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Research Article

Output Regulation for a Class of MIMO Uncertain Stochastic Nonlinear Systems by Active Disturbance Rejection Control Approach

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In this paper, the active disturbance rejection control (ADRC) approach is applied to a class of multi-input multioutput (MIMO) uncertain stochastic nonlinear systems. An extended state observer (ESO) is first designed for estimation of both unmeasured states and stochastic total disturbance of each subsystem which represents the total effects of internal unmodeled stochastic dynamics and external stochastic disturbance with unknown statistical property. The ADRC controller based on the states of ESO is further designed to achieve the closed-loop system's output regulation performance including practical mean square reference signals tracking, disturbance attenuation, and practical mean square stability when the reference signals are zero avoiding solving any partial differential equations in the conventional output regulation theory. Some numerical simulations are presented to demonstrate the effectiveness of the proposed ADRC approach.

1. Introduction

During the past couple of years, there have been existing representative control approaches to cope with uncertainties in controlled plants such as the internal model principle [1–3], the robust control [4], and the adaptive control [5]. Nevertheless, a majority of these control methods pay attention to the worst situation so that the design of controller is comparatively conservative. The active disturbance rejection control (ADRC), as a nontraditional control strategy, was first put forward by Han in leading paper [6]. The disturbance coped with by ADRC is much more general which can be the total coupling effects of internal unmodeled system dynamics and external disturbance. The most noteworthy feature of ADRC is that an extended state observer (ESO), as its key part, is designed to estimate the disturbance in real time so that the disturbance can be cancelled in the ESO-based feedback loop. This estimation/cancellation strategy leads to less control energy consumption in the control engineering practice [7].

Many industry practitioners have been paying more attention to the ADRC approach as presented in recent survey type paper [8]. Specific applications in different fields emerge in large numbers including synchronous motors [9], DC-DC power converters [10], control system in superconducting RF cavities [11], and flight vehicles control [12]. On the other hand, there have been lots of theoretical researches like the convergence analysis of ADRC for uncertain nonlinear systems [13–16] and the stabilization and output tracking problem for distributed parameter systems by the ADRC approach [17–21].

However, tardy theoretical progress has been made about ADRC for stochastic systems. Some progress could be found such as the practical mean square convergence analysis of ESO for the open-loop of a class of MIMO stochastic nonlinear systems [22] and the practical mean square convergence analysis of ADRC for both single-input single-output stochastic nonlinear systems [23] and lower triangular stochastic nonlinear systems [24]. The practical mean square stability of MIMO stochastic nonlinear systems by the ADRC approach is addressed in [25].

Although there have been some works on ADRC for stochastic nonlinear systems like [22, 24, 25] where the considered external disturbance is the bounded stochastic noise with unknown statistical characteristics existing widely in practical systems [26–28], the output regulation problem for this class of stochastic nonlinear systems receives less attention. In [22], the authors focus on the convergence of practical mean square estimation errors by ESO for the open-loop of systems without considering the performances of the closed-loop system under the ESO-based feedback. The practical mean square stability of the ADRC's closed-loop systems is investigated for lower triangular stochastic nonlinear systems in [24] and MIMO stochastic nonlinear ones in [25] where the output-feedback stabilization problem is just a special case of the output regulation one in this paper. As the continuous research of [22–25], the ADRC approach is applied in the output regulation problem for a class of MIMO uncertain stochastic nonlinear systems with large stochastic uncertainties including unknown nonlinear system functions, external stochastic disturbance with unknown statistical property, unknown stochastic inverse dynamics, uncertain nonlinear coupling effects between subsystems, and uncertainties caused by the partially unknown input gains, where the output regulation performance of the resulting closed-loop systems includes practical mean square reference tracking, disturbance attenuation, and practical mean square stability when the reference signals are zero. To be specific, in this paper we consider the output regulation problem for the partial exact feedback linearizable MIMO system [29] subject to vast stochastic uncertainties as follows:

$$\begin{aligned} dx_i &= A_{n_i} x_i dt + B_{n_i}\left[f_i(t,x,\zeta,w) + \sum_{l=1}^{m} b_{il}u_l\right]dt, \\ d\zeta &= g_1(t,x,\zeta,w)\,dt + g_2(t,x,\zeta,w)\,dW_1(t), \\ y_i &= C_{n_i}x_i, \quad i = 1,2,\ldots,m, \end{aligned} \tag{1}$$

where $x = (x_1^\top, \ldots, x_m^\top)^\top \in \mathbb{R}^n$ with $x_i = (x_{i1}, \ldots, x_{in_i})^\top$ and $n = n_1 + \cdots + n_m$, $u = (u_1, \ldots, u_m)^\top \in \mathbb{R}^m$, and $y = (y_1, \ldots, y_m)^\top \in \mathbb{R}^m$ are the state, control, and measured output of the system, respectively; $\zeta = (\zeta_1, \ldots, \zeta_s)^\top \in \mathbb{R}^s$ denotes the state of stochastic inverse dynamics; the functions $f_i : [0,\infty) \times \mathbb{R}^{n+s+1} \longrightarrow \mathbb{R}$, $g_1 : [0,\infty) \times \mathbb{R}^{n+s+1} \longrightarrow \mathbb{R}^s$, and $g_2 : [0,\infty) \times \mathbb{R}^{n+s+1} \longrightarrow \mathbb{R}^{s\times p}$ are unknown; the constants b_{il} $(i, l = 1,2,\ldots,m)$ are the partially unknown control coefficients with some known nominal values b_{il}^* satisfying Assumption (A3); $\{W_1(t)\}_{t\geq 0}$ is a p-dimensional standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with Ω being a sample space, $\mathcal{F}$ a σ-field, $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration, and P the probability measure; $w(t) \triangleq \psi(t, W_2(t)) \in \mathbb{R}$ is the external stochastic disturbance where $\psi(\cdot) : [0,\infty) \times \mathbb{R}^q \longrightarrow \mathbb{R}$ is an unknown bounded function satisfying Assumption (A1) and $\{W_2(t)\}_{t\geq 0}$ is a q-dimensional standard Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ as well and is mutually independent with $\{W_1(t)\}_{t\geq 0}$; in addition, we denote

$$\begin{aligned} A_{n_i} &= \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix}_{n_i\times n_i}, \\ B_{n_i} &= (0, \ldots, 0, 1)^\top_{n_i\times 1}, \\ C_{n_i} &= (1, 0, \ldots, 0)_{1\times n_i}. \end{aligned} \tag{2}$$

For each $1 \leq i \leq m$, the stochastic total disturbance of each i-subsystem is defined as follows:

$$x_{i(n_i+1)} \triangleq f_i(t,x,\zeta,w) + \sum_{l=1}^{m}\left(b_{il} - b_{il}^*\right)u_l, \tag{3}$$

which represents the total effects of unknown nonlinear system functions, external stochastic disturbance with unknown statistical property, unknown stochastic inverse dynamics, uncertain nonlinear coupling effects between subsystems, and uncertainties caused by the partially unknown input gains. So the stochastic uncertainties in the considered system are very complex.

For given, bounded, deterministic reference signals $v_i(t)$ $(i = 1,2,\ldots,m)$ whose derivatives $\dot{v}_i(t), \ddot{v}_i(t), \ldots, v_i^{(n_i+1)}(t)$ are assumed to be bounded, the control objective is to design an output-feedback control such that for any initial states the output y_i converges practically to $v_i(t)$ in mean square sense and at the same time x_{ij} converge practically to $v_i^{(j-1)}(t)$ in mean square sense for all $j = 2, \ldots, n_i$. The output-feedback stabilization problem for the class of MIMO uncertain stochastic nonlinear systems is covered by letting $v_i(t) \equiv 0$ for all $t \geq 0$.

On the basis of a partial exact feedback linearizable MIMO system [29] widely investigated in control theory and the fact that stochastic uncertainties are ubiquitous in practical control engineering and often cause disadvantageous effects on control performance, the ADRC approach is addressed for the output regulation problem of system (1) including practical mean square reference signals tracking, disturbance attenuation, and practical mean square stability when the reference signals are zero in this paper. It should be noticed that system (1) representing a partial exact feedback linearizable MIMO system [29] with vast stochastic uncertainties is quite general and has physical and engineering background. Firstly, the SISO nonlinear systems and MIMO nonlinear ones widely addressed by the ADRC approach in available literatures [6, 7, 13, 14, 21, 30] are covered as special cases of system (1) when $\psi(\cdot)$ is the function of time variable t only: $w(t) \triangleq \psi(t)$ and $g_2(\cdot) \equiv 0$. Secondly, the external stochastic disturbance $w(t)$ is quite general in the sense that it is not required to know its statistical characteristics since the function $\psi(\cdot)$ can be unknown and the bounded stochastic noise investigated in [26–28] in many practical systems is also covered as its special case. Finally, system (1) covers some stochastic systems considered in the aforementioned literatures like SISO stochastic nonlinear systems in [31] when $m = 1$, $\zeta(\cdot) \equiv 0$, and $b_{11} = 1$.

