-delay system (4) is said to be regular and causal, if the pair (E, A) is regular and causal. (ii) System (4) is said to be stable if for any scalar $\epsilon > 0$, there exists a scalar $\delta(\epsilon) > 0$ such that, for any compatible initial conditions $\varphi(k)$ satisfying $\sup_{-d_2\le k\le 0}\|\varphi(k)\| \le \delta(\epsilon)$, the solution $x(k)$ to system (4) satisfies $\|x(k)\| \le \epsilon$ for any $k \ge 0$; moreover, $\lim_{k\longrightarrow\infty} x(k) = 0$. (iii) System (4) is said to be admissible if it is regular, causal, and stable.

Moreover, for the pair $(E, A + \Delta A)$, there exist invertible matrices G and H, such that

$$\begin{aligned} \overline{E} &:= GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \\ \widetilde{A} &= \overline{A} + \Delta\overline{A} = G(A + \Delta A)H = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \end{aligned} \tag{5}$$

where $A_{11} \in R^{r\times r}$.

$$\widetilde{A}_d = \overline{A}_d + \Delta\overline{A}_d = G\left(A_d + \Delta A_d\right)H = \begin{bmatrix} A_{d11} & 0 \\ 0 & A_{d22} \end{bmatrix}. \tag{6}$$

In this paper, the state feedback controller with the following form is used:

$$u(k) = Kx(k), \tag{7}$$

where K is the controller gains to be determined. The closed-loop system is given by

$$Ex(k+1) = (A + \Delta A + BK)x(k) + \left(A_d + \Delta A_d\right)x(k-d(k)). \tag{8}$$

Now, we will use some lemmas in the proof which should be introduced first.

Lemma 1. *Let M, N, and $F(k)$ be real matrices of appropriate dimensions with $F(k)^T F(k) \le I$. Then, for any matrix U, we have*

$$MF(k)N + N^T F(k)^T M^T \le MUM^T + N^T U^{-1} N. \tag{9}$$

Lemma 2 (see [12]). *Let* $\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}$, *where* Ω_{11}, Ω_{12}, Ω_{21}, *and* Ω_{22} *are any real matrices with appropriate dimensions such that* Ω_{22} *is invertible and* $\Omega + \Omega^T < 0$. *Then, we have*

$$\Omega_{11} + \Omega_{11}^T - \Omega_{12}\Omega_{22}^{-1}\Omega_{21} - \Omega_{21}^T\Omega_{22}^{-T}\Omega_{12}^T < 0. \tag{10}$$

Lemma 3 (see [44]). *Consider the following inequality in the variable X:*

$$BXC + (BXC)^T + \Omega < 0, \tag{11}$$

which has a solution X if and only if

$$\begin{aligned} N_B\Omega N_B^T < 0, \\ N_C^T\Omega N_C < 0, \end{aligned} \tag{12}$$

where N_B *and* N_C *denote bases of the null spaces of B and C, respectively.*

Lemma 4 (see [45]). *For any matrix* $R > 0$, *integer* $r_1 < r_2$, *and vector function* $x(i) : [k - r_2 + 1, k - r_1] \longrightarrow \mathbb{R}^n$, *there holds*

$$\begin{aligned} &\sum_{j=-r_2+1}^{-r_1} (x(j) - x(j-1))^T R (x(j) - x(j-1)) \\ &\geq \frac{1}{r}(x(-r_1) - x(-r_2))^T R (x(-r_1) - x(-r_2)) \\ &+ \frac{3}{r}\left(x(-r_1) + x(-r_2) - \frac{2}{r+1}\sum_{-r_2+1}^{-r_1} x(j) \right)^T \\ &\times R\left(x(-r_1) + x(-r_2) - \frac{2}{r+1}\sum_{-r_2+1}^{-r_1} x(j) \right). \end{aligned} \tag{13}$$

Lemma 5 (see [46]). *For matrices Z, W, vectors* η_1, η_2, *and real scalars* $\mu \geq 0$, $\nu \geq 0$ *satisfying* $\begin{bmatrix} Z & W \\ * & Z \end{bmatrix} \geq 0$ *and* $\mu + \nu = 1$, *the following inequality holds*

$$-\frac{1}{\mu}\eta_1^T Z \eta_1 - \frac{1}{\nu}\eta_2^T Z \eta_2 \leq -\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}^T \begin{bmatrix} Z & W \\ * & Z \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}. \tag{14}$$

3. Main Results

In this section, a new condition of admissibility for discrete-time singular systems with time-varying delay is derived and it is extended to design the state feedback controllers. For simplicity, in rest of the section, we will consider the transformation of system (1) as

$$\begin{aligned} \overline{E}\overline{x}(k+1) &= \widetilde{A}\overline{x}(k) + \widetilde{A}_d\overline{x}(k - d(k)) + \overline{B}u(k) \\ &= (\overline{A} + \Delta\overline{A})\overline{x}(k) + (\overline{A}_d + \Delta\overline{A}_d)\overline{x}(k - d(k)) \\ &\quad + \overline{B}u(k). \end{aligned} \tag{15}$$

Now, let $\overline{x}(k) = \begin{bmatrix} \overline{x}_1(k) \\ \overline{x}_2(k) \end{bmatrix}$, where $\overline{x}(k) = H^{-1}x(k)$, $\overline{x}_1(k) \in \mathbb{R}^r$, $\overline{x}_2(k) \in \mathbb{R}^{n-r}$, $\overline{B} = GB$, $\overline{M} = GM$, $\overline{M_d} = GM_d$, $\overline{N} = NH$, and $\overline{N_d} = N_dH$.

We first consider the stability of system (15). If A_{22} is invertible, by (15) and (5), the singular delay system can be decomposed as

$$\begin{aligned} \overline{x}_1(k+1) &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)\overline{x}_1(k) + A_{d11}\overline{x}_1(k - d(k)) \\ &\quad - A_{12}A_{22}^{-1}A_{d22}\overline{x}_2(k - d(k)) \\ &\quad + \left(B_1 - A_{12}A_{22}^{-1}B_2\right)u(k), \end{aligned} \tag{16}$$

$$\begin{aligned} 0 &= A_{21}\overline{x}_1(k) + A_{22}\overline{x}_2(k) + A_{d22}\overline{x}_2(k - d(k)) \\ &\quad + B_2u(k). \end{aligned} \tag{17}$$

It is easy to see that the stability of the singular delay system (15) is equivalent to that of system (16).

