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Research Article

Spectrum and Stability of a 1-*d* Heat-Wave Coupled System with Dynamical Boundary Control

Xue-Lian Jin,[1] Yan Li,[2] and Fu Zheng[2]

[1] *Science College, Liaoning University of Technology, Jinzhou, 121001, China*
[2] *College of Mathematics and Physics, Bohai University, Jinzhou, 121013, China*

Correspondence should be addressed to Fu Zheng; zhengfu@amss.ac.cn

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In this paper, the negative proportional dynamic feedback is designed in the right boundary of the wave component of the 1-*d* heat-wave system coupled at the interface and the long-time behavior of the system is discussed. The system is formulated into an abstract Cauchy problem on the energy space. The energy of the system does not increase because the semigroup generated by the system operator is contracted. In the meanwhile, the asymptotic stability of the system is derived in light of the spectral configuration of the system operator. Furthermore, the spectral expansions of the system operator are precisely investigated and the asymptotical stability is not exponential and is shown in view of the spectral expansions.

1. Introduction

This paper is devoted to analyzing 1-*d* heat-wave coupled system with a dynamical boundary feedback (see [1–3]). More precisely, we study the following coupled system:

$$\begin{aligned}
y_t(t,x) - y_{xx}(t,x) &= 0, \quad x \in (-1,0), \ t > 0, \\
z_{tt}(t,x) - z_{xx}(t,x) &= 0, \quad x \in (0,1), \ t > 0, \\
y(t,-1) &= 0, \\
z_x(t,1) &= -\eta(t) \\
\eta_t(t) - z_t(t,1) &= -\beta\eta(t) \\
y(t,0) &= z_t(t,0), \\
y_x(t,0) &= z_x(t,0) \\
y(0,x) &= y_0(x), \\
\eta(0) &= \eta_0, \\
z(0,x) &= z_0(x), \\
z_t(0,x) &= z_1(x),
\end{aligned} \tag{1}$$

where β is a positive constant, η denotes the dynamical control, and $-\beta\eta(t)$ is the negative proportional feedback. This system consists of a wave equation, arising on the interval $(0,1)$ with state (z, z_t) and a heat equation that holds on the interval $(-1,0)$ with state y. The wave and heat components are coupled through an interface, the point $x = 0$, with transmission conditions imposing the continuity of (y, z_t) and (y_x, z_x).

The wave component in (1) is similar to that of [2] and the coupled heat-wave model without dynamical control comes from [1]. However, the coupled heat-wave model described by (1) has a slight difference with that of [1]. The zero Dirichlet boundary at the end point of the wave component was considered in [1]. But the Neumann boundary control is imposed at the same position in (1). This means that a dynamic boundary controller is applied at the free end of the system.

In recent years, the hyperbolic-parabolic coupled models have been studied extensively due to their applications in analyzing fluid-structure interactions, which are crucial in many scientific and engineering areas, such as air flow along the aircraft, deformation of heart valves, and the process of mixing [1, 4–6].

Some researchers studied the decay rates for the heat-wave coupled systems with various boundary conditions. In 2003, Zhang and Zuazua discussed the polynomial decay and controllability of the systems similar to (1) by the theory of operator semigroup and Riesz basis method in [7, 8]. In 2007, they analyzed the long-time behavior of a coupled heat-wave system in which a wave and a heat equation evolve in two bounded domains of $\mathbb{R}^n$, with natural transmission conditions at a common interface in [9]. In 2009, Dalsen considered the question of stabilization of a fluid-structure model that describes the interaction between a 3D incompressible fluid and a 2D plate, the interface, which coincides with a flat flexible part of the surface of the vessel containing the fluid through a frequency domain approach in [10].

In 2013, using the same approach, Avalos and Triggiani gave the rational decay rate for a PDE heat-structure interaction in [11]. In 2016, by the frequency domain approach together with multipliers, Han and Zuazua discussed the large time decay rate of a transmission problem coupled heat and wave equations on a planar network in [12]. In 2018, Peralta and Kunisch studied a coupled parabolic-hyperbolic system of partial differential equations modeling the interaction of a structure submerged in a fluid in [13]. The system incorporates delay in the interaction on the interface between the fluid and the solid. They discussed the stability property of the interaction model under suitable assumptions between the competing strength of the delay and the feedback control. For more results on the heat-wave coupled systems, one can also refer to [14, 15] and references therein.

The stabilities or stabilizations were obtained through static boundary conditions or feedbacks in the literatures above. However, the dynamical boundary controls were widely used in the beam equations or the wave equations [16–19]. Morgül proposed a dynamic boundary controller applied at the free end of the 1-d linear wave equation in [17]. The stabilities of the system were analyzed in light of the transfer function. For example, the transfer function of the wave equation controlled by the dynamic boundary controller in [2] is $g(s) = 1/(s+\beta)$; then the stabilization is asymptotic but not exponential since the transfer function $g(s)$ is strictly positive real. Furthermore, under some conditions, the exponential stability of the wave equation with similar dynamic boundary controller was obtained in [18]. In fact, the dynamical control forms a part of indirect mechanism proposed by Russell (see [19–21] and the references therein). More recently, Wehbe studied the exact controllability of a wave equation with dynamical boundary control in [3]. In 2015, Feng and Guo systematically investigated stability equivalence between dynamic output feedback and static output feedback for a class of second-order systems in [22].