The main contributions of this paper can be summarized as follows: (a) The ADRC approach is systematically proposed to solve the output regulation problem for a class of MIMO uncertain stochastic nonlinear systems without difficulty in solving any partial differential equations compared with the conventional output regulation theory. (b) The stochastic uncertainties dealt with by ADRC in this paper are very general including unknown nonlinear system functions, external stochastic disturbance with unknown statistical property, unknown stochastic inverse dynamics, uncertain nonlinear coupling effects between subsystems, and uncertainties caused by the partially unknown input gains. (c) Most available output-feedback controls for stochastic nonlinear systems are designed to guarantee the global asymptotic stability in probability provided that the noise vector field vanishes at the origin [32] or only the noise-to-state (or input-to-state) stability in probability [33] otherwise. In this paper, however, the practical mean square convergence is obtained by the ADRC approach without assuming that the noise vector field should be vanishing at the origin.

The rest of this paper is presented as follows. In Section 2, both ESO and ESO-based feedback control are designed for the x-subsystem of (1), the assumptions of the main result are stated, and the output regulation performance of the closed-loop is summarized as Theorem 2. In Section 3, a rigorous proof of Theorem 2 is given. Finally, in Section 4, some numerical simulations are presented to illustrate the effectiveness of the proposed ADRC approach.

The following notations are used throughout this paper. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ represents the space of all real $n\times m$-matrices; for a vector or matrix K, $K^\top$ denotes its transpose; for a square matrix K, $\mathrm{Tr}(K)$ denotes its trace; $I_{n\times n}$ denotes the $n\times n$ unit matrix; $\lambda_{\min}(K)$ and $\lambda_{\max}(K)$ denote the minimal and maximal eigenvalues of the symmetric real matrix K, respectively; $\|K\|$ denotes the Euclidean norm of the vector K and the corresponding induced norm when K is a matrix; $(a^{(ij)})_{m\times n}$ denotes an $m\times n$ matrix with entries $a^{(ij)}$; for a differentiable function $f:\mathbb{R}^n\longrightarrow\mathbb{R}$, $\partial f/\partial z \triangleq (\partial f/\partial z_1,\ldots,\partial f/\partial z_n)^\top$ for $z=(z_1,\ldots,z_n)^\top$; for a twice differentiable function $f:\mathbb{R}^n\longrightarrow\mathbb{R}$, $\partial^2 f/\partial z^2 \triangleq (\partial^2 f/\partial z_i z_j)_{n\times n}$ $(i,j=1,2,\ldots,n)$ for $z=(z_1,\ldots,z_n)^\top$; for a matrix valued function $f:\mathbb{R}^n\longrightarrow\mathbb{R}^{m\times s}$, $f\triangleq(f^{(ij)})_{m\times s}$ $(i=1,2,\ldots,m,\ j=1,2,\ldots,s)$.

2. Main Results

By analogy with [30], the one-parameter tuning linear ESO for the x-subsystem of (1) is designed as follows:

$$\begin{aligned}
\dot{\hat{x}}_{i1} &= \hat{x}_{i2} + a_{i1} r\left(y_i - \hat{x}_{i1}\right),\\
\dot{\hat{x}}_{i2} &= \hat{x}_{i3} + a_{i2} r^2\left(y_i - \hat{x}_{i1}\right),\\
&\ \ \vdots\\
\dot{\hat{x}}_{in_i} &= \hat{x}_{i(n_i+1)} + a_{in_i} r^{n_i}\left(y_i - \hat{x}_{i1}\right) + \sum_{l=1}^{m} b_{il}^* u_l,\\
\dot{\hat{x}}_{i(n_i+1)} &= a_{i(n_i+1)} r^{n_i+1}\left(y_i - \hat{x}_{i1}\right),\quad i=1,2,\ldots,m,
\end{aligned}\tag{4}$$

where a_{ij} $(i=1,2,\ldots,m,\ j=1,2,\ldots,n_i+1)$ are designed parameters such that the following matrix is Hurwitz:

$$E_i = \begin{pmatrix} -a_{i1} & 1 & 0 & \cdots & 0\\ -a_{i2} & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ -a_{in_i} & 0 & 0 & \ddots & 1\\ -a_{i(n_i+1)} & 0 & 0 & \cdots & \end{pmatrix}_{(n_i+1)\times(n_i+1)},\tag{5}$$

$r>0$ is the gain parameter to be tuned, and b_{il}^* is the nominal value of b_{il} satisfying Assumption (A3). ESO 2 is designed to estimate both stochastic total disturbance and unmeasured states by choosing suitable parameters a_{ij} and tuning the gain parameter r. Specially, $\hat{x}_{i(n_i+1)}$ is the estimate of the stochastic total disturbance $x_{i(n_i+1)}$ defined by (3). It should be noted that we only need to tune one parameter r in ESO 2 based on the estimation accuracy and the variation of the stochastic total disturbance. Generally, the higher the estimation accuracy is needed and the faster the stochastic total disturbance varies, the larger the parameter r needs to be set. Here and throughout the paper, we always drop r for the solution of 2 by abuse of notation without confusion.

For all $1\le i\le m$, let

$$\left(v_{i1}, v_{i2},\ldots,v_{i(n_i+1)}\right) = \left(v_i,\dot{v}_i,\ldots,v_i^{(n_i)}\right)\tag{6}$$

and

$$\varrho_i\left(z_{i1},\ldots,z_{in_i}\right) = k_{i1}z_{i1} + k_{i2}z_{i2} + \cdots + k_{in_i}z_{in_i},\quad \forall\left(z_{i1},\ldots,z_{in_i}\right)^\top\in\mathbb{R}^{n_i}.\tag{7}$$

ESO 2 based output-feedback control is designed as

$$\begin{aligned}
u_i = \sum_{l=1}^{m}\hat{b}_{il}^*\Big(&\varrho_l\left(\hat{x}_{l1}-v_{l1}(t), \hat{x}_{l2}-v_{l2}(t),\ldots,\hat{x}_{ln_l}\right.\\
&\left.-v_{ln_l}(t)\right) - \hat{x}_{l(n_l+1)} + v_{l(n_l+1)}(t)\Big),\quad 1\le i\le m,
\end{aligned}\tag{8}$$

where $\hat{b}_{il}^*$ are defined in (13) and the feedback gain parameters k_{ij} $(i=1,2,\ldots,m,\ j=1,2,\ldots,n_i)$ are chosen such that the following matrix is Hurwitz:

$$F_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \ddots & 1\\ k_{i1} & k_{i2} & \cdots & k_{i(n_i-1)} & k_{in_i}\end{pmatrix}_{n_i\times n_i}.\tag{9}$$

To obtain practical mean square convergence of the closed-loop of the x-subsystem of (1) under ESO 2 based output-feedback control (8) including ESO's estimation of unmeasured states and stochastic total disturbance, practical mean square reference signals tracking, disturbance attenuation, and practical mean square stability, the following assumptions are needed.

Assumption (A1) is about the unknown function $\psi(\cdot)$ defining the external stochastic disturbance.

Assumption (A1). $\psi(t,\theta)\ :\ [0,\infty)\times\mathbb{R}^q \longrightarrow \mathbb{R}$ is continuously differentiable and twice continuously differentiable with respect to t and θ, respectively, and there exists a known constant $D_1 > 0$ such that, for all $\theta = (\theta_1,\ldots,\theta_q)^\top \in \mathbb{R}^q$,

$$\left|\psi(t,\theta)\right| + \left|\frac{\partial\psi(t,\theta)}{\partial t}\right| + \left\|\frac{\partial\psi(t,\theta)}{\partial\theta}\right\| + \sum_{i=1}^{q}\left|\frac{\partial^2\psi(t,\theta)}{\partial\theta_i^2}\right| \le D_1. \tag{10}$$

Remark 1. Roughly speaking, Assumption (A1) indicates that both the external stochastic disturbance and its "variation" should be bounded which is reasonable since it is as a part of the stochastic total disturbance estimated by ESO.

Assumption (A2) is a prior assumption about the unknown functions $f_i(\cdot)$, $g_1(\cdot)$, and $g_2(\cdot)$ in system (1).