The closed-loop system is

$$\overline{E}\overline{x}(k+1) = (\overline{A} + \Delta\overline{A} + \overline{BK})\overline{x}(k) + (\overline{A}_d + \Delta\overline{A}_d)\overline{x}(k - d(k)), \tag{18}$$

where $\overline{K} = KH$.

For system (15) with $u(k) = 0$, we have the following theorem.

Theorem 1. *The discrete-time delay singular system (15) with* $u(k) = 0$ *is admissible, if there exist matrices* $P > 0$, $U_1 > 0$, $U_2 > 0$, $Q_1 = \begin{bmatrix} Q_{11} & * \\ Q_{12} & Q_{13} \end{bmatrix} > 0$, $Q_2 = \begin{bmatrix} Q_{21} & * \\ Q_{22} & Q_{23} \end{bmatrix} > 0$, $Q_3 = \begin{bmatrix} Q_{31} & * \\ Q_{32} & Q_{33} \end{bmatrix} > 0$, $R_1 = \begin{bmatrix} R_{11} & * \\ R_{12} & R_{13} \end{bmatrix} > 0$, $R_2 = \begin{bmatrix} R_{21} & * \\ R_{22} & R_{23} \end{bmatrix} > 0$, S, T, W_1, W_2, Y_1, Y_2, Y_3, *and* Y_4, *such that*

$$\begin{bmatrix} -\frac{1}{2}\left(Y_4+Y_4^T\right) & * & * & * & * & * & * \\ -\frac{1}{2}\left(Y_3+Y_2^T\right) & -\frac{1}{2}\left(Y_1+Y_1^T\right) & * & * & * & * & * \\ \Pi_1 & \Pi_2 & X & * & * & * & * \\ -Y_3-\frac{1}{2}Y_2^T & -Y_1-\frac{1}{2}Y_1^T+P & \Pi_2^T & -Y_1-Y_1^T & * & * & * \\ -Y_4-\frac{1}{2}Y_4^T+d_1^2R_{11}+(d_2-d_1)^2R_{21} & -Y_2-\frac{1}{2}Y_3^T & \Pi_1^T & -Y_2-Y_3^T & -Y_4-Y_4^T & * & * \\ \overline{M}^T\Pi_3 & \overline{M}^T\Pi_4 & \overline{M}^T\Pi_5 & \overline{M}^T\Pi_4 & \overline{M}^T\Pi_3 & -U_1 & * \\ \overline{M}_d^T\Pi_3 & \overline{M}_d^T\Pi_4 & \overline{M}_d^T\Pi_5 & \overline{M}_d^T\Pi_4 & \overline{M}_d^T\Pi_3 & 0 & -U_2 \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} \operatorname{diag}\{R_2, 3R_2\} & * \\ T^T & \operatorname{diag}\{R_2, 3R_2\} \end{bmatrix} > 0, \tag{20}$$

where

$$\begin{aligned} X &= e_1^TQ_1e_1 - e_2^TQ_2e_2 + e_1^TQ_2e_1 - e_4^TQ_2e_4 + (d_2-d_1+1)e_1^TQ_3e_1 - e_3^TQ_3e_3 - (e_1-e_2)^T\overline{E}^TR_1\overline{E}(e_1-e_2) - 3\left(\overline{E}e_1+\overline{E}e_2-2e_5\right)^T \\ &\quad \cdot R_1\left(\overline{E}e_1+\overline{E}e_2-2e_5\right) - \begin{bmatrix} \overline{E}e_1-\overline{E}e_3 \\ \overline{E}e_2+\overline{E}e_3-2e_6 \\ \overline{E}e_3-\overline{E}e_4 \\ \overline{E}e_3+\overline{E}e_4-2e_7 \end{bmatrix}^T \begin{bmatrix} \operatorname{diag}\{R_2, 3R_2\} & T \\ T^T & \operatorname{diag}\{R_2, 3R_2\} \end{bmatrix} \begin{bmatrix} \overline{E}e_2-\overline{E}e_3 \\ \overline{E}e_2+\overline{E}e_3-2e_6 \\ \overline{E}e_3-\overline{E}e_4 \\ \overline{E}e_3+\overline{E}e_4-2e_7 \end{bmatrix}^T \\ &\quad + e_1^T\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}\left(\overline{A}e_1+\overline{A}_de_3\right) + \left(\overline{A}e_1+\overline{A}_de_3\right)^T\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}e_1 - e_1^T\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}e_1 + \left(\overline{N}e_1\right)^TU_1\left(\overline{N}e_1\right) + \left(\overline{N}_de_3\right)^TU_2\left(\overline{N}_de_3\right). \\ \Pi_1 &= \left(e_1^T\overline{A}^T + e_3^T\overline{A}_d^T\right)\left[\, Y_3+Y_4 \;\; W_1 \,\right]^T - e_1^T\overline{E}^T\left[\, Y_4 \;\; 0 \,\right]^T, \\ \Pi_2 &= \left(e_1^T\overline{A}^T + e_3^T\overline{A}_d^T\right)\left[\, Y_1+Y_2 \;\; W_2 \,\right]^T - e_1^T\overline{E}^T\left[\, Y_2 \;\; 0 \,\right]^T, \\ \Pi_3 &= \left[\, Y_3+Y_4 \;\; W_1 \,\right]^T, \\ \Pi_4 &= \left[\, Y_1+Y_2 \;\; W_2 \,\right]^T, \\ \Pi_5 &= \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}^T e_1, \\ e_i &= \left[0_{n\times(i-1)n} I_n 0_{n\times(n-i)n}\right], \quad i = 1, 2, \ldots, 7. \end{aligned} \tag{21}$$