The main contributions of this note are as follows. When the heat equation is not present, the system (with boundary feedback) was discussed in [2], and the polynomial (but not exponential) stability was shown. In the same time, when the heat equation was present (without dynamical boundary feedback), the system was analyzed in [1] and the asymptotical (but not exponential) stability was shown again. We combine these two results and show that the similar results hold in this note. But this extension is not trivial and should give strict proof. Moreover, we give the asymptotic expansions of the spectrum of the system operator. In light of the spectral analysis, we show that the system is not exponential stabilization. Though we believe that it is not hard to obtain the rational decay rate of the system (1) by the frequency domain method (see [23–25]) or Riesz basis method (see [2, 7, 8, 12, 26]), we do not explicitly elaborate because the information reflected by the spectrum is enough in some cases (see [2, 7, 8, 14, 15, 27, 28]).

The outline of this paper is as follows. In Section 2, we transform (1) into an abstract Cauchy problem on the energy space. The well-posedness and asymptotic stability of system (1) are given through the theory of operator semigroup and the spectral configuration of the system operator. In Section 3, the spectral expansions, which imply nonexponential decay rate of energy of system (1), of system operator are given. In the last section, we make a concise concluding remark.

2. Well-Posedness and Asymptotic Stability

In this section, we shall transform (1) into an abstract Cauchy problem on a suitable state space. For this purpose, let $H^m(a,b)$ be the Sobolev spaces of positive integer order m on the open interval (a,b). The energy space of (1) is the Hilbert space

$$\mathscr{H} := L^2(-1,0) \times H^1(0,1) \times L^2(0,1) \times \mathbb{C} \tag{2}$$

with the norm

$$\begin{aligned} \|(y,z,w,\eta)\|_{\mathscr{H}}^2 &= \int_{-1}^{0} |y(x)|^2\,dx + \int_0^1 |z_x(x)|^2\,dx \\ &\quad + \int_0^1 |w(x)|^2\,dx + |\eta|^2, \\ &\qquad \forall (y,z,w,\eta) \in \mathscr{H}. \end{aligned} \tag{3}$$

Introduce the operator A in $\mathscr{H}$ as follows:

$$D(A) = \left\{ \begin{pmatrix} y \\ z \\ w \\ \eta \end{pmatrix} \in \mathscr{H} \;\middle|\; \begin{array}{c} y \in H^2(-1,0),\, z \in H^2(0,1), \\ w \in H^1(0,1),\, y(-1) = 0, \\ y(0) = w(0),\, y_x(0) = z_x(0), \\ z_x(1) = -\eta \end{array} \right\}, \tag{4}$$

$$Au = \begin{pmatrix} y_{xx} \\ w \\ z_{xx} \\ w(1) - \beta\eta \end{pmatrix}, \quad \forall u = \begin{pmatrix} y \\ z \\ w \\ \eta \end{pmatrix} \in D(A). \tag{5}$$

Setting $u = (y, z, w, \eta)^{\top} \in \mathscr{H}$, then we formally transform system (1) into an evolution equation on $\mathscr{H}$

$$\begin{aligned} u'(t) &= Au(t), \\ u(0) &= u_0 \in \mathscr{H}. \end{aligned} \tag{6}$$

Now we give one of the main results of this section.

Theorem 1. *The operator $(A, D(A))$ generates a contracted C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathscr{H}$.*

Proof. It is easy to see that the operator A is densely defined and closed. It follows from a straightforward computation that

$$\operatorname{Re}\langle Au, u\rangle_{\mathscr{H}} = -\int_{-1}^{0} |y_x(x)|^2\,dx - \beta|\eta|^2 \leq 0, \quad \forall u \in D(A). \tag{7}$$

This means that the operator A is dissipative.

Moreover, for any $u = (y, z, w, \eta)^{\perp} \in D(A)$ and $\tilde{u} = (\tilde{y}, \tilde{z}, \tilde{w}, \tilde{\eta})^{\perp} \in \mathscr{H}$, we have