Assumption (A2). $f_i(\cdot)$ $(i = 1,\ldots,m)$ are continuously differentiable and twice continuously differentiable with respect to t and other arguments, respectively, and $g_i(\cdot)$ $(i = 1,2)$ are locally Lipschitz continuous in (x,ζ,w) uniformly in $t \in [0,\infty)$. There exist known constants $D_i \ge 0$ $(i = 2,3,4)$ and a nonnegative continuous function $\varsigma \in C(\mathbb{R};\mathbb{R})$ such that, for all $t \ge 0$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^s$, $w \in \mathbb{R}$, $i = 1,2,\ldots,m$,

$$\left|\frac{\partial f_i(t,x,\zeta,w)}{\partial t}\right| + \left\|g_1(t,x,\zeta,w)\right\| \le D_2 + D_3\|x\| + \varsigma(w), \tag{11}$$

$$\begin{aligned}&\left\|\frac{\partial f_i(t,x,\zeta,w)}{\partial x}\right\| + \left\|\frac{\partial f_i(t,x,\zeta,w)}{\partial \zeta}\right\| \\ &\quad + \sum_{j,l=1}^{s}\left|\frac{\partial^2 f_i(t,x,\zeta,w)}{\partial\zeta_j\partial\zeta_l}\right| + \left|\frac{\partial f_i(t,x,\zeta,w)}{\partial w}\right| \\ &\quad + \left|\frac{\partial^2 f_i(t,x,\zeta,w)}{\partial w^2}\right| + \sum_{j=1}^{p}\sum_{i=1}^{s}\left|g_2^{(ij)}(t,x,\zeta,w)\right| \le D_4 \\ &\quad + \varsigma(w).\end{aligned} \tag{12}$$

Assumption (A3) is about the prior estimates b_{il}^* $(i,l = 1,2,\ldots,m)$ for the unknown control parameters b_{il} $(i,l = 1,2,\ldots,m)$ in system (1).

Assumption (A3). The matrix with entries b_{il}^* $(i,l = 1,2,\ldots,m)$ in 2 is invertible with the inverse matrix given by

$$\begin{pmatrix} b_{11}^* & b_{12}^* & \cdots & b_{1m}^* \\ b_{21}^* & b_{22}^* & \cdots & b_{2m}^* \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}^* & b_{m2}^* & \cdots & b_{mm}^* \end{pmatrix}^{-1} = \begin{pmatrix} \widehat{b}_{11}^* & \widehat{b}_{12}^* & \cdots & \widehat{b}_{1m}^* \\ \widehat{b}_{21}^* & \widehat{b}_{22}^* & \cdots & \widehat{b}_{2m}^* \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{b}_{m1}^* & \widehat{b}_{m2}^* & \cdots & \widehat{b}_{mm}^* \end{pmatrix}, \tag{13}$$

and the nominal values b_{il}^* of b_{il} satisfy

$$\Xi \triangleq 1 - \sum_{i,l,d=1}^{m} 2\lambda_{max}(Q_i)\left|(b_{il} - b_{il}^*)\widehat{b}_{ld}^* a_{d(n_d+1)}\right| > 0, \tag{14}$$

where Q_i are the positive definite matrix solution satisfying $Q_iE_i + E_i^\top Q_i = -I_{(n_i+1)\times(n_i+1)}$.

The main result on practical mean square convergence of the closed-loop of the x-subsystem of (1), 2, and (8), which includes ESO's practical mean square estimation of both unmeasured states and stochastic total disturbance and output regulation performance, is summarized in Theorem 2.

Theorem 2. *Under Assumptions (A1)–(A3) and supposing that* $\sup_{t\ge0}\|(v_i(t),\dot{v}_i(t),\ldots,v_i^{(n+1)}(t))\| \le M$ *for some constant* $M \ge 0$*, then the closed-loop of the x-subsystem of (1), 2, and (8) is practically mean square convergent in the sense that there are a constant $r^* > 0$ (specified by (32) later) and a r-dependent constant $t_r^* > 0$ with $r > r^*$ such that, for any initial values and all $t \ge t_r^*$,*

$$\mathbb{E}\left|x_{ij}(t) - \widehat{x}_{ij}(t)\right|^2 \le \frac{\Gamma}{r^{2n_i+3-2j}}, \quad 1 \le i \le m,\ 1 \le j \le n_i + 1, \tag{15}$$

and

$$\mathbb{E}\left|x_{ij}(t) - v_{ij}(t)\right|^2 \le \frac{\Gamma}{r}, \quad 1 \le i \le m,\ 1 \le j \le n_i, \tag{16}$$

where $\Gamma > 0$ is a r-independent constant.

Remark 3. It should be noticed that the practical mean square convergence addressed in this paper includes ESO's practical mean square estimation of both unmeasured states and stochastic total disturbance and output regulation performance for the closed-loop system, but the practical mean square convergence addressed in [22] only refers to ESO's practical mean square estimation of both unmeasured states and stochastic total disturbance for the open-loop system. In addition, there does not exist an essential difference between the upper bound of the estimation error of ESO in practical mean square sense in this paper and the one in [22] since the high gain parameter in ESO is denoted by r in this paper and by $1/\varepsilon$ in [22], respectively.

3. Proof of the Main Result

Proof of Theorem 2. For all $1 \le i \le m, 1 \le j \le n_i + 1$, we set

$$\begin{aligned}
\eta_{ij} &= r^{n_i+1-j}\left(x_{ij} - \widehat{x}_{ij}\right), \\
\eta_i &= \left(\eta_{i1}, \ldots, \eta_{i(n_i+1)}\right)^\top, \\
\eta &= \left(\eta_1^\top, \ldots, \eta_m^\top\right)^\top, \\
\kappa_{ij} &= x_{ij} - v_{ij}(t), \\
\kappa_i &= \left(\kappa_{i1}, \ldots, \kappa_{in_i}\right)^\top, \\
\kappa &= \left(\kappa_1, \ldots, \kappa_m\right)^\top,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\Delta_i &= \varrho_i\left(\widehat{x}_{i1} - v_{i1}(t), \widehat{x}_{i2} - v_{i2}(t), \ldots, \widehat{x}_{in_i} - v_{in_i}(t)\right) \\
&\quad - \varrho_i(\kappa_i),
\end{aligned} \tag{18}$$

$$\begin{aligned}
\Phi_i(x_i) &= \left(x_{i2}, x_{i3}, \cdots,\right. \\
&\quad \varrho_i\left(\widehat{x}_{i1} - v_{i1}(t), \widehat{x}_{i2} - v_{i2}(t), \ldots, \widehat{x}_{in_i} - v_{in_i}(t)\right) \\
&\quad \left. + x_{i(n_i+1)} - \widehat{x}_{i(n_i+1)}\right)^\top, \\
\Phi(x) &= \left(\Phi_1(x_1)^\top, \Phi_2(x_2)^\top, \ldots, \Phi_m(x_m)^\top\right)^\top.
\end{aligned} \tag{19}$$

By Itô's formula, we can obtain that

$$\begin{aligned}
df_i(t,x,\zeta,w)\Big|_{\text{along}(1)} &= \left\{\frac{\partial f_i(t,x,\zeta,w)}{\partial t}\right. \\
&\quad + \left(\frac{\partial f_i(t,x,\zeta,w)}{\partial x}\right)^\top \Phi(x) + \left(\frac{\partial f_i(t,x,\zeta,w)}{\partial \zeta}\right)^\top \\
&\quad \cdot g_1(t,x,\zeta,w) + \frac{1}{2} \\
&\quad \cdot \operatorname{Tr}\left\{g_2^\top(t,x,\zeta,w)\frac{\partial^2 f_i(t,x,\zeta,w)}{\partial \zeta^2} g_2(t,x,\zeta,w)\right\} \\
&\quad + \frac{\partial f_i(t,x,\zeta,w)}{\partial w}\left(\frac{\partial \psi(t,W_2(t))}{\partial t} + \frac{1}{2}\right. \\
&\quad \left.\cdot \sum_{i=1}^{q} \frac{\partial^2 \psi(t,W_2(t))}{\partial \theta_i^2}\right) + \frac{1}{2}\frac{\partial^2 f_i(t,x,\zeta,w)}{\partial w^2} \\
&\quad \left.\cdot \sum_{i=1}^{q}\left(\frac{\partial \psi(t,W_2(t))}{\partial \theta_i}\right)^2\right\} dt \\
&\quad + \left(\frac{\partial f_i(t,x,\zeta,w)}{\partial \zeta}\right)^\top g_2(t,x,\zeta,w)\, dW_1(t) \\
&\quad + \frac{\partial f_i(t,x,\zeta,w)}{\partial w}\left(\frac{\partial \psi(t,W_2(t))}{\partial \theta}\right)^\top dW_2(t) \\
&\triangleq \vartheta_{i1} dt + \vartheta_{i2} dW_1(t) + \vartheta_{i3} dW_2(t),
\end{aligned} \tag{20}$$

where we set

$$\begin{aligned}
\vartheta_{i2} &= \left(\vartheta_{i2,j}\right)_{1\times p}, \\
\vartheta_{i2,j} &= \sum_{l=1}^{s} \frac{\partial f_i(t,x,\zeta,w)}{\partial \zeta_l} g_2^{(lj)}(t,x,\zeta,w), \\
\vartheta_{i3} &= \left(\vartheta_{i3,j}\right)_{1\times q}, \\
\vartheta_{i3,j} &= \frac{\partial f_i(t,x,\zeta,w)}{\partial w}\frac{\partial \psi(t,W_2(t))}{\partial \theta_j}.
\end{aligned} \tag{21}$$