Proof. By Lemma 1 and equations (2) and (3), we obtain that

$$\begin{aligned}&(\Delta\overline{A}e_1+\Delta\overline{A}_de_3)^T[\,\Pi_3\ \ \Pi_4\ \ \Pi_5\ \ \Pi_4\ \ \Pi_3\,]+(*)\\&\le(\overline{N}e_1)^TU_1(\overline{N}e_1)+[\,\Pi_3\ \ \Pi_4\ \ \Pi_5\ \ \Pi_4\ \ \Pi_3\,]^T\overline{M}U_1^{-1}\overline{M}^T\\&\quad\cdot[\,\Pi_3\ \ \Pi_4\ \ \Pi_5\ \ \Pi_4\ \ \Pi_3\,]+(\overline{N}_de_3)^TU_2(\overline{N}_de_3)\\&\quad+[\,\Pi_3\ \ \Pi_4\ \ \Pi_5\ \ \Pi_4\ \ \Pi_3\,]^T\overline{M}_dU_2^{-1}\overline{M}_d^T[\,\Pi_3\ \ \Pi_4\ \ \Pi_5\ \ \Pi_4\ \ \Pi_3\,],\end{aligned}\tag{22}$$

where $U_1>0$ and $U_2>0$. By (22) and Schur's complement, (19) gives

$$\begin{bmatrix}-\frac{1}{2}\left(Y_4+Y_4^T\right) & * & * & * & *\\ -\frac{1}{2}\left(Y_3+Y_2^T\right) & -\frac{1}{2}\left(Y_1+Y_1^T\right) & * & * & *\\ \overline{\Pi}_1 & \overline{\Pi}_2 & \overline{X} & * & *\\ -Y_3-\frac{1}{2}Y_2^T & -Y_1-\frac{1}{2}Y_1^T+P & \overline{\Pi}_2^T & -Y_1-Y_1^T & *\\ -Y_4-\frac{1}{2}Y_4^T+d_1^2R_1+(d_2-d_1)^2R_2 & -Y_2-\frac{1}{2}Y_3^T & \overline{\Pi}_1^T & -Y_2-Y_3^T & -Y_4-Y_4^T\end{bmatrix}<0,\tag{23}$$

where

$$\begin{aligned}\overline{\Pi}_1&=\left(e_1^T\tilde{A}^T+e_3^T\tilde{A}_d^T\right)[\,Y_3+Y_4\ \ W_1\,]^T-e_1^T\overline{E}^T[\,Y_4\ \ 0\,]^T,\\\overline{\Pi}_2&=\left(e_1^T\tilde{A}^T+e_3^T\tilde{A}_d^T\right)[\,Y_1+Y_2\ \ W_2\,]^T-e_1^T\overline{E}^T[\,Y_2\ \ 0\,]^T,\\\overline{X}&=X+e_1^T\begin{bmatrix}0&0\\0&S\end{bmatrix}(\Delta\overline{A}e_1+\Delta\overline{A}_de_3)\\&\quad+(\Delta\overline{A}e_1+\Delta\overline{A}_de_3)^T\begin{bmatrix}0&0\\0&S\end{bmatrix}^Te_1\\&\quad-(\overline{N}e_1+\overline{N}_de_3)^TU(\overline{N}e_1+\overline{N}_de_3).\end{aligned}\tag{24}$$

Multiplying (23) left and right by $\left[0_{(n-r)\times(3r)}\ \ I_{n-r}\ \ 0_{(n-r)\times(6+2r)}\right]$ and its transpose, we obtain that $Q_{13}+Q_{23}+(d_2-d_1+1)Q_{33}+A_{22}^TS^T+SA_{22}<0$, which implies that $A_{22}^TS^T+SA_{22}<0$. Therefore, system (15) with $u(k)=0$ is regular and causal.

Next, we will consider the stability of the system. We choose the Lyapunov–Krasovskii functional as follows:

$$\begin{aligned}V(k)&=V_1(k)+V_2(k)+V_3(k),\\V_1(k)&=\overline{x}_1^T(k)P\overline{x}_1(k),\\V_2(k)&=\sum_{j=k-d_1}^{k-1}\overline{x}^T(j)Q_1\overline{x}(j)+\sum_{j=k-d_2}^{k-1}\overline{x}^T(j)Q_2\overline{x}(j)\\&\quad+\sum_{j=-d_2+1}^{-d_1+1}\sum_{i=k-1+j}^{k-1}\overline{x}^T(i)Q_3\overline{x}(i),\\V_3(k)&=d_1\sum_{j=-d_1+1}^{0}\sum_{i=k-1+j}^{k-1}\eta^T(i)\overline{E}^TR_1\overline{E}\eta(i)\\&\quad+(d_2-d_1)\sum_{j=-d_2+1}^{-d_1}\sum_{i=k-1+j}^{k-1}\eta^T\overline{E}^TR_2\overline{E}\eta(i),\end{aligned}\tag{25}$$

where $\eta(i)=\overline{x}(k+1)-\overline{x}(k)$, $P>0$, $Q_i>0$, $i=1,2,3$, and $R_i>0$, $i=1,2$.

Calculating the forward difference of $V(k)$, we obtain

$$\Delta V(k)=V(k+1)-V(k)=\Delta V_1(k)+\Delta V_2(k)+\Delta V_3(k),\tag{26}$$

$$\Delta V_1(k)=\overline{x}_1^T(k+1)P\overline{x}_1(k+1)-\overline{x}_1^T(k)P\overline{x}_1(k),\tag{27}$$

$$\begin{aligned}\Delta V_2(k) \le{}& \overline{x}^T(k)Q_1\overline{x}(k) - \overline{x}^T(k-d_1)Q_1\overline{x}(k-d_1)\\ &+ \overline{x}^T(k)Q_2\overline{x}(k) - \overline{x}^T(k-d_2)Q_1\overline{x}(k-d_2)\\ &+ (d_2-d_1+1)\overline{x}^T(k)Q_3\overline{x}(k) - \overline{x}^T(k-d(k))\\ &\cdot Q_3\overline{x}(k-d(k)),\end{aligned} \tag{28}$$

$$\begin{aligned}\Delta V_3(k) ={}& d_1^2\eta(k)^T\overline{E}^T R_1\overline{E}\eta(k) + (d_2-d_1)^2\eta(k)^T\overline{E}^T R_2\overline{E}\eta(k)\\ &- d_1\sum_{j=k-d_1}^{k-1}\eta(j)^T\overline{E}^T R_1\overline{E}\eta(j)\\ &- (d_2-d_1)\sum_{j=k-d_2}^{k-d_1-1}\eta(j)^T\overline{E}^T R_2\overline{E}\eta(j).\end{aligned} \tag{29}$$