$$\begin{aligned} \langle Au, \tilde{u}\rangle_{\mathscr{H}} &= \int_{-1}^{0} y_{xx}(x)\,\tilde{y}(x)\,dx + \int_0^1 w_x(x)\,\tilde{z}(x)\,dx \\ &\quad + \int_0^1 z_{xx}(x)\,\tilde{w}(x)\,dx + (w(1) - \beta\eta)\,\tilde{\eta} \\ &= \int_{-1}^{0} y(x)\,\tilde{y}_{xx}(x)\,dx - \int_0^1 w(x)\,\tilde{z}_x(x)\,dx \\ &\quad - \int_0^1 z_x(x)\,\tilde{w}_x(x)\,dx + (w(1) - \beta\eta)\,\tilde{\eta} \\ &\quad + y_x(x)\,\tilde{y}(x)\big|_{-1}^{0} - y(x)\,\tilde{y}_x(x)\big|_{-1}^{0} \\ &\quad + w(x)\,\tilde{z}_x(x)\big|_0^1 + z_x(x)\,\tilde{w}(x)\big|_0^1 \\ &= \int_{-1}^{0} y(x)\,\tilde{y}_{xx}(x)\,dx - \int_0^1 w(x)\,\tilde{z}_x(x)\,dx \\ &\quad - \int_0^1 z_x(x)\,\tilde{w}_x(x)\,dx - \eta\,(\tilde{w}(1) + \beta\tilde{\eta}) \\ &\quad + y_x(0)\left[\tilde{y}(0) - \tilde{w}(0)\right] \\ &\quad - y(0)\left[\tilde{y}_x(0) + \tilde{z}_x(0)\right] \\ &\quad + w(1)\left[\tilde{z}_x(1) + \tilde{\eta}\right] - y_x(-1)\,\tilde{y}(-1) \end{aligned} \tag{8}$$

It follows from the above identities that the adjoint A^* of the operator A is given as

$$\begin{aligned} D(A^*) &= \left\{ \begin{pmatrix} y \\ z \\ w \\ \eta \end{pmatrix} \in \mathscr{H} \;\middle|\; \begin{array}{c} y \in H^2(-1,0),\, z \in H^2(0,1), \\ w \in H^1(0,1),\, y(-1) = 0, \\ y(0) = w(0),\, y_x(0) = -z_x(0), \\ z_x(1) = -\eta \end{array} \right\}, \\ A^* u &= \begin{pmatrix} y_{xx} \\ -w \\ -z_{xx} \\ -w(1) - \beta\eta \end{pmatrix}, \quad \forall u = \begin{pmatrix} y \\ z \\ w \\ \eta \end{pmatrix} \in D(A^*). \end{aligned} \tag{9}$$

Similarly, we also have

$$\operatorname{Re}\langle A^* u, u\rangle_{\mathscr{H}} = -\int_{-1}^{0} |y_x(x)|^2\,dx - \beta|\eta|^2 \leq 0, \quad \forall u \in D(A^*). \tag{10}$$

This shows that the operator A^* is dissipative as well. It follows form Pro.3.1.11 of [29] that the operator A is further m-dissipative. Thus the Lumer-Phillips theorem (Theorem 3.8.4. of [29]) implies that the statement of the theorem holds. □

To show the asymptotic stability of system (1), we should analyze the spectral configuration of the operator A.

Lemma 2. *Let A and $\mathscr{H}$ be defined as above; then we have $0 \in \rho(A)$. Furthermore, A^{-1} is compact on $\mathscr{H}$. Therefore, the spectrum of A consists only of the isolated eigenvalues of finite multiplicity, i.e., $\sigma(A) = \sigma_p(A)$.*

Proof. For any fixed $\tilde{u} \in \mathscr{H}$, we consider the solvability of the resolvent equation $Au = \tilde{u}$, which is equivalent to

$$\begin{aligned} y_{xx}(x) &= \tilde{y}(x), \quad x \in (-1, 0); \\ w(x) &= \tilde{z}(x), \quad x \in (0, 1); \\ z_{xx}(x) &= \tilde{w}(x), \quad x \in (0, 1); \\ w(1) - \beta\eta &= \tilde{\eta}; \end{aligned} \tag{11}$$

with the known conditions

$$\begin{aligned} y(-1) &= 0, \\ z_x(1) &= -\eta, \\ y_x(0) &= z_x(0), \\ y(0) &= w(0). \end{aligned} \tag{12}$$

By (11) and (12), a straightforward computation yields

$$\begin{aligned} y(x) &= \int_{-1}^{x}\int_{0}^{s}\widetilde{y}(r)\,dr\,ds - (x+1)\int_{0}^{1}\widetilde{w}(r)\,dr \\ &\quad - \frac{\widetilde{z}(1)-\widetilde{\eta}}{\beta}(x+1), \quad x \in (-1,0); \\ z(x) &= \int_{0}^{x}\int_{1}^{s}\widetilde{w}(r)\,dr\,ds - \frac{\widetilde{z}(1)-\widetilde{\eta}}{\beta}x, \quad x \in (0,1); \\ w(x) &= \widetilde{z}(x) \quad x \in (0,1); \\ \eta &= \frac{\widetilde{z}(1)-\widetilde{\eta}}{\beta}. \end{aligned} \tag{13}$$

From the expressions of $y(x)$, $z(x)$, and $w(x)$ in (13), we see that $u = (y, z, w, \eta) \in D(A)$ and $Au = \widetilde{u}$. This means that there exists a nonzero element $u \in D(A)$ such that $Au = \widetilde{u}$. Moreover, it is easy to see that $\ker(A) = \{0\}$. Therefore, by the inverse operator theorem, $0 \in \rho(A)$. According to the Sobolev embedding theorem, we have that $D(A)$ is a compact subspace of $\mathscr{H}$, which implies that A^{-1} is compact on $\mathscr{H}$. Hence, A is a discrete operator in $\mathscr{H}$. Therefore, the spectrum of A consists only of isolated eigenvalues of finite multiplicity (see [27, 28]). The proof of the lemma is complete. □

Furthermore, we have the following spectral distribution result.