By Assumptions (A1)-(A2), we can easily conclude that there exist r-independent positive constants $\delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4}$ such that

$$\begin{aligned}
\left|\vartheta_{i1}\right| &\le \delta_{i1} + \delta_{i2}\|\eta\| + \delta_{i3}\|\kappa\|, \\
\left\|\vartheta_{i2}\right\|^2 + \left\|\vartheta_{i3}\right\|^2 &\le \delta_{i4}.
\end{aligned} \tag{22}$$

Moreover,

$$\begin{aligned}
&\frac{d}{dt}\Big|_{\text{along}(4)}\left[\sum_{l=1}^{m}\left(b_{il} - b_{il}^*\right)u_l\right] = \frac{d}{dt}\Big|_{\text{along}(4)} \\
&\quad \cdot \left[\sum_{l,d=1}^{m}\left(b_{il} - b_{il}^*\right)\widehat{b}_{ld}^*\right. \\
&\quad \cdot \left(\varrho_d\left(\widehat{x}_{d1} - v_{d1}(t), \ldots, \widehat{x}_{dn_d} - v_{dn_d}(t)\right) - \widehat{x}_{d(n_d+1)}\right. \\
&\quad \left.\left. + v_{d(n_d+1)}(t)\right)\right] = \sum_{l,d=1}^{m}\left(b_{il} - b_{il}^*\right)\widehat{b}_{ld}^* \\
&\quad \cdot \left\{\sum_{j=1}^{n_d-1} k_{dj}\left(\widehat{x}_{d(j+1)} - v_{d(j+1)}(t) + \frac{a_{dj}}{r^{n_d-j}}\eta_{d1}\right)\right\} \\
&\quad + \sum_{l,d=1}^{m}\left(b_{il} - b_{il}^*\right)\widehat{b}_{ld}^*\left\{k_{dn_d}\left(\widehat{x}_{d(n_d+1)} - v_{d(n_d+1)}(t)\right.\right. \\
&\quad \left.\left. + a_{dn_d}\eta_{d1} + \sum_{k=1}^{m} b_{dk}^* u_k\right)\right\} + \sum_{l,d=1}^{m}\left(b_{il} - b_{il}^*\right)
\end{aligned}$$

$$\cdot \hat{b}_{ld}^{*}\left(-ra_{d(n_d+1)}\eta_{d1}\right) + \sum_{l,d=1}^{m}\left(b_{il} - b_{il}^{*}\right)\hat{b}_{ld}^{*}\dot{v}_{d(n_d+1)}(t) \triangleq \vartheta_{i4}. \tag{23}$$

Suppose that $r > 1$. It follows that there exist r-independent positive constants $\delta_{i5}, \delta_{i6}, \xi_i$ such that

$$\left|\vartheta_{i4}\right| \leq \delta_{i5}\|\eta\| + \delta_{i6}\|\kappa\| + r\xi_i\|\eta\|, \tag{24}$$

where

$$\xi_i = \sum_{l,d=1}^{m}\left|\left(b_{il} - b_{il}^{*}\right)\hat{b}_{ld}^{*}a_{d(n_d+1)}\right|. \tag{25}$$

Thus, it follows that the closed-loop of the x-subsystem of (1), 2, and (8) is equivalent to

$$\begin{aligned}
d\kappa_i &= A_{n_i}\kappa_i dt + B_{n_i}\left[\varrho_i(\kappa_i) + \Delta_i + \eta_{i(n_i+1)}\right]dt,\\
d\eta_i &= rA_{n_i+1}\eta_i dt - r\begin{pmatrix} a_{i1}\eta_{i1}\\ \cdots \\ a_{i(n_i+1)}\eta_{i1}\end{pmatrix}dt\\
&\quad + B_{n_i+1}\left(\vartheta_{i1} + \vartheta_{i4}\right)dt + B_{n_i+1}\vartheta_{i2}dW_1(t)\\
&\quad + B_{n_i+1}\vartheta_{i3}dW_2(t), \quad 1 \leq i \leq m,
\end{aligned} \tag{26}$$

where Δ_i is defined as that in (18). It follows from the definition of Δ_i in (18) that

$$\begin{aligned}
&\left|\Delta_i\right|^2\\
&\leq \left(\max_{1\leq j\leq n_i}\left|k_{ij}\right|\right)^2\left[\left(x_{i1} - \hat{x}_{i1}\right)^2 + \cdots + \left(x_{in_i} - \hat{x}_{in_i}\right)^2\right]\\
&= \left(\max_{1\leq j\leq n_i}\left|k_{ij}\right|\right)^2\left[\frac{1}{r^{2n_i}}\left|\eta_{i1}\right|^2 + \cdots + \frac{1}{r^2}\left|\eta_{in_i}\right|^2\right]\\
&\leq \left(\max_{1\leq j\leq n_i}\left|k_{ij}\right|\right)^2\left\|\eta_i\right\|^2, \quad i = 1, 2, \ldots, m.
\end{aligned} \tag{27}$$

The remaining proof is arranged in the following three steps.

Step 1. We prove that the solution (κ, η) of system (26) is practically mean square bounded.

We first define the positive definite functions $V_{1i}(\cdot) : \mathbb{R}^{n_i} \longrightarrow \mathbb{R}$, $V_{2i}(\cdot) : \mathbb{R}^{n_i+1} \longrightarrow \mathbb{R}$, and $V(\cdot) : \mathbb{R}^{2n+m} \longrightarrow \mathbb{R}$ as follows:

$$\begin{aligned}
V_{1i}(\kappa_i) &= \kappa_i^{\top}H_i\kappa_i,\\
V_{2i}(\eta_i) &= \eta_i^{\top}Q_i\eta_i,\\
V(\kappa, \eta) &= \sum_{i=1}^{m}V_{1i}(\kappa_i) + \sum_{i=1}^{m}V_{2i}(\eta_i),
\end{aligned} \tag{28}$$

where H_i and Q_i are the positive definite matrix solutions satisfying the Lyapunov equations $H_iF_i + F_i^{\top}H_i = -I_{n_i\times n_i}$ and $Q_iE_i + E_i^{\top}Q_i = -I_{(n_i+1)\times(n_i+1)}$, respectively.

It is easy to obtain that

$$\begin{aligned}
&\lambda_{\min}(H_i)\left\|\kappa_i\right\|^2 \leq V_{1i}(\kappa_i) \leq \lambda_{\max}(H_i)\left\|\kappa_i\right\|^2,\\
&\left|\frac{\partial V_{1i}(\kappa_i)}{\partial \kappa_{in_i}}\right| \leq 2\lambda_{\max}(H_i)\left\|\kappa_i\right\|,\\
&\left|\frac{\partial^2 V_{1i}(\kappa_i)}{\partial \kappa_{n_i}^2}\right| \leq 2\lambda_{\max}(H_i),\\
&\lambda_{\min}(Q_i)\left\|\eta_i\right\|^2 \leq V_{2i}(\eta_i) \leq \lambda_{\max}(Q_i)\left\|\eta_i\right\|^2,\\
&\left|\frac{\partial V_{2i}(\eta_i)}{\partial \eta_{i(n_i+1)}}\right| \leq 2\lambda_{\max}(Q_i)\left\|\eta_i\right\|,\\
&\left|\frac{\partial^2 V_{2i}(\eta_i)}{\partial \eta_{in_i}^2}\right| \leq 2\lambda_{\max}(Q_i),\\
&\left|\frac{\partial^2 V_{2i}(\eta_i)}{\partial \eta_{i(n_i+1)}^2}\right| \leq 2\lambda_{\max}(Q_i),\\
&\left|\frac{\partial^2 V_{2i}(\eta_i)}{\partial \eta_{in_i}\partial \eta_{i(n_i+1)}}\right| \leq 2\lambda_{\max}(Q_i).
\end{aligned} \tag{29}$$