By Lemmas 4 and 5, the cross terms in (29) are bounded as

$$\begin{aligned}&-d_1\sum_{j=k-d_1}^{k-1}\eta(j)^T\overline{E}^T R_1\overline{E}\eta(j)\\ &\le -(\overline{x}(k)-\overline{x}(k-d_1))^T\overline{E}^T R_1\overline{E}(\overline{x}(k)-\overline{x}(k-d_1))\\ &\quad -3\left(\overline{x}(k)+\overline{x}(k-d_1)-\frac{2}{d_1+1}\sum_{j=k-d_1}^{k}\overline{x}(j)\right)^T\overline{E}^T R_1\overline{E}\left(\overline{x}(k)+\overline{x}(k-d_1)-\frac{2}{d_1+1}\sum_{j=k-d_1}^{k}\overline{x}(j)\right),\end{aligned} \tag{30}$$

$$\begin{aligned}&-(d_2-d_1)\sum_{j=k-d_2}^{k-d_1-1}\eta(j)^T\overline{E}^T R_2\overline{E}\eta(j)\\ &\le -\begin{bmatrix}\overline{x}(k-d_1)-\overline{x}(k-d(k))\\ \overline{x}(k-d_1)+\overline{x}(k-d(k))-\dfrac{2}{d(k)-d_1+1}\displaystyle\sum_{j=k-d(k)}^{k-d_1}\overline{x}(j)\\ \overline{x}(k-d(k))-\overline{x}(k-d_2)\\ \overline{x}(k-d(k))+\overline{x}(k-d_2)-\dfrac{2}{d_2-d(k)+1}\displaystyle\sum_{j=k-d_2}^{k-d(k)}\overline{x}(j)\end{bmatrix}^T\begin{bmatrix}\operatorname{diag}\{R_2,3R_2\} & *\\ T^T & \operatorname{diag}\{R_2,3R_2\}\end{bmatrix}\\ &\quad\times\begin{bmatrix}\overline{x}(k-d_1)-\overline{x}(k-d(k))\\ \overline{x}(k-d_1)+\overline{x}(k-d(k))-\dfrac{2}{d(k)-d_1+1}\displaystyle\sum_{j=k-d(k)}^{k-d_1}\overline{x}(j)\\ \overline{x}(k-d(k))-\overline{x}(k-d_2)\\ \overline{x}(k-d(k))+\overline{x}(k-d_2)-\dfrac{2}{d_2-d(k)+1}\displaystyle\sum_{j=k-d_2}^{k-d(k)}\overline{x}(j)\end{bmatrix},\end{aligned} \tag{31}$$

where

$$\begin{bmatrix} \mathrm{diag}\{R_2, 3R_2\} & * \\ T^T & \mathrm{diag}\{R_2, 3R_2\} \end{bmatrix} > 0. \tag{32}$$

Noted that

$$\overline{x}^T(k)\begin{bmatrix} 0 & 0 \\ 0 & \overline{S} \end{bmatrix}\overline{E}\overline{x}(k+1) + \overline{x}^T(k+1)\overline{E}^T\begin{bmatrix} 0 & 0 \\ 0 & \overline{S} \end{bmatrix}^T\overline{x}(k) = 0. \tag{33}$$

Combining (26)–(33), we have

$$\Delta V(k) \le \xi^T(k)\Xi\xi(k), \tag{34}$$

where

$$\begin{aligned}
\Xi = {} & \mathscr{A}^T P\mathscr{A} - e_{11}^T P e_{11} + (\mathscr{A} - e_{11})^T\left(d_1^2 R_{11} + (d_2 - d_1)^2 R_{21}\right)(\mathscr{A} - e_{11}) \\
& + e_1^T\begin{bmatrix} 0 & 0 \\ 0 & \overline{S} \end{bmatrix}\left(\widetilde{A}e_1 + \widetilde{A}_d e_3\right) + \left(\widetilde{A}e_1 + \widetilde{A}_d e_3\right)^T\begin{bmatrix} 0 & 0 \\ 0 & \overline{S} \end{bmatrix}^T e_1 \\
& + e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_1^T Q_2 e_1 - e_4^T Q_2 e_4 + (d_2 - d_1 + 1)e_1^T Q_3 e_1 - e_3^T Q_3 e_3 \\
& - \left(\overline{E}e_1 - \overline{E}e_2\right)^T R_1\left(\overline{E}e_1 - \overline{E}e_2\right) - 3\left(\overline{E}e_1 + \overline{E}e_2 - 2e_5\right)^T R_1\left(\overline{E}e_1 + \overline{E}e_2 - 2e_5\right) \\
& \cdot \begin{bmatrix} \overline{E}e_2 - \overline{E}e_3 \\ \overline{E}e_2 + \overline{E}e_3 - 2e_6 \\ \overline{E}e_3 - \overline{E}e_4 \\ \overline{E}e_3 + \overline{E}e_4 - 2e_7 \end{bmatrix}^T \begin{bmatrix} \mathrm{diag}\{R_2, 3R_2\} & * \\ T^T & \mathrm{diag}\{R_2, 3R_2\} \end{bmatrix} \begin{bmatrix} \overline{E}e_2 - \overline{E}e_3 \\ \overline{E}e_2 + \overline{E}e_3 - 2e_6 \\ \overline{E}e_3 - \overline{E}e_4 \\ \overline{E}e_3 + \overline{E}e_4 - 2e_7 \end{bmatrix},
\end{aligned} \tag{35}$$

$$\mathscr{A} = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)\left[I_r 0_{r\times(n-r)}\right]e_{11} + \left[A_{d11} - A_{22}^{-1}A_{d22}\right]e_3,$$

$$e_{11} = \left[\,\overline{E}\ \ 0_{n\times 6n}\,\right],$$

$$\xi(k)^T = \begin{bmatrix} \overline{x}^T(k)\overline{x}^T(k-d_1)\overline{x}^T(k-d(k))\overline{x}^T(k-d_2) & \dfrac{1}{d_1+1}\sum_{j=k-d_1}^{k}\overline{x}^T(j)\overline{E}^T \\ \dfrac{1}{d(k)-d_1+1}\sum_{j=k-d(k)}^{k-d_1}\overline{x}^T(j)\overline{E}^T & \dfrac{1}{d_2-d(k)+1}\sum_{j=k-d_2}^{k-d(k)}\overline{x}^T(j)\overline{E}^T \end{bmatrix}.$$

It is known that $\Delta V(k) < 0$ if the following inequality holds:

$$\Xi < 0. \tag{36}$$

Hence, there exists a scalar $\epsilon > 0$ such that $\Delta V(k) \le -\epsilon\|\overline{x}(k)\|^2$. Then, we have $-V(0) \le V(k+1) - V(0) = \sum_{i=0}^{k}\Delta V(i) \le -\epsilon\sum_{i=0}^{k}\|\overline{x}(i)\|^2 \le 0$, which implies $0 \le \sum_{i=0}^{k}\Delta V(i) \le (V(0)/\epsilon)$. Thus, the series $\sum_{i=0}^{k}\Delta V(i)$ converge, which implies that $\lim_{k\longrightarrow\infty}\overline{x}(k) = 0$. According to Definition 2, system (15) with $u(k) = 0$ is stable.