Lemma 3. *There is no eigenvalue of A on the imaginary axis.*

Proof. We use a contradiction argument and assume that the conclusion of the lemma is false. That is to say there exist $0 \neq b \in \mathbb{R}$ and $0 \neq u = (y, z, w, \eta) \in D(A)$ such that $Au = ibu$. It is easy to see that

$$\operatorname{Re}\langle Au, u\rangle = \operatorname{Re}\langle ibu, u\rangle = 0, \tag{14}$$

which together with (7) implies that $y_x(x) = 0$ and $\eta = 0$. On the other hand, $Au = ibu$ is equivalent to

$$\begin{aligned} y_{xx}(x) &= iby(x), \\ w(x) &= ibz(x), \\ z_{xx}(x) &= -bz(x) \end{aligned} \tag{15}$$

with the known conditions

$$\begin{aligned} y(-1) = y_x(0) = z_x(0) = z_x(1) = w(1) = z(1) = 0, \\ w(0) = y(0). \end{aligned} \tag{16}$$

Obviously, we have

$$z(x) = ce^{ibx} + de^{-ibx} \tag{17}$$

and

$$\begin{aligned} ce^{ib} + de^{-ib} &= 0, \\ ce^{ib} - de^{-ib} &= 0. \end{aligned} \tag{18}$$

This means that $c = d = 0$ and $z(x) = 0$. By the same argument, we have $y(x) = 0$ and $w(x) = 0$. Thus we show that $u = 0$. This contradicts $u \neq 0$. Therefore, the statement of the lemma holds. □

Theorem 1 and Lemmas 2 and 3 imply that system (1) is asymptotically stable. Certainly, one can also obtain the same result by using LaSalle's invariance principle (see [17] or [30]).

3. Spectral Expansion of the System Operator A

Firstly, we reduce the eigenvalue problem of A to the problem of finding nonzero root of a transcendental function. For convenience, we introduce the following functions:

$$\begin{aligned} r(\lambda) &= \frac{\lambda+\beta+1}{\lambda+\beta-1}e^{2\lambda}, \\ s(\lambda) &= \frac{\lambda+\beta-1}{\lambda+\beta+1}e^{-2\lambda}. \end{aligned} \tag{19}$$

Lemma 4. *The set consisting of the eigenvalues of the operator A is given by*

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \mid f(\lambda) = 0\} \tag{20}$$

where

$$\begin{aligned} &f(\lambda) \\ &= (\lambda+\beta+1)e^{2\lambda}\left[\sqrt{\lambda}\left(e^{2\sqrt{\lambda}}+1\right)+\left(e^{2\sqrt{\lambda}}-1\right)\right] \\ &\quad + (\lambda+\beta-1)\left[\sqrt{\lambda}\left(e^{2\sqrt{\lambda}}+1\right)-\left(e^{2\sqrt{\lambda}}-1\right)\right]. \end{aligned} \tag{21}$$

Moreover, for any $\lambda \in \sigma_p(A)$, the corresponding eigenvector is

$$a(y, z, w, \eta), \quad 0 \neq a \in \mathbb{C}, \tag{22}$$

in which

$$\begin{aligned} y &= y(x,\lambda) = \frac{\lambda(1+s(\lambda))}{1-e^{2\sqrt{\lambda}}}\left[e^{-\sqrt{\lambda}x} - e^{\sqrt{\lambda}(x+2)}\right], \\ &\qquad x \in (-1,0), \\ z &= z(x,\lambda) = s(\lambda)e^{\lambda x} + e^{-\lambda x}, \quad x \in (0,1), \\ w &= w(x,\lambda) = \lambda\left[s(\lambda)e^{\lambda x} + e^{-\lambda x}\right], \quad x \in (0,1), \\ \eta &= \eta(\lambda) = -\lambda\left(s(\lambda)e^{\lambda} - e^{-\lambda}\right). \end{aligned} \tag{23}$$

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and $(y, z, w, \eta) \in D(A) \setminus \{0\}$ the corresponding eigenvector; then it is easy to check that

$$\begin{aligned} y_{xx}(x) &= \lambda y(x) \quad x \in (-1, 0), \\ z_{xx}(x) &= \lambda^2 z(x), \quad x \in (0, 1), \\ w(x) &= \lambda z(x), \quad x \in (0, 1), \\ w(1) - \beta\eta &= \lambda\eta, \end{aligned} \tag{24}$$

$$\begin{aligned} y(-1) &= 0, \\ z_x(1) &= -\eta, \\ y(0) &= w(0), \\ z_x(0) &= y_x(0). \end{aligned} \tag{25}$$