Apply Itô's formula to $V(\kappa, \eta)$ with respect to t along the solution (κ, η) of system (26) to obtain

$$\begin{aligned}
dV(\kappa, \eta) &= \sum_{i=1}^{m}\left[\sum_{j=1}^{n_i-1}\frac{\partial V_{1i}(\kappa_i)}{\partial \kappa_{ij}}\kappa_{i(j+1)} + \frac{\partial V_{1i}(\kappa_i)}{\partial \kappa_{in_i}}\varrho_i(\kappa_i)\right.\\
&\quad \left. + \frac{\partial V_{1i}(\kappa_i)}{\partial \kappa_{in_i}}\Delta_i + \frac{\partial V_{1i}(\kappa_i)}{\partial \kappa_{in_i}}\eta_{i(n_i+1)}\right]dt\\
&\quad + r\sum_{i=1}^{m}\left[\sum_{j=1}^{n_i}\frac{\partial V_{2i}(\eta_i)}{\partial \eta_{ij}}\left(\eta_{i(j+1)} - a_{ij}\eta_{i1}\right)\right.\\
&\quad \left. - \frac{\partial V_{2i}(\eta_i)}{\partial \eta_{i(n_i+1)}}a_{i(n_i+1)}\eta_{i1}\right]dt + \sum_{i=1}^{m}\frac{\partial V_{2i}(\eta_i)}{\partial \eta_{i(n_i+1)}}(\vartheta_{i1}\\
&\quad + \vartheta_{i4})dt + \sum_{i=1}^{m}\frac{1}{2}\frac{\partial^2 V_{2i}(\eta_i)}{\partial \eta_{i(n_i+1)}^2}\left[\left\|\vartheta_{i2}\right\|^2 + \left\|\vartheta_{i3}\right\|^2\right]dt\\
&\quad + \sum_{i=1}^{m}\frac{\partial V_{2i}(\eta_i)}{\partial \eta_{i(n_i+1)}}\vartheta_{i2}dW_1(t) + \sum_{i=1}^{m}\frac{\partial V_{2i}(\eta_i)}{\partial \eta_{i(n_i+1)}}\vartheta_{i3}dW_2(t).
\end{aligned} \tag{30}$$

It is easy to conclude that there exist $0 < \mu < 1$ and $r_1 > 0$ such that

$$-\frac{r_1\Xi}{2} + \frac{4}{\mu}\max_{1\le i\le m}\lambda_{\max}^2(H_i)\max_{1\le i\le m}\left(\max_{1\le j\le n_i}\left|k_{ij}\right|\right)^2 + \frac{4}{\mu}\max_{1\le i\le m}\lambda_{\max}^2(H_i) + 1 + \sum_{i=1}^m 2\lambda_{\max}(Q_i)\delta_{i2} + \frac{4\max_{1\le i\le m}\left(\lambda_{\max}^2(Q_i)\delta_{i3}^2\right)}{\mu} + \sum_{i=1}^m 2\lambda_{\max}(Q_i)\delta_{i5} + \frac{4\max_{1\le i\le m}\left(\lambda_{\max}^2(Q_i)\delta_{i6}^2\right)}{\mu} < 0, \tag{31}$$

where Ξ is given in (14). Now we suppose that

$$r > r^* \triangleq \max\left\{1, r_1, \frac{2(1-\mu)}{\Xi}\right\}. \tag{32}$$

Set

$$\beta_1 = \max\left\{\max_{1\le i\le m}\{\lambda_{\max}(H_i)\}, \max_{1\le i\le m}\{\lambda_{\max}(Q_i)\}\right\}. \tag{33}$$

It follows from (22), (24), (27), (31), and Young's inequality that

$$\begin{aligned}\frac{d\mathbb{E}V(\kappa,\eta)}{dt} &\le -\mathbb{E}\|\kappa\|^2 + \sum_{i=1}^m 2\lambda_{\max}(H_i)\max_{1\le j\le n_i}\left|k_{ij}\right| \cdot \mathbb{E}\left(\|\kappa_i\|\cdot\|\eta_i\|\right) + \sum_{i=1}^m 2\lambda_{\max}(H_i)\mathbb{E}\left(\|\kappa_i\|\cdot\left|\eta_{i(n_i+1)}\right|\right) \\ &\quad - r\mathbb{E}\|\eta\|^2 + \sum_{i=1}^m 2\lambda_{\max}(Q_i)\mathbb{E}\{\|\eta_i\|\cdot(\delta_{i1} + \delta_{i2}\|\eta\| + \delta_{i3}\|\kappa\| + \delta_{i5}\|\eta\| + \delta_{i6}\|\kappa\| + r\xi_i\|\eta\|)\} \\ &\quad + \sum_{i=1}^m \lambda_{\max}(Q_i)\delta_{i4} \le -\mathbb{E}\|\kappa\|^2 + \frac{\mu}{4}\mathbb{E}\|\kappa\|^2 + \frac{4}{\mu} \cdot \max_{1\le i\le m}\lambda_{\max}^2(H_i)\cdot\max_{1\le i\le m}\left(\max_{1\le j\le n_i}\left|k_{ij}\right|\right)^2\mathbb{E}\|\eta\|^2 + \frac{\mu}{4} \\ &\quad \cdot \mathbb{E}\|\kappa\|^2 + \frac{4}{\mu}\max_{1\le i\le m}\lambda_{\max}^2(H_i)\mathbb{E}\|\eta\|^2 - r\mathbb{E}\|\eta\|^2 + \mathbb{E}\|\eta\|^2 + \sum_{i=1}^m\lambda_{\max}^2(Q_i)\delta_{i1}^2 + \sum_{i=1}^m 2\lambda_{\max}(Q_i) \\ &\quad \cdot \delta_{i2}\mathbb{E}\|\eta\|^2 + \frac{\mu}{4}\mathbb{E}\|\kappa\|^2 + \frac{4\max_{1\le i\le m}\left(\lambda_{\max}^2(Q_i)\delta_{i3}^2\right)}{\mu}\mathbb{E}\|\eta\|^2 + \sum_{i=1}^m 2\lambda_{\max}(Q_i)\delta_{i5}\mathbb{E}\|\eta\|^2 + \frac{\mu}{4}\mathbb{E}\|\kappa\|^2 \\ &\quad + \frac{4\max_{1\le i\le m}\left(\lambda_{\max}^2(Q_i)\delta_{i6}^2\right)}{\mu}\mathbb{E}\|\eta\|^2 + r\sum_{i=1}^m 2\lambda_{\max}(Q_i)\xi_i\mathbb{E}\|\eta\|^2 + \sum_{i=1}^m\lambda_{\max}(Q_i)\delta_{i4} \\ &\le -(1-\mu)\mathbb{E}\|\kappa\|^2 - \frac{r\Xi}{2}\mathbb{E}\|\eta\|^2 + \beta_2 \le -\frac{1-\mu}{\beta_1} \cdot \mathbb{E}V(\kappa,\eta) + \beta_2,\end{aligned} \tag{34}$$

where $\beta_2 \triangleq \sum_{i=1}^m\lambda_{\max}^2(Q_i)\delta_{i1}^2 + \sum_{i=1}^m\lambda_{\max}(Q_i)\delta_{i4}$ and Ξ is given in (14). It is easy to conclude that $\mathbb{E}V(\kappa(0),\eta(0)) \le M_1 r^{2\max_{1\le i\le m}n_i}$ for some r-independent constant M_1. Hence for any $r > r^*$ and any $\epsilon > 0$, there exists $t_r \triangleq r^\epsilon$ such that, for all $t \ge t_r$, we have

$$\begin{aligned}\mathbb{E}V(\kappa,\eta) &\le e^{-((1-\mu)/\beta_1)t}\mathbb{E}V(\kappa(0),\eta(0)) + \beta_2\int_0^t e^{-((1-\mu)/\beta_1)(t-s)}ds \\ &\le e^{-((1-\mu)/\beta_1)t_r}\mathbb{E}V(\kappa(0),\eta(0)) + \frac{\beta_1\beta_2}{1-\mu} \\ &\le M_2,\end{aligned} \tag{35}$$

for some r-independent constant $M_2 > 0$.

Step 2. We prove the convergence of the estimation errors of ESO for both unmeasured states and stochastic total disturbance in practical mean square sense.