Next, we will decompose (23) by (5). Multiplying (23) left and right by

$$\begin{bmatrix} I_{2r} & & & \\ & I_r & & \\ & & I_{6n+2r} & \\ & & & I_{n-r} \end{bmatrix} \tag{37}$$

and its transpose, we obtain that

$$\Phi + \Phi^T < 0, \tag{38}$$

where

$$\Phi = \begin{bmatrix} -\frac{1}{2}Y_4 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}Y_3 + Y_2^T & -\frac{1}{2}Y_1 & 0 & 0 & 0 & 0 \\ \overline{\Pi}_{11} & \overline{\Pi}_{21} & \frac{1}{2}\overline{X}_1 & \overline{\Pi}_{21} & \overline{\Pi}_{11} & \Pi_6 \\ -Y_3 - \frac{1}{2}Y_2^T & -Y_1 - \frac{1}{2}Y_1^T + P & 0 & -Y_1 & 0 & 0 \\ -Y_4 - \frac{1}{2}Y_4 + d_1^2 R_1 + (d_2 - d_1)^2 R_2 & -Y_2 - \frac{1}{2}Y_3^T & 0 & 0 & -Y_4 & 0 \\ \overline{\Pi}_{12} & \overline{\Pi}_{22} & 0 & \overline{\Pi}_{22} & \overline{\Pi}_{12} & \Pi_7 \end{bmatrix},$$

$$\overline{\Pi}_{11} = \overline{e}_{11}^T\left(A_{11}^T (Y_3 + Y_4)^T + A_{21}^T W_1^T\right) + \overline{e}_3^T \widetilde{A}_d^T [\, Y_3 + Y_4 \;\; W_1 \,]^T - \overline{e}_{11}^T [\, Y_4 \;\; 0 \,]^T,$$

$$\overline{\Pi}_{21} = \overline{e}_{11}^T\left(A_{11}^T (Y_1 + Y_2)^T + A_{21}^T W_2^T\right) + \overline{e}_3^T \widetilde{A}_d^T [\, Y_1 + Y_2 \;\; W_2 \,]^T - \overline{e}_{11}^T [\, Y_2 \;\; 0 \,]^T,$$

$$\overline{\Pi}_{12} = A_{12}^T (Y_3 + Y_4)^T + A_{22}^T W_1^T,$$

$$\overline{\Pi}_{22} = A_{12}^T (Y_1 + Y_2)^T + A_{22}^T W_2^T,$$

$$\begin{aligned} \overline{X}_1 = {} & \overline{e}_{11}^T Q_{11} \overline{e}_{11} - \overline{e}_2^T Q_1 \overline{e}_2 + \overline{e}_{11}^T Q_{21} \overline{e}_{11} - \overline{e}_4^T Q_2 \overline{e}_4 + (d_2 - d_1 + 1)\overline{e}_{11}^T Q_{31} \overline{e}_{11} - \overline{e}_3^T Q_3 \overline{e}_3 - 4\overline{e}_{11}^T R_{11} \overline{e}_{11} \\ & - \overline{e}_2^T \overline{E}^T R_1 \overline{E}\overline{e}_2 - 2\overline{e}_{11}^T [I_r \; 0] R_1 \overline{E}\overline{e}_2 - 2\overline{e}_2^T \overline{E}^T R_1 [I_r \; 0]^T \overline{e}_{11} - 6\overline{e}_{11}^T [I_r \; 0] R_1 \overline{e}_5 - 6\overline{e}_5^T R_1 [I_r \; 0]^T \overline{e}_{11} - 3\left(\overline{E}\overline{e}_2 - 2\overline{e}_5\right)^T R_1 \left(\overline{E}\overline{e}_2 - 2\overline{e}_5\right), \\ & - \begin{bmatrix} \overline{E}\overline{e}_2 - \overline{E}\overline{e}_3 \\ \overline{E}\overline{e}_2 + \overline{E}\overline{e}_3 - 2\overline{e}_6 \\ \overline{E}\overline{e}_3 - \overline{E}\overline{e}_4 \\ \overline{E}\overline{e}_3 + \overline{E}\overline{e}_4 - 2\overline{e}_7 \end{bmatrix}^T \begin{bmatrix} \operatorname{diag}\{R_2, 3R_2\} & T \\ T^T & \operatorname{diag}\{R_2, 3R_2\} \end{bmatrix} \begin{bmatrix} \overline{E}\overline{e}_2 - \overline{E}\overline{e}_3 \\ \overline{E}\overline{e}_2 + \overline{E}\overline{e}_3 - 2\overline{e}_6 \\ \overline{E}\overline{e}_3 - \overline{E}\overline{e}_4 \\ \overline{E}\overline{e}_3 + \overline{E}\overline{e}_4 - 2\overline{e}_7 \end{bmatrix} - \overline{e}_{11}^T P \overline{e}_{11}, \end{aligned}$$

$$\Pi_6 = \overline{e}_{11}^T\left(Q_{12}^T + Q_{22}^T + (d_2 - d_1 + 1)Q_{23}^T\right) + \overline{e}_{11}^T A_{21}^T S^T + \overline{e}_3^T [0 \; A_{d22}]^T S^T,$$

$$\Pi_7 = A_{22}^T S^T + \frac{1}{2}\left(Q_{13} + Q_{23} + (d_2 - d_1 + 1)Q_{33}\right),$$

$$\overline{e}_{11} = \left[I_r \;\; 0_{r\times(n-r)} \;\; 0_{r\times 6n}\right],$$

$$\overline{e}_i = \left[0_{n\times r} \;\; 0_{n\times(i-1)} \;\; I_n \;\; 0_{n\times(n-i)}\right], \quad i = 2, 3, \ldots, 6. \tag{39}$$