By the first two equations in (24), we conclude that there exist four constants a, b, c, and d such that

$$\begin{aligned} y(x) &= ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}, \quad x \in (-1, 0,) \\ z(x) &= ce^{\lambda x} + de^{-\lambda x}, \quad x \in (0, 1). \end{aligned} \tag{26}$$

Using $y(-1) = 0$, we have $a = -be^{2\sqrt{\lambda}}$ and

$$y(x) = b\left[e^{-\sqrt{\lambda}x} - e^{\sqrt{\lambda}(x+2)}\right]. \tag{27}$$

Using other known conditions in (24) and (26)-(27), we get

$$\begin{aligned} \lambda c + \lambda d - \left(1 - e^{2\sqrt{\lambda}}\right) b &= 0, \\ \sqrt{\lambda} c - \sqrt{\lambda} d + \left(1 + e^{2\sqrt{\lambda}}\right) b &= 0, \\ (\lambda + \beta + 1) e^{\lambda} c - (\lambda + \beta - 1) e^{-\lambda} d &= 0. \end{aligned} \tag{28}$$

System (28) is seemed as the equations on the unknown parameters b, c, and d. It is easy to see that (y, z, w, η) is nonzero vector if and only if the coefficient determinant of (28) is zero, i.e.,

$$\triangle(\lambda) := \begin{vmatrix} \lambda & \lambda & e^{2\sqrt{\lambda}} - 1 \\ \sqrt{\lambda} & -\sqrt{\lambda} & 1 + e^{2\sqrt{\lambda}} \\ (\lambda + \beta + 1) e^{\lambda} & -(\lambda + \beta - 1) e^{-\lambda} & 0 \end{vmatrix} = 0. \tag{29}$$

However, it is easy to see that $\lambda \in \mathbb{C}$ is a root of $\triangle(\lambda)$ if and only if it is that of $f(\lambda)$, which is defined in (21). Thus the first statement of the lemma holds. In the moreover part, we observe that c and d cannot be zero simultaneously. Without loss of generality, we may choose $c = 1$. By (28) and some simple calculations, we obtain $b = \lambda(1 + r(\lambda))/(1 - e^{2\sqrt{\lambda}})$ and $d = r(\lambda)$. Substituting them into the corresponding equations of (24) and (26), we get (23). The proof of the lemma is complete. □

We now mainly focus on analyzing the asymptotic behavior of "large" roots λ of $f(\lambda)$ since this is essential to both the Riesz bases property of generalized eigenvectors and polynomial stability of system (1) (see [1, 5, 14, 15]). When λ is sufficiently large, the leading term of $f(\lambda)$ is

$$L(\lambda) = \sqrt{\lambda}\left[(\lambda + \beta + 1) e^{2\lambda} + (\lambda + \beta - 1)\right]\left(1 + e^{2\sqrt{\lambda}}\right). \tag{30}$$

It is easy to see that $L(\lambda)$ has two classes of nonzero roots, i.e., the roots of "$1 + e^{2\sqrt{\lambda}}$" and "$(\lambda + \beta + 1)e^{2\lambda} + (\lambda + \beta - 1)$", respectively.

Obviously, "$1 + e^{2\sqrt{\lambda}} = 0$" has the roots

$$\lambda_l^0 = -\left(l - \frac{1}{2}\right)^2 \pi^2, \quad l = 1, 2, \cdots \tag{31}$$

Note that $\{\lambda_l^0 : l = 1, 2, \cdots\}$ are typically the eigenvalues of the classical heat equation with mixed Neumann-Dirichlet boundary conditions (see Ex.2.6.10 in [29]). In the meanwhile, the roots of "$(\lambda+\beta+1)e^{2\lambda} + (\lambda+\beta-1) = 0$" are just the eigenvalues of the wave equation with dynamical boundary control (see pp.360 of [2]). According to lemma 3.3 of [2] we know the roots of "$(\lambda + \beta + 1)e^{2\lambda} + (\lambda + \beta - 1)$" have the asymptotic expansions:

$$\lambda_n = i\left(\frac{1}{2} + n\right)\pi + O\left(\frac{1}{n}\right), \quad \text{as } n \longrightarrow \infty. \tag{32}$$

If we let

$$\lambda_k^1 = i\left(\frac{1}{2} + k\right)\pi, \quad k \in \mathbb{Z}, \tag{33}$$

then we have the following asymptotic expansions of the roots of $f(\lambda)$.

Lemma 5. *There exist $l_1 \in \mathbb{N}$ and $k_1 \in \mathbb{N}$ such that $f(\lambda)$ has two sequences of roots, $\{\lambda_l^p\}_{l=l_1}^{\infty}$ and $\{\lambda_k^h\}_{|k|=k_1}^{\infty}$, which satisfy*

$$\sqrt{\lambda_l^p} \in B_{l^{-1}}\left(\sqrt{\lambda_l^0}\right), \quad l \geq l_1 \tag{34}$$

and

$$\lambda_k^h \in B_{|k|^{-1/2}}\left(\lambda_k^1\right), |k| \geq k_1. \tag{35}$$

where $B_r(z)$ denotes the closed disk in $\mathbb{C}$ centered at z and with radius r.