Similar to the operations used in (34), it follows from (35) that, for all $t \ge t_r$, we have

$$\begin{aligned}\frac{d\mathbb{E}V_2(\eta)}{dt} &\le -\frac{r\Xi}{2}\mathbb{E}\|\eta\|^2 + \frac{\mu}{2}\mathbb{E}\|\kappa\|^2 + \beta_2 \\ &\le -\frac{r\Xi}{2\max_{1\le i\le m}\{\lambda_{\max}(Q_i)\}}\mathbb{E}V_2(\eta) + \frac{\mu M_2}{2\min_{1\le i\le m}\{\lambda_{\min}(H_i)\}} + \beta_2.\end{aligned} \tag{36}$$

Set

$$\beta_3 = \frac{\Xi}{2\max_{1\le i\le m}\{\lambda_{\max}(Q_i)\}},$$
$$\beta_4 = \frac{\mu M_2}{2\min_{1\le i\le m}\{\lambda_{\min}(H_i)\}} + \beta_2. \tag{37}$$

Then

$$\mathbb{E}V_2(\eta) \le e^{-r\beta_3(t-t_r)}\mathbb{E}V_2(\eta(t_r)) + \beta_4\int_{t_r}^{t} e^{-r\beta_3(t-s)}ds. \tag{38}$$

We can see from (35) that the first term of the right-hand side of (38) is bounded by $e^{-r\beta_3}$ multiplied by a r-independent constant and the second term is bounded by $1/r$ multiplied by a r-independent constant when $t \ge t_r + 1$; thus there exists a r-independent constant $\beta_5 > 0$ such that, for all $t \ge t_r + 1$,

$$\mathbb{E}V_2(\eta) \le \frac{\beta_5}{r}, \tag{39}$$

and thus

$$\mathbb{E}\|\eta_i\|^2 \le \frac{\mathbb{E}V_2(\eta)}{\lambda_{\min}(Q_i)} \le \frac{\beta_5}{r\lambda_{\min}(Q_i)}. \tag{40}$$

Therefore, for all $t \ge t_r + 1$, $1 \le i \le m$, $1 \le j \le n_i + 1$, we have

$$\mathbb{E}\left|x_{ij} - \widehat{x}_{ij}\right|^2 = \frac{1}{r^{2n_i+2-2j}}\mathbb{E}\left|\eta_{ij}\right|^2 \le \frac{1}{r^{2n_i+2-2j}}\mathbb{E}\|\eta_i\|^2$$
$$\le \frac{\beta_5}{\lambda_{\min}(Q_i)\,r^{2n_i+3-2j}}. \tag{41}$$

Step 3. We prove the convergence of the reference signals tracking errors in practical mean square.

For any $r > r^*$ and all $t \ge t_r + 1$, it follows from (34) and (40) that

$$\frac{d\mathbb{E}V_1(\kappa)}{dt} \le -\left(1-\frac{\mu}{2}\right)\mathbb{E}\|\kappa\|^2 + \left(\frac{4}{\mu}\max_{1\le i\le m}\lambda_{\max}^2(H_i)\right.$$
$$\left.\cdot\max_{1\le i\le m}\left(\max_{1\le j\le n_i}k_{ij}\right)^2 + \frac{4}{\mu}\max_{1\le i\le m}\lambda_{\max}^2(H_i)\right)\mathbb{E}\|\eta\|^2$$
$$\le -\left(1-\frac{\mu}{2}\right)\mathbb{E}\|\kappa\|^2 + \left(\frac{4}{\mu}\max_{1\le i\le m}\lambda_{\max}^2(H_i)\right. \tag{42}$$
$$\left.\cdot\max_{1\le i\le m}\left(\max_{1\le j\le n_i}k_{ij}\right)^2 + \frac{4}{\mu}\max_{1\le i\le m}\lambda_{\max}^2(H_i)\right)$$
$$\cdot\frac{m\beta_5}{r\min_{1\le i\le m}\lambda_{\min}(Q_i)} \le -\beta_6\mathbb{E}V_1(\kappa) + \frac{\beta_7}{r},$$

where we set

$$\beta_6 = \frac{1-\mu/2}{\max_{1\le i\le m}\{\lambda_{\max}(H_i)\}} \tag{43}$$

and

$$\beta_7 = \left(\frac{4}{\mu}\max_{1\le i\le m}\lambda_{\max}^2(H_i)\max_{1\le i\le m}\left(\max_{1\le j\le n_i}k_{ij}\right)^2\right.$$
$$\left.+\frac{4}{\mu}\max_{1\le i\le m}\lambda_{\max}^2(H_i)\right)\frac{m\beta_5}{\min_{1\le i\le m}\lambda_{\min}(Q_i)}. \tag{44}$$

Therefore, for any $r > r^*$ and all $t \ge t_r + 1$, we have

$$\mathbb{E}\|\kappa\|^2$$
$$\le \frac{1}{\min_{1\le i\le m}\{\lambda_{\min}(H_i)\}}e^{-\beta_6(t-t_r-1)}\mathbb{E}V_1(\kappa(t_r+1))$$
$$+\frac{\beta_7}{r\min_{1\le i\le m}\{\lambda_{\min}(H_i)\}}\int_{t_r+1}^{t} e^{-\beta_6(t-s)}ds. \tag{45}$$

It follows from (35) that the first term of the right-hand side of (45) is bounded by $e^{-\beta_6 r^{\epsilon}}$ multiplied by a r-independent constant and the second term is bounded by $1/r$ multiplied by a r-independent constant when for all $t \ge 2t_r + 1$ so that there exist $t_r^* \ge 2t_r + 1$ and $\Gamma \ge \beta_5/\min_{1\le i\le m}\lambda_{\min}(Q_i) > 0$ such that, for all $1 \le i \le m$, $1 \le j \le n_i$, we have

$$\mathbb{E}\left|x_{ij} - v_{ij}\right|^2 = \mathbb{E}\|\kappa_i\|^2 \le \mathbb{E}\|\kappa\|^2 \le \frac{\Gamma}{r}, \quad \forall t \ge t_r^*. \tag{46}$$

This completes the proof of Theorem 2. □

4. Numerical Simulations

In this section, we aim to verify the validity of the ADRC approach by considering the following MIMO uncertain stochastic system:

$$dx_{11} = x_{12}dt,$$
$$dx_{12} = \left[c_1x_{12} + c_2x_{21} + c_3\zeta + c_4\sin(c_5t + c_6W_2(t))\right.$$
$$\left.+u_1 + u_2\right]dt,$$
$$dx_{21} = x_{22}dt,$$
$$dx_{22} = \left[c_7x_{21} + c_8x_{12} + c_9\zeta + c_{10}\sin(c_5t + c_6W_2(t))\right.$$
$$\left.+u_1 - u_2\right]dt \tag{47}$$
$$d\zeta = \left[c_{11}\sin(\zeta)\cdot x_{11} + c_{12}\cos(\zeta)\cdot x_{22}\right]dt + c_{13}\sin(\zeta)$$
$$\cdot\sin(c_5t + c_6W_2(t))\,dW_1(t),$$
$$y_1 = x_{11},$$
$$y_2 = x_{21},$$

where c_i $(i = 1, 2, \ldots, 13)$ are unknown parameters satisfying $|c_i| \le M$ $(i = 1, 2, \ldots, 13)$ for a given known constant $M > 0$. $w(t) \triangleq \sin(c_5t + c_6W_2(t))$ is a bounded nonwhite noise that exists in many practical dynamical systems like the motion of oscillators [26, 27], where c_5 and c_6^2 denote the central frequency and strength of frequency disturbance, respectively. In this case, $m = 2, n_1 = n_2 = 2, n = n_1 + n_2 = 4, s = 1, p = q = 1, b_{11} = b_{12} = b_{21} = 1, b_{22} = -1$. In addition, the constants D_i $(i = 1,2,3,4)$ and the function $\varsigma(w)$ in Assumptions (A1)-(A2) can be specified as $D_1 = \max\{1, c_5, c_6, c_6^2\}$, $D_2 = 0$, $D_3 = \max\{c_{11}, c_{12}\}$, $D_4 = \max\{c_1, c_2, c_3, c_4, c_7, c_8, c_9, c_{10}\}$, and $\varsigma(w) = c_{13}|w|$. We can easily check that Assumptions (A1)-(A2) hold in this case.