For the special structure of the matrix Φ, it is easy to find out matrices $\Gamma_1 \in \mathbb{R}^{3r\times 3r}$, $\Gamma_2 \in \mathbb{R}^{(6n+2r)\times(6n+2r)}$, and Z, such that

$$\Phi + \Phi^T = \Sigma_1 + \Sigma_2 < 0, \tag{40}$$

$$\Sigma_1 := \overline{\Phi} + \overline{\Phi}^T - \operatorname{diag}\{\Gamma_1, \Gamma_2, 0_{n-r}\} < 0, \tag{41}$$

$$\Sigma_2 := \operatorname{diag}\{\Gamma_1, \Gamma_2, 0_{n-r}\} + \begin{bmatrix} 0_{6n+5r} & * \\ \overline{\Pi}_{62}^T & \overline{\Pi}_{72} \end{bmatrix} < 0, \tag{42}$$

where

$$\overline{\Phi} = \begin{bmatrix} -\frac{1}{2}Y_4 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}Y_3 + Y_2^T & -\frac{1}{2}Y_1 & 0 & 0 & 0 & 0 \\ \overline{\Pi}_{11} & \overline{\Pi}_{21} & \frac{1}{2}\overline{X}_1 & \overline{\Pi}_{21} & \overline{\Pi}_{11} & \overline{\Pi}_{61} \\ -Y_3 - \frac{1}{2}Y_2^T & -Y_1 - \frac{1}{2}Y_1^T + P & 0 & -Y_1 & 0 & 0 \\ -Y_4 - \frac{1}{2}Y_4 + d_1^2 R_1 + (d_2 - d_1)^2 R_2 & -Y_2 - \frac{1}{2}Y_3^T & 0 & 0 & -Y_4 & 0 \\ \overline{\Pi}_{12} & \overline{\Pi}_{22} & 0 & \overline{\Pi}_{22} & \overline{\Pi}_{12} & \overline{\Pi}_{71} \end{bmatrix}, \tag{43}$$

$$\overline{\Pi}_{61} = \overline{e}_{11}^T A_{21}^T S^T Z^T + \overline{e}_3^T [0 \ A_{d22}]^T S^T Z^T,$$

$$\overline{\Pi}_{62} = \left[0_{(n-r)\times 2r} \ (Q_{12} + Q_{22} + (d_2 - d_1 + 1)Q_{32})\overline{e}_{11} + \overline{e}_{11}^T A_{21}^T S^T (I - Z)^T + \overline{e}_3^T [0 \ A_{d22}]^T S^T (I - Z)^T 0_{(n-r)\times 2r}\right],$$

$$\overline{\Pi}_{71} = A_{22}^T S^T Z^T,$$

$$\overline{\Pi}_{72} = A_{22}^T S^T (I - Z)^T + Q_{13} + Q_{23} + [(d_2 - d_1 + 1)Q_{33}].$$

Since the matrix A_{22} is invertible, applying Lemma 2 to (41) and combining (42), we obtain that

$$\begin{bmatrix} -\frac{1}{2}\left(Y_4 + Y_4^T\right) & * & * & * & * \\ -\frac{1}{2}\left(Y_3 + Y_2^T\right) & -\frac{1}{2}\left(Y_1 + Y_1^T\right) & * & * & * \\ \widetilde{\Pi}_1 & \widetilde{\Pi}_2 & \widetilde{X} & * & * \\ -Y_3 - \frac{1}{2}Y_2^T & -Y_1 - \frac{1}{2}Y_1^T + P & \widetilde{\Pi}_2^T & -Y_1 - Y_1^T & * \\ -Y_4 - \frac{1}{2}Y_4^T + d_1^2 R_{11} + (d_2 - d_1)^2 R_{21} & -Y_2 - \frac{1}{2}Y_3^T & \widetilde{\Pi}_1^T & -Y_2 - Y_3^T & -Y_4 - Y_4^T \end{bmatrix} < 0, \tag{44}$$

where

$$\begin{aligned}
\widetilde{X} &= e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_1^T Q_2 e_1 - e_4^T Q_2 e_4 + (d_2 - d_1 + 1) e_1^T Q_3 e_1 - e_3^T Q_3 e_3 - (e_1 - e_2)^T \overline{E}^T R_1 \overline{E} (e_1 - e_2) \\
&\quad - 3(\overline{E}e_1 + \overline{E}e_2 - 2e_5)^T R_1 (\overline{E}e_1 + \overline{E}e_2 - 2e_5) - \begin{bmatrix} \overline{E}e_2 - \overline{E}e_3 \\ \overline{E}e_2 + \overline{E}e_3 - 2e_6 \\ \overline{E}e_3 - \overline{E}e_4 \\ \overline{E}e_3 + \overline{E}e_4 - 2e_7 \end{bmatrix} \begin{bmatrix} \operatorname{diag}\{R_2, 3R_2\} & T \\ T^T & \operatorname{diag}\{R_2, 3R_2\} \end{bmatrix} \begin{bmatrix} \overline{E}e_2 - \overline{E}e_3 \\ \overline{E}e_2 + \overline{E}e_3 - 2e_6 \\ \overline{E}e_3 - \overline{E}e_4 \\ \overline{E}e_3 + \overline{E}e_4 - 2e_7 \end{bmatrix} \\
&\quad + e_1^T \begin{bmatrix} 0 & 0 \\ 0 & \overline{S} \end{bmatrix} (\widetilde{A}e_1 + \widetilde{A}_d e_3) + (\widetilde{A}e_1 + \widetilde{A}_d e_3)^T \begin{bmatrix} 0 & 0 \\ 0 & \overline{S} \end{bmatrix}^T e_1 - e_1^T \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} e_1. \\
\widetilde{\Pi}_1 &= \mathscr{A} [\, Y_3 + Y_4 \;\; 0 \,]^T - e_1^T \overline{E}^T [\, Y_4 \;\; 0 \,]^T, \\
\Pi_2 &= \mathscr{A} [\, Y_1 + Y_2 \;\; 0 \,]^T - e_1^T \overline{E}^T [\, Y_2 \;\; 0 \,]^T, \\
\overline{S} &= ZS.
\end{aligned} \tag{45}$$

Next, we will give that inequality (36) can be satisfied if (44) holds.