Proof. The first case is when "$\operatorname{Re}\lambda \longrightarrow -\infty$". In this case, we shall show that $f(\lambda)$ has a sequence of roots $\{\lambda_l^p\}$ satisfying (34). To this end, we set

$$h(\lambda) = \left(1 + e^{2\sqrt{\lambda}}\right) + \frac{[r(\lambda) - 1]\left(e^{2\sqrt{\lambda}} - 1\right)}{\sqrt{\lambda}[r(\lambda) + 1]}. \tag{36}$$

Obviously, λ is a root of $f(\lambda)$ iff it is a root of $h(\lambda)$ since the roots of $(\lambda + \beta + 1)e^{2\lambda} + (\lambda + \beta - 1)$ satisfy (32). For any fixed $l \in \mathbb{N}$, let $\mu = \sqrt{\lambda} - (l - 1/2)\pi i$; then we have

$$
\begin{aligned}
h(\lambda) &= h\left(\mu^2 + 2\mu\left(l-\frac{1}{2}\right)\pi i - \left(l-\frac{1}{2}\right)^2\pi^2\right)\\
&= \left(1-e^{2\mu}\right) - \frac{\left[r\left(\mu^2+2\mu(l-1/2)\pi i-(l-1/2)^2\pi^2\right)-1\right]\left(1+e^{2\mu}\right)}{\left[\mu+(l-1/2)\pi i\right]\left[r\left(\mu^2+2\mu(l-1/2)\pi i-(l-1/2)^2\pi^2\right)+1\right]}
\end{aligned} \tag{37}
$$

Now, let us define a function $H(\mu) = \mu + (1/2)h(\mu^2 + 2\mu(l - 1/2)\pi i - (l - 1/2)^2\pi^2)$. For any $\mu \in B_{l^{-1}}(0)$, then it is easy to see that

$$
\begin{aligned}
|H(\mu)| &\le \left|\mu + \frac{1}{2}\left(1-e^{2\mu}\right)\right| + \frac{1}{2}\left|\frac{\left(2+O\left(l^{-1}\right)\right)}{\left[(l-1/2)\pi i + O\left(l^{-1}\right)\right]}\right|\\
&= \left|\mu\int_0^1 1-e^{2\mu s}ds\right| + \frac{1}{2}\left|\frac{\left(2+O\left(l^{-1}\right)\right)}{\left[(l-1/2)\pi i + O\left(l^{-1}\right)\right]}\right|
\end{aligned} \tag{38}
$$

By the similar method of Proof of Lem.2.2 in [1], we have

$$
\left|\mu\int_0^1 1-e^{2\mu s}ds\right| = O\left(l^{-2}\right), \tag{39}
$$

which implies that $|H(\mu)| \le 1/(l - 1/2)\pi$. It follows from the last inequality that there exists a sufficiently large $l_1 \in \mathbb{N}$ such that $H(\mu) \in B_{l^{-1}}(0)$, $\forall\mu \in B_{l^{-1}}(0)$, $l \ge l_1$. Therefore, by means of Brouwer fixed point theorem, we conclude that there exists a $u_l^0 \in B_{l^{-1}}(0)$ such that $H(u_l^0) = u_l^0$. This implies that $\lambda_l^p = (\mu_l^0)^2 + 2\mu_l^0(l-1/2)\pi i - (l-1/2)^2\pi^2$ is a root of $h(\lambda)$ and (34) holds.

The second case is when "Im$\lambda \longrightarrow \infty$". In this case, we shall show that $f(\lambda)$ has a sequence of roots $\{\lambda_k^h\}$ satisfying (35). To this end, we set

$$
g(\lambda) = (1 + r(\lambda)) + \frac{[r(\lambda)-1]\left(e^{2\sqrt{\lambda}}-1\right)}{\sqrt{\lambda}\left(e^{2\sqrt{\lambda}}+1\right)}. \tag{40}
$$

Obviously, λ is a root of $f(\lambda)$ iff it is a root of $g(\lambda)$ since the roots of $e^{2\sqrt{\lambda}}+1$ satisfy (31). For any fixed $k \in \mathbb{Z}$, let $\mu = \lambda - \lambda_k^1$; in this case, we have

$$
\begin{aligned}
g(\lambda) &= g\left(\mu+\lambda_k^1\right)\\
&= 1 + r\left(\mu+\lambda_k^1\right) - \frac{\left[r\left(\mu+\lambda_k^1\right)+1\right]\left(e^{2\sqrt{\mu+\lambda_k^1}}-1\right)}{\sqrt{\mu+\lambda_k^1}\left(e^{2\sqrt{\mu+\lambda_k^1}}+1\right)}.
\end{aligned} \tag{41}
$$