The stochastic total disturbance $(x_{13}, x_{23})^\top$ is defined by

$$\begin{aligned} x_{13} &= c_1 x_{12} + c_2 x_{21} + c_3 \zeta + c_4 \sin\left(c_5 t + c_6 W_2(t)\right), \\ x_{23} &= c_7 x_{21} + c_8 x_{12} + c_9 \zeta + c_{10} \sin\left(c_5 t + c_6 W_2(t)\right). \end{aligned} \tag{48}$$

An ESO (49) is designed for system (47) as follows:

$$\begin{aligned} \dot{\hat{x}}_{11} &= \hat{x}_{12} + 3r\left(y_1 - \hat{x}_{11}\right), \\ \dot{\hat{x}}_{12} &= \hat{x}_{13} + 3r^2\left(y_1 - \hat{x}_{11}\right) + u_1 + u_2, \\ \dot{\hat{x}}_{13} &= r^3\left(y_1 - \hat{x}_{11}\right), \\ \dot{\hat{x}}_{21} &= \hat{x}_{22} + 3r\left(y_2 - \hat{x}_{21}\right), \\ \dot{\hat{x}}_{22} &= \hat{x}_{23} + 3r^2\left(y_2 - \hat{x}_{21}\right) + u_1 - u_2, \\ \dot{\hat{x}}_{23} &= r^3\left(y_2 - \hat{x}_{21}\right). \end{aligned} \tag{49}$$

In this case the corresponding matrices in (5) become

$$E_1 = E_2 = \begin{pmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \tag{50}$$

where all eigenvalues are -1 and thus these two matrices are Hurwitz. Choose $\varrho_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $(i = 1,2)$ in (7) as follows:

$$\begin{aligned} \varrho_1\left(z_{11}, z_{12}\right) &= -2z_{11} - 3z_{12}, \\ \varrho_2\left(z_{21}, z_{22}\right) &= -2z_{21} - 2z_{22} \end{aligned} \tag{51}$$

with the corresponding matrices in (9)

$$\begin{aligned} F_1 &= \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \\ F_2 &= \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \end{aligned} \tag{52}$$

being Hurwitz. ESO (49) based feedback control is designed as

$$\begin{aligned} u_1 &= \frac{1}{2}\left(\varrho_1\left(\hat{x}_{11} - v_{11}, \hat{x}_{12} - v_{12}\right) - \hat{x}_{13} + v_{13}\right) \\ &\quad + \frac{1}{2}\left(\varrho_2\left(\hat{x}_{21} - v_{21}, \hat{x}_{22} - v_{22}\right) - \hat{x}_{23} + v_{23}\right), \\ u_2 &= \frac{1}{2}\left(\varrho_1\left(\hat{x}_{11} - v_{11}, \hat{x}_{12} - v_{12}\right) - \hat{x}_{13} + v_{13}\right) \\ &\quad - \frac{1}{2}\left(\varrho_2\left(\hat{x}_{21} - v_{21}, \hat{x}_{22} - v_{22}\right) - \hat{x}_{23} + v_{23}\right), \end{aligned} \tag{53}$$

where in this case $\hat{b}_{il}^*$ $(i, l = 1,2)$ in (8) are specified as $\hat{b}_{11}^* = \hat{b}_{12}^* = \hat{b}_{21}^* = 1/2$, $\hat{b}_{22}^* = -1/2$. Both system (47) and system (49) are discretized by the Milstein approximation method proposed in [34]. In Figures 1 and 2, the initial values are

$$\begin{aligned} x(0) &= (1, -1, 1, -1), \\ \zeta(0) &= 0, \\ \hat{x}(0) &= (0, 0, 0, 0, 0, 0), \end{aligned} \tag{54}$$

the time discrete step is taken as

$$\Delta t = 0.001, \tag{55}$$

and the gain r in (49) is taken as

$$r = 100. \tag{56}$$

The uncertain parameters and reference signals in Figures 1 and 2 are taken as

$$\begin{aligned} c_1 &= 1, \\ c_2 &= 2, \\ c_3 &= 1, \\ c_4 &= \frac{1}{3}, \\ c_i &= 1 \quad (i = 5, 6, \ldots, 13), \\ v_{11}(t) &= \sin(t+1), \\ v_{21}(t) &= \cos(t+1), \end{aligned} \tag{57}$$

and

$$\begin{aligned} c_1 &= 1, \\ c_2 &= 2, \\ c_3 &= 2, \\ c_4 &= 1, \\ c_5 &= 1, \\ c_6 &= 2, \\ c_7 &= 1, \\ c_8 &= 1.5, \\ c_i &= 2 \quad (i = 9, 10, 11, 12, 13), \\ v_{11}(t) &= \sin(2t+1), \\ v_{21}(t) &= \cos(2t+1), \end{aligned} \tag{58}$$

respectively.

It is observed from both Figures 1 and 2 that ESO (49) is very valid to estimate both the state $x = (x_{11}, x_{12}, x_{21}, x_{22})^\top$ and the stochastic total disturbance $x = (x_{13}, x_{23})^\top$ defined

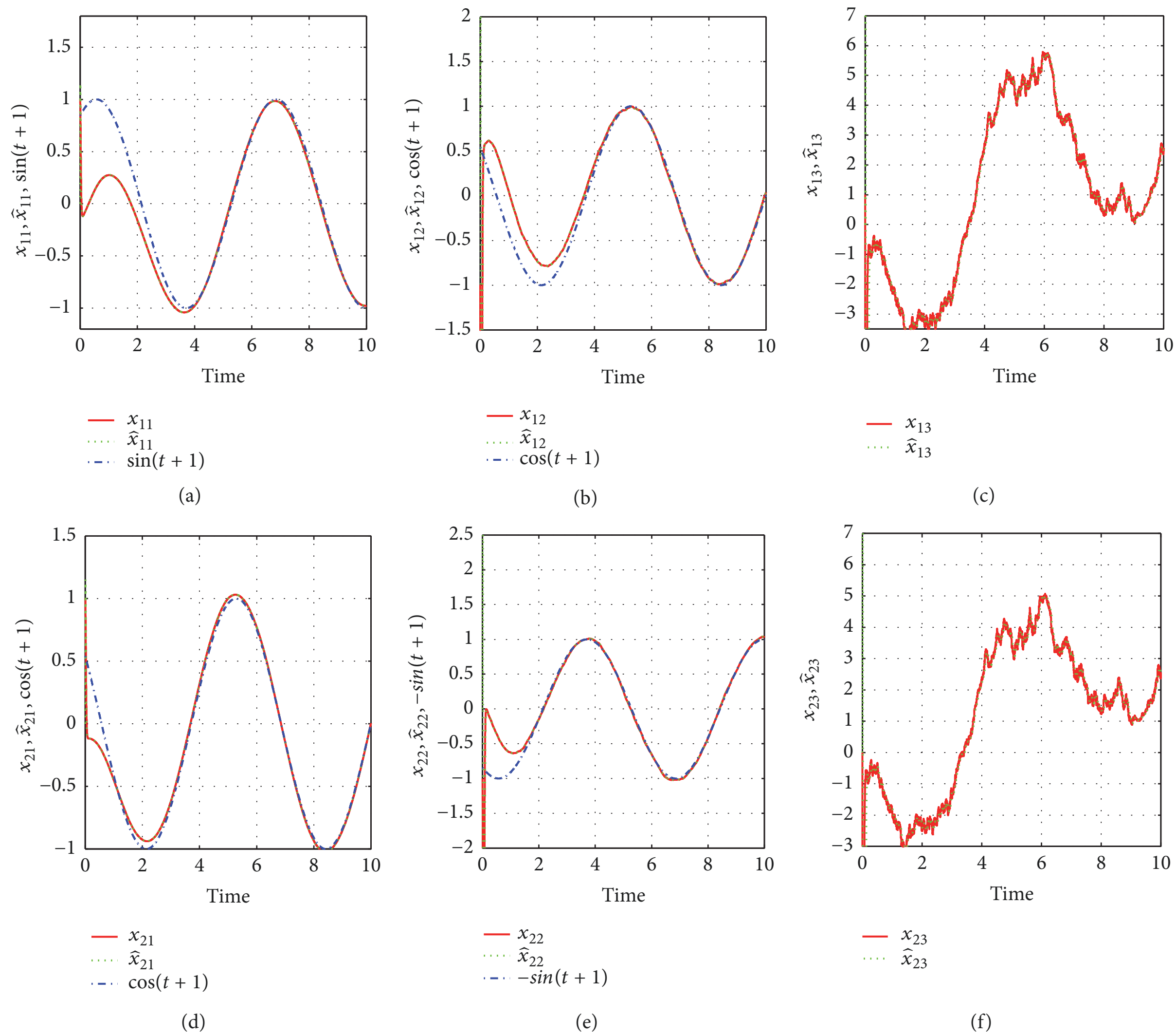

FIGURE 1: Estimation of unmeasured states and stochastic total disturbance of the ADRC's closed-loop system and the closed-loop reference signal tracking with the uncertain parameters and the reference signal given in (57).

in (48), despite varying system parameters and reference signals between (57) and (58). In addition, it is seen from Figures 1 and 2 that the tracking effects of $x_{i1} (i = 1, 2)$ to $v_{i1}(t)$ $(i = 1, 2)$ and x_{i2} $(i = 1, 2)$ to $v_{i2}(t)$ $(i = 1, 2)$ are very satisfactory. Figure 2 shows that the estimation effects of ESO and the performance of reference signals tracking are still very effective regardless of the larger parameters than the one given in (57), which validates the good robust performance of ADRC.