Noted that equation (44) can be rewritten as

$$\Upsilon + \zeta^T \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}^T \psi + \psi^T \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \zeta < 0, \tag{46}$$

where

$$\begin{aligned}
\Upsilon &= \begin{bmatrix} 0 & 0 & 0 & 0 & d_1^2 R_1 + (d_2 - d_1)^2 R_2 \\ 0 & 0 & 0 & P & 0 \\ 0 & 0 & \widetilde{X} & 0 & 0 \\ 0 & P & 0 & 0 & 0 \\ d_1^2 R_1 + (d_2 - d_1)^2 R_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\zeta &= \begin{bmatrix} 0 & -\frac{1}{2} I_r & \mathscr{A} & -I_r & 0 \\ -\frac{1}{2} I_r & 0 & \mathscr{A} - e_{11} & 0 & -I_r \end{bmatrix}, \\
\psi &= \begin{bmatrix} 0 & I_r & 0 & I_r & 0 \\ I_r & 0 & 0 & 0 & I_r \end{bmatrix}.
\end{aligned} \tag{47}$$

By Lemma 3, (46) is equivalent to

$$\begin{aligned}
\chi_\zeta^T \Upsilon \chi_\zeta &< 0, \\
\chi_\psi^T \Upsilon \chi_\psi &< 0,
\end{aligned} \tag{48}$$

where

$$\chi_{\zeta}^{T}=\begin{bmatrix} I_r & 0 & 0 & 0 & -\frac{1}{2}I_r \\ 0 & I_r & 0 & -\frac{1}{2}I_r & 0 \\ 0 & 0 & I_{7n} & \mathscr{A} & \mathscr{A}-e_{11} \end{bmatrix},$$
$$\chi_{\psi}^{T}=\begin{bmatrix} 0 & 0 & I_{7n} & 0 & 0 \\ 0 & -I_r & 0 & I_r & 0 \\ -I_r & 0 & 0 & 0 & I_r \end{bmatrix}, \tag{49}$$

which can be rewritten as

$$\begin{bmatrix} \widetilde{X} & & \\ & -2P & \\ & & -2\left(d_1^2R_1+(d_2-d_1)^2R_2\right) \end{bmatrix}<0, \tag{50}$$

$$\begin{bmatrix} -\left(d_1^2R_1+(d_2-d_1)^2R_2\right) & 0 & \left(d_1^2R_1+(d_2-d_1)^2R_2\right)(\mathscr{A}-e_{11}) \\ 0 & -P & P\mathscr{A} \\ (\mathscr{A}-e_{11})^T\left(d_1^2R_1+(d_2-d_1)^2R_2\right) & \mathscr{A}^TP & -\widetilde{X} \end{bmatrix}<0. \tag{51}$$

By Schur's complement, (51) is equivalent to $\Xi<0$, which ensures the stability of the system. This completes the proof.

Remark 1. Theorem 1 gives a new criterion for discrete-time singular systems with time-varying delay. By Lemmas 2 and 3, the admissible condition (36) can be solved by (19). Compared with condition (36) and some of the existing conditions, some auxiliary matrices Y_1, Y_2, Y_3, and Y_4 are added in (19) in addition to the Lyapunov matrix P. The auxiliary matrices bring extra relaxation for the criterion.

Next, we will further deal with the state feedback control problem for discrete-time systems (15) and the following theorem is obtained.

Theorem 2. *The closed-loop system (18) is admissible, if there exist matrices $P>0$, $U_1>0$, $U_2>0$, $Q_1>0$, $Q_2>0$, $Q_3>0$, $R_1>0$, $R_2>0$, S, T, W, L, Y_1, Y_2, Y_4, and ε, such that*

$$\begin{bmatrix} -\frac{1}{2}\left(Y_4+Y_4^T\right) & * & * & * & * & * & * \\ -\frac{1}{2}\left(Y_3+Y_2^T\right) & -\frac{1}{2}\left(Y_1+Y_1^T\right) & * & * & * & * & * \\ \widetilde{\Pi}_1 & \widetilde{\Pi}_2 & \widetilde{X} & * & * & * & * \\ -Y_3-\frac{1}{2}Y_2^T & -Y_1-\frac{1}{2}Y_1^T+P & \widetilde{\Pi}_2^T & -Y_1-Y_1^T & * & * & * \\ -Y_4-\frac{1}{2}Y_4^T+d_1^2R_{11}+(d_2-d_1)^2R_{21} & -Y_2-\frac{1}{2}Y_3^T & \widetilde{\Pi}_1^T & -Y_2-Y_3^T & -Y_4-Y_4^T & * & * \\ \overline{N}\Pi_3 & \overline{N}\Pi_4 & \overline{N}\Pi_5 & \overline{N}\Pi_4 & \overline{N}\Pi_3 & -U_1 & * \\ \overline{N}_d\Pi_3 & \overline{N}_d\Pi_4 & \overline{N}_d\Pi_5 & \overline{N}_d\Pi_4 & \overline{N}_d\Pi_3 & 0 & -U_2 \end{bmatrix}<0, \tag{52}$$

$$\begin{bmatrix} \mathrm{diag}\{R_2,3R_2\} & * \\ T^T & \mathrm{diag}\{R_2,3R_2\} \end{bmatrix}>0, \tag{53}$$

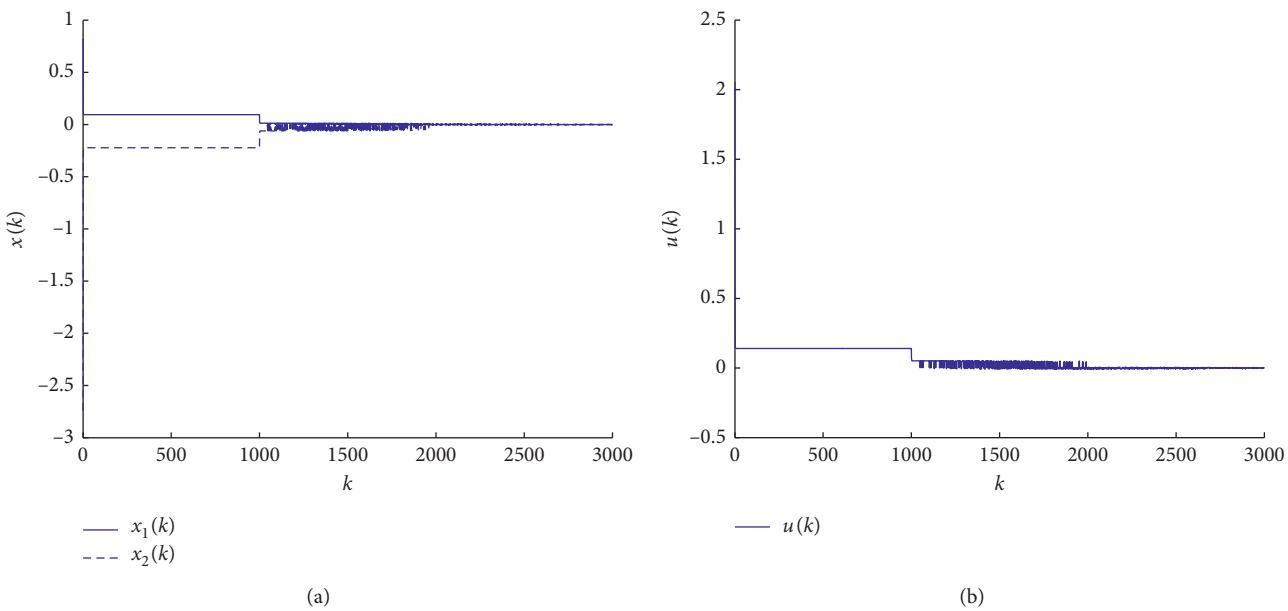

Figure 1: Control results of example 1. (a) State response curves for the closed-loop system. (b) Input signals.