Define a function $G(\mu) = \mu + g(\mu + \lambda_k^1)$. For any $\mu \in B_{|k|^{-1/2}}(0)$, by the definition of the function $r(\lambda)$ in (19), then we have

$$
\begin{aligned}
r\left(\mu+\lambda_k^1\right) &= -\frac{\mu+\lambda_k^1+\beta+1}{\mu+\lambda_k^1+\beta-1}e^{2\mu} = -\left(1 - \frac{2}{(2k+1)\pi}\frac{1}{1-2(\mu-\beta+1)/(2k+1)\pi}\right)e^{2\mu}\\
&= -\left[1-O\left(k^{-1}\right)\left(1+O\left(k^{-1}\right)\right)\right]\left(1+O\left(|k|^{-1/2}\right)\right)\\
&= -1 + O\left(k^{-1}\right)
\end{aligned} \tag{42}
$$

in which $1/(1-z) = 1 + z + O(|z|^2)$, $e^z = 1 + z + O(|z|^2)$ when $|z| < 1$ and $\lambda_k^1 = (k + 1/2)\pi i$ are used. Moreover, by proposition 2.1 and its proof of [1], we know

$$
\begin{aligned}
\sqrt{\mu+\lambda_k^1} &= \sqrt{\lambda_k^1} + O\left(k^{-1}\right)\\
\frac{e^{2\sqrt{\mu+\lambda_k^1}}-1}{e^{2\sqrt{\mu+\lambda_k^1}}+1} &= \operatorname{sgn}(k) + O\left(k^{-2}\right)
\end{aligned} \tag{43}
$$

Now, combining (42) and (43), we get

$$
\begin{aligned}
|G(\mu)| &\le \left|\mu + O\left(k^{-1}\right)\right| + \left|\frac{\left[2+O\left(k^{-1}\right)\right]\left(\operatorname{sgn}(k)+O\left(k^{-2}\right)\right)}{\sqrt{\lambda_k^1}+O\left(k^{-1}\right)}\right|\\
&\le O\left(|k|^{-1/2}\right) + \frac{2}{\sqrt{|k|\pi}}
\end{aligned} \tag{44}
$$

It follows from the last inequality the there exists a sufficiently large $k_1 \in \mathbb{N}$ such that

$$
G(\mu) \in B_{|k|^{-1/2}}(0), \quad \forall\mu \in B_{|k|^{-1/2}}(0), \ |k| \ge k_1. \tag{45}
$$

Thus, by means of Brouwer fixed point theorem, we conclude that there exists a $u_k^1 \in B_{|k|^{-1/2}}(0)$ such that $G(u_k^1) = u_k^1$. This implies that $\lambda_k^h = \mu_k^1 + (k + 1/2)\pi i$ is a root of $g(\lambda)$ and (35) holds. □

Furthermore, we are able to obtain more precisely asymptotic expansions on λ_k^h and λ_l^p.

Lemma 6. *The following asymptotic expansions hold:*

$$\sqrt{\lambda_l^p} = \sqrt{\lambda_l^0} + \frac{1}{\sqrt{\lambda_l^0}} + O\left(l^{-2}\right), \quad l \longrightarrow \infty, \tag{46}$$

$$\lambda_k^h = \lambda_k^1 - \frac{\operatorname{sgn}(k)}{\sqrt{\lambda_k^1}} + O\left(k^{-1}\right), \quad |k| \longrightarrow \infty, \tag{47}$$

Proof. From (36), we know that λ_l^p satisfies

$$e^{2\sqrt{\lambda_l^p}} = -1 - \frac{\left[r\left(\lambda_l^p\right) - 1\right]\left(e^{2\sqrt{\lambda_l^p}} - 1\right)}{\sqrt{\lambda_l^p}\left[r\left(\lambda_l^p\right) + 1\right]}. \tag{48}$$

By proposition 2.2 in [1], we have

$$\begin{aligned} e^{2\lambda_l^p} &= O\left(l^{-2}\right), \\ e^{2\sqrt{\lambda_l^p}} &= -1 + O\left(l^{-1}\right). \end{aligned} \tag{49}$$

Moreover, it is easy to see that

$$\frac{\lambda_l^p + \beta + 1}{\lambda_l^p + \beta - 1} = 1 + O\left(l^{-2}\right), \quad r\left(\lambda_l^p\right) = O\left(l^{-2}\right). \tag{50}$$

Thus, by (48)-(50), we have

$$\begin{aligned} e^{2(\sqrt{\lambda_l^p} - \sqrt{\lambda_l^0})} &= 1 + \frac{\left[r\left(\lambda_l^p\right) - 1\right]\left(e^{2\sqrt{\lambda_l^p}} - 1\right)}{\sqrt{\lambda_l^p}\left[r\left(\lambda_l^p\right) + 1\right]} \\ &= 1 + \frac{\left[-1 + O\left(l^{-2}\right)\right]\left[-2 + O\left(l^{-1}\right)\right]}{\left(\sqrt{\lambda_l^0} + O\left(l^{-1}\right)\left[1 + O\left(l^{-2}\right)\right]\right)} \\ &= 1 + \frac{2}{\sqrt{\lambda_l^0}} + O\left(l^{-2}\right). \end{aligned} \tag{51}$$

Taking logarithm in the above identity and noting that $\ln(1 + z) = z + O(z^2)$ when $|z| < 1$, we conclude that (46) holds for any sufficiently large l.