5. Concluding Remarks

In this paper, the active disturbance rejection control (ADRC) approach is applied to the output regulation problem for a class of multi-input multioutput (MIMO) uncertain stochastic nonlinear systems subject to vast stochastic uncertainties. The stochastic uncertainties of each subsystem including unknown nonlinear system functions, external stochastic disturbance with unknown statistical property, unknown stochastic inverse dynamics, uncertain nonlinear coupling effects between subsystems, and uncertainties caused by the partially unknown input gains are first regarded as the stochastic total disturbance of each subsystem. The stochastic total disturbance is then estimated by a linear ESO in real time and cancelled in the ESO-based output feedback loop. The output regulation performance of the resulting ADRC's closed-loop system is obtained with rigorous theoretical proof including practical mean square reference signals tracking, disturbance attenuation, and practical mean square stability when the reference signals are zero. Finally, the validity of the estimation performance of ESO, efficient reference tracking of ADRC, and good robust performance of ADRC are shown by some numerical simulations.

Data Availability

No data were used to support this study.

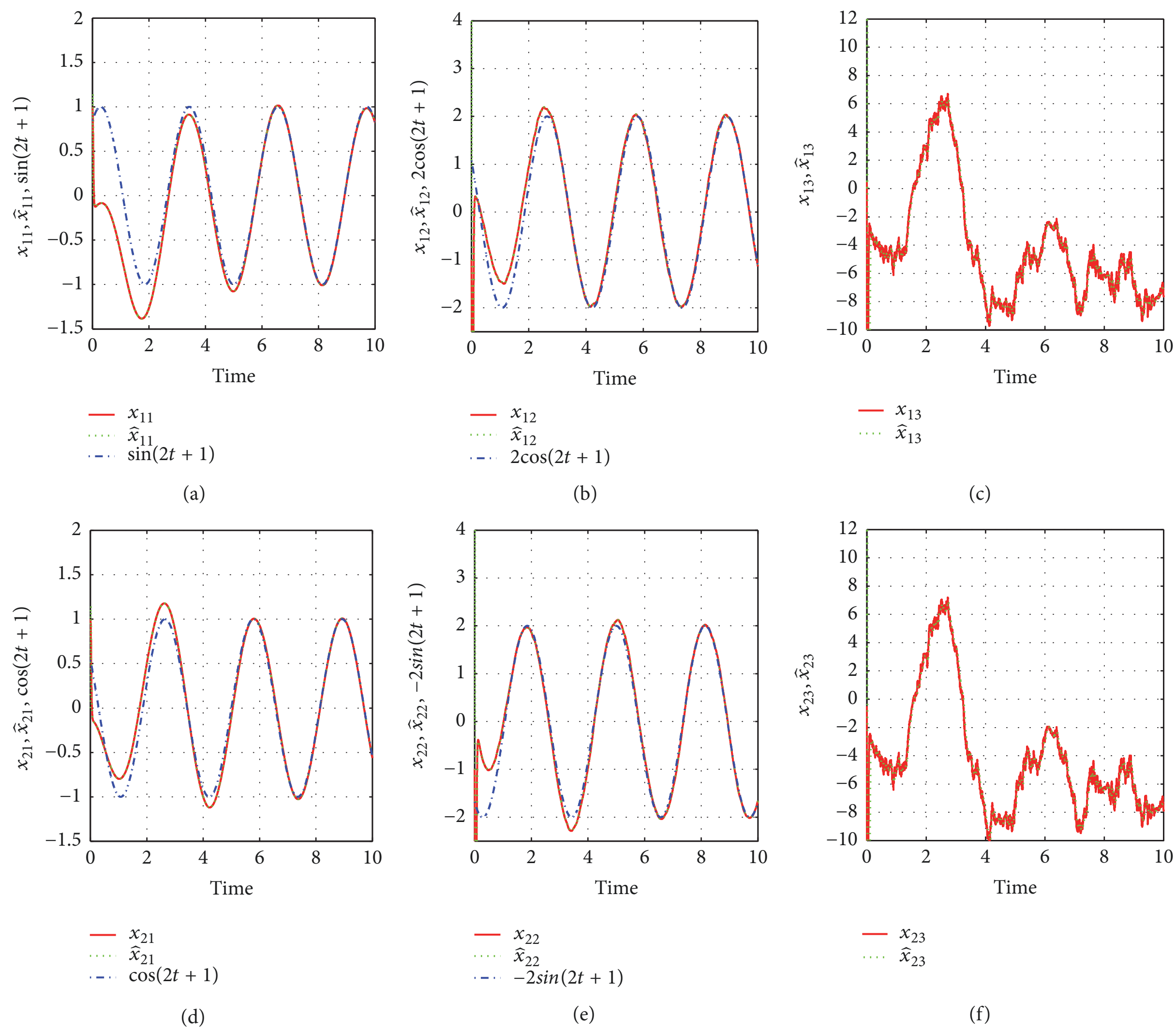

Figure 2: Estimation of unmeasured states and stochastic total disturbance of the ADRC's closed-loop system and the closed-loop reference signal tracking with the uncertain parameters and the reference signal given in (58).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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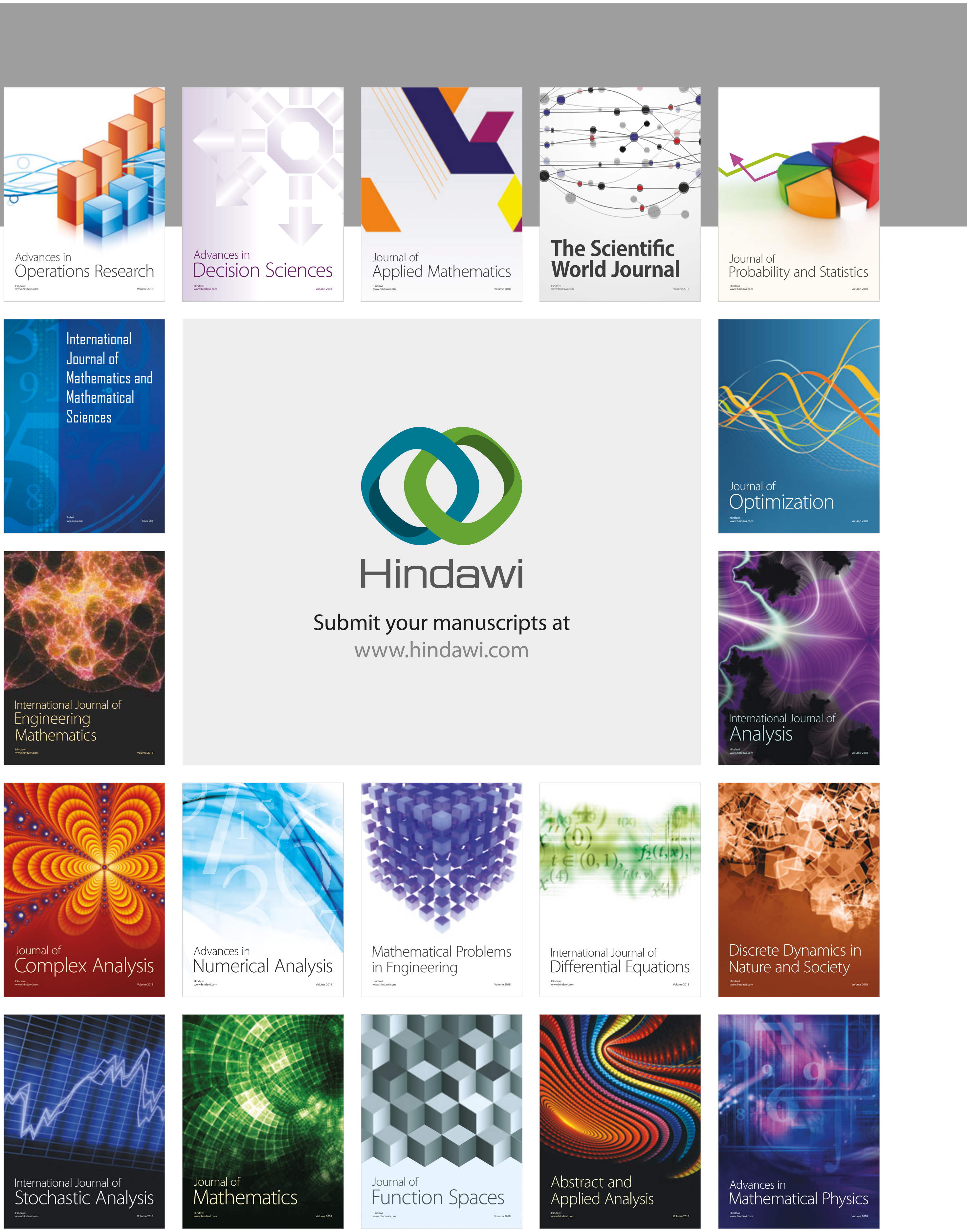
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