where

$$\begin{aligned}
\widetilde{X} &= e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_1^T Q_2 e_1 - e_4^T Q_2 e_4 + (d_2 - d_1 + 1) e_1^T Q_3 e_1 - e_3^T Q_3 e_3 - (e_1 - e_2)^T \overline{E}^T R_1 \overline{E} (e_1 - e_2) \\
&\quad - 3(\overline{E}e_1 + \overline{E}e_2 - 2e_5)^T R_1 (\overline{E}e_1 + \overline{E}e_2 - 2e_5) - \begin{bmatrix} \overline{E}e_2 - \overline{E}e_3 \\ \overline{E}e_2 + \overline{E}e_3 - 2e_6 \\ \overline{E}e_3 - \overline{E}e_4 \\ \overline{E}e_3 + \overline{E}e_4 - 2e_7 \end{bmatrix}^T \begin{bmatrix} \operatorname{diag}\{R_2, 3R_2\} & T \\ T^T & \operatorname{diag}\{R_2, 3R_2\} \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} \overline{E}e_2 - \overline{E}e_3 \\ \overline{E}e_2 + \overline{E}e_3 - 2e_6 \\ \overline{E}e_3 - \overline{E}e_4 \\ \overline{E}e_3 + \overline{E}e_4 - 2e_7 \end{bmatrix} + e_1^T \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} (\overline{A}e_1 + \overline{A}_d e_3)^T + (\overline{A}e_1 + \overline{A}_d e_3) \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}^T e_1 + e_1^T \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} L^T B^T e_1 \\
&\quad + e_1^T BL \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} e_1 - e_1^T \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} e_1 + e_1^T \overline{M} U_1 \overline{M}^T e_1 + e_3^T \overline{M}_d U_2 \overline{M}_d^T e_3, \\
\widetilde{\Pi}_1 &= \left(e_1^T \overline{A} + e_3^T \overline{A}_d\right) \varepsilon \left[(Y_1 + Y_2) \;\; W \right]^T + \varepsilon e_1^T BL [I_r \;\; 0]^T - e_1^T \overline{E}^T [Y_4 \;\; 0]^T, \\
\widetilde{\Pi}_2 &= \left(e_1^T \overline{A} + e_3^T \overline{A}_d\right) [Y_1 + Y_2 \;\; W]^T + e_1^T BL [I_r \;\; 0]^T - e_1^T \overline{E}^T [Y_2 \;\; 0]^T, \\
Y_3 &= \varepsilon (Y_1 + Y_2) - Y_4, \Pi_3, \Pi_4, \Pi_5, e_i, \quad i = 1, 2, \ldots, 7,
\end{aligned} \tag{54}$$

are defined in Theorem 1. If it is the case, the controller gains can be solved as

$$K = L \begin{bmatrix} Y_1 + Y_2 & W \\ 0 & S \end{bmatrix}^{-T} H^{-1}. \tag{55}$$

Proof. Replacing A^T, ΔA^T, A_d^T, and ΔA_d^T into $A + BK$, ΔA, A_d, and ΔA_d, respectively, and letting $Y_3 = \varepsilon(Y_1 + Y_2) - Y_4$, $W_2 = W$, $W_1 = \varepsilon W$, and $L = K \begin{bmatrix} Y_1 + Y_2 & W \\ 0 & S \end{bmatrix}^T$. Using the similar procedure as the proof in Theorem 1, this theorem can be easily proved.

Remark 2. Usually, an additional equation $(Y_1 x(k) + Y_2 \overline{E} x(k+1))^T (-\overline{E} x(k+1) + (\widetilde{A} + BK) x(k) + \widetilde{A}_d x(k - d(k))) + (*) = 0$ is added in the derivations of criteria for the state feedback controller design for discrete-time singular systems with time delay [34, 36, 38, 42]. It is obvious that it leads to a nonlinear problem for controller synthesis. The usual way in the existing controller design is to set $Y_2 = \varepsilon Y_1$, where ε is fixed, such that the conditions can be transformed into

LMIs. However, the dealing methods bring much conservatism and the results depend extremely on the parameter ε. In our method, by introducing auxiliary matrices Y_1, Y_2, Y_3, and Y_4 in Theorems 1 and 2 and letting $Y_3 = \varepsilon(Y_1 + Y_2) - Y_4$, the dependence on ε is weakened and obviously the conservatism is reduced. In general, the value of ε can be taken in the interval of $[0, 2]$.

Remark 3. The auxiliary matrices which are added in the criteria can really increase the computational complexity.

4. Numerical Example

Consider the discrete time-delay singular system (1) with

$$\begin{aligned} E &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ A &= \begin{bmatrix} 1 & 1 \\ 1 & 0.3 \end{bmatrix}, \\ A_d &= \begin{bmatrix} 0.1 & 0 \\ 0.3 & 0.1 \end{bmatrix}, \\ B &= \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \\ M &= M_d = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.3 \end{bmatrix}, \\ N &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \end{aligned} \tag{56}$$

It is proved in [33] that the system is unstable at least for a particular choice of $d(k)$. If $1 \le d(k) \le 1000$, the method in [33, 38] is infeasible. By Theorem 2, set $\varepsilon = 1$, the controller gain to make the closed-loop system robustly admissible is designed as $K = [0.48240.5194]$. Figure 1 depicts the closed-loop system responses with the control law $u(k)$. It is observed that the closed-loop system is robust stable with the uncertainties.

5. Conclusion

In this paper, the state feedback admissibilization for discrete-time singular systems with time-varying delay is under consideration. By using the Seuret summation inequality, the reciprocally convex inequality, and some relaxation techniques, a new admissible condition for the considered system is obtained. And the state feedback controller design problem can be solved by the linear matrix inequalities (LMIs) such that the closed-loop system is admissible. Finally, a simulation is provided to demonstrate the effectiveness of the proposed methods. Furthermore, research topics can include the extension of our results to more complex cases with performance control and performance analysis and to further reduce the conservatism.

Data Availability

Our manuscript involves the theory work and is not related to the data. So, we do not provide the information about the data availability statement.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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