Similarly, from (40) and (19), we know that λ_k^h satisfies

$$\begin{aligned} & e^{2\lambda_k^h} \\ &= \frac{\lambda_k^h + \beta - 1}{\lambda_k^h + \beta + 1}\left(-1 - \frac{\left[r\left(\lambda_k^h\right) - 1\right]\left(e^{2\sqrt{\lambda_k^h}} - 1\right)}{\sqrt{\lambda_k^h}\left(e^{2\sqrt{\lambda_k^h}} + 1\right)}\right). \end{aligned} \tag{52}$$

By (43), we get

$$\begin{aligned} \sqrt{\lambda_k^h} &= \sqrt{\lambda_k^1} + O\left(k^{-1}\right) \\ \frac{e^{2\sqrt{\lambda_k^h}} - 1}{e^{2\sqrt{\lambda_h^1}} + 1} &= \operatorname{sgn}(k) + O\left(k^{-2}\right) \end{aligned} \tag{53}$$

It is easy to see from proposition 2.2 in [1] that

$$\begin{aligned} \frac{\lambda_k^h + \beta - 1}{\lambda_k^h + \beta + 1} &= 1 + O\left(k^{-1}\right), \\ r\left(\lambda_k^h\right) &= -1 + O\left(|k|^{-1/2}\right). \end{aligned} \tag{54}$$

Thus, by (52)-(54), we have

$$\begin{aligned} & e^{2(\lambda_k^h - \lambda_k^1)} \\ &= \frac{\lambda_k^h + \beta - 1}{\lambda_k^h + \beta + 1}\left(1 + \frac{\left[r\left(\lambda_k^h\right) - 1\right]\left(e^{2\sqrt{\lambda_k^h}} - 1\right)}{\sqrt{\lambda_k^h}\left(e^{2\sqrt{\lambda_k^h}} + 1\right)}\right) \\ &= \left(1 + O\left(k^{-1}\right)\right) \\ &\quad \cdot \left[1 + \frac{\left(\operatorname{sgn}(k) + O\left(k^{-2}\right)\right)\left(-2 + O\left(|k|^{-1/2}\right)\right)}{\sqrt{\lambda_k^1} + O\left(k^{-1}\right)}\right] \\ &= 1 - \left[\frac{2\operatorname{sgn}(k)}{\sqrt{\lambda_k^1}} + O\left(k^{-1}\right)\right]. \end{aligned} \tag{55}$$

Taking logarithm in the above identity, and noting that $\ln(1 - z) = -z + O(z^2)$ when $|z| < 1$, we conclude that (47) holds for any sufficiently large $|k|$. □

Theorem 7. *System (1) or (6) is not exponentially stable.*

Proof. By (46) and (31), we get the asymptotic expansions on λ_l^p:

$$\lambda_l^p = -\left(l - \frac{1}{2}\right)^2 \pi^2 + 2 + O\left(l^{-1}\right). \tag{56}$$

However, by (33) and some simple computations, we know

$$\frac{1}{\sqrt{\lambda_k^1}} = \begin{cases} \dfrac{1}{\sqrt{|1 + 2k|\pi}}(1 - i), & \text{if } k \ge 0; \\ -\dfrac{1}{\sqrt{|1 + 2k|\pi}}(1 + i), & \text{if } k < -1. \end{cases} \tag{57}$$

Thus, inserting $1/\sqrt{\lambda_k^1}$ into (47), we obtain the asymptotic estimates on λ_k^h:

$$\begin{aligned} \lambda_k^h = & -\frac{1}{\sqrt{|1 + 2k|\pi}} + \left(\frac{1}{2} + k\right)\pi i + \frac{\operatorname{sgn}(k)}{\sqrt{|1 + 2k|\pi}} i \\ & + O\left(|k|^{-1}\right). \end{aligned} \tag{58}$$

This shows that the spectral bound is zero. The growth bound of the semigroup corresponding to (6) is also zero because the growth bound of the semigroup is greater than or equal to the spectral bound of its generator. However, the exponential stability of the semigroup is equivalent to the fact that its growth bound is negative [29, 30]. Therefore, system (1) or (6) is not exponentially stable. □

4. Conclusion

In this note, the spectrum and the asymptotic stability of the heat-wave system with dynamical boundary control are investigated through the theory of operator semigroup. The solution to the system is shown to be uniquely existed. The energy of the solution to the system does not increase because the semigroup generated by the system operator is contracted. In the meanwhile, the asymptotic stability, which implies that the energy of the system approaches zero as time tends to infinity, of the semigroup is derived in light of the spectral configuration. However, the spectral expansions of the system operator are investigated and the energy of the system does not exponentially decay to zero and is shown in view of the spectral expansions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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