Hindawi
Mathematical Problems in Engineering
Volume 2019, Article ID 5851739, 9 pages
https://doi.org/10.1155/2019/5851739

Research Article

Robust Performance and Observer Based Control for Periodic Discrete-Time Uncertain Systems

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Received 31 October 2018; Revised 11 January 2019; Accepted 30 January 2019; Published 11 February 2019

Academic Editor: Maria L. Gandarias

This paper presents an approach for the synthesis of observer-based controllers for discrete-time periodic linear systems. The $\mathscr{H}_2$ performance criterion has been employed to design both the observer and the controller. For the periodic observer design, two conditions in the form of Linear Matrix Inequalities (LMIs) are proposed, which stem from the Lyapunov Theory applied over the dynamics of the estimation error. The LMI condition obtained for the periodic state-feedback controller results from the application of the duality principle over the periodic system, under the assumption that only the estimated states are available to be used in the state-feedback compensation. Numerical experiments illustrate the potential of the proposed observer-based control technique.

1. Introduction

Periodic systems have received considerable attention over the last decades, due to their wide utilization in distinct areas [1, 2]. Among the reasons that justify the interest in such class of systems, one may highlight their applicability in modeling systems with repetitive tasks such as satellites or helicopters [3]; the quality of periodic models obtained through multirate sample data systems [4]; multiagent systems [5] and periodic models obtained from nonlinear systems, which describe more accurately the nonlinear behaviors when compared to the time-invariant linear approximations [6–8].

A great amount of papers proposing techniques, based on the solution of Linear Matrix Inequalities (LMIs) [9], for the synthesis of controllers for discrete-time periodic systems consider state-feedback structures. For instance, in [7], a method to design periodically-varying state-feedback controllers to stabilize periodic discrete-time systems affected by polytopic uncertainties is proposed. The resultant controller guarantees a bound for the $\mathscr{H}_2$ norm of the controlled system. In [10] the controller is a memory state-feedback controller, depending on the states stored from a predefined number of past samples. A general overview of other techniques can be found, for instance, in [6] and the references therein.

A drawback of the above cited techniques is that the system states are usually not entirely available for feedback, thus preventing the use of state-feedback controllers in this case. A possible way to deal with this problem is to determine output-feedback gains, as *e.g.* presented in [11]. In [11], an LMI based approach is presented to design periodic state and output-feedback gains for periodic discrete-time uncertain systems. However, the presence of equality constraints imposes some conservatism to the conditions. Another usual approach for the absence of the states' information is to design observer-based controllers. The technique presented in [12] proposes an observer-based controller through the application of a lifting technique over the periodic system, followed by a pole placement procedure using the monodromy matrix. In [13] a similar approach is used with the addition of a parametrization for the computation of the observer. In [14], an observer-based method is proposed for a class of nonlinear fractional-order uncertain systems with time-varying parameters. Finally, an LMI-based method is presented in [15] to compute periodic observer-based controllers capable of minimizing the $\mathscr{H}_\infty$ norm of the controlled system, in order to assure the desired robustness.

The main contribution of this paper is the proposition of an observer-based controller synthesis technique for uncertain discrete-time periodic systems. Differently from the technique presented by [7], the proposed approach makes use of observer-based state feedback controllers, dealing with a more realistic scenario, which is the case when it is not possible to access the real states of the periodic system. In the proposed method, both the observer and the state-feedback gains are designed to minimize a bound for the $\mathscr{H}_2$ norm of the closed-loop system, guaranteeing certain performance specifications. More specifically, the observer is synthesized through the resolution of a set of LMI conditions, and the controller stems from the application of the principle of duality [16] over the original system, following the lines presented in [15]. The adaptation of the technique to deal with polytopic uncertainties is also presented.

The paper is organized as follows: Section 2 introduces the problem formulation and some preliminary results. The main results are introduced in Section 3 for the observer design problem and in Section 4 considering the state feedback design. Section 5 is devoted to numerical examples and Section 6 concludes the paper.

Notation. For two symmetric matrices of same dimensions X and Y, $X > Y$ means that $X - Y$ is positive definite. $\mathbb{Z}^+$ is the set of positive integers numbers. Identity matrices are denoted by I and null matrices are denoted by 0. The symbol $\star$ indicates a symmetric block in matrices. The trace of a matrix M is denoted by $\mathrm{Tr}(M)$.

2. Problem Formulation and Preliminaries

Consider the following linear discrete-time periodic system Σ:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_{u_k} u_k + B_{w_k} w_k \\ y_k &= C_k x_k + D_{u_k} u_k + D_{w_k} w_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector, $u_k \in \mathbb{R}^{n_u}$ is the control input, $w_k \in \mathbb{R}^{n_w}$ is the noise input, and $y_k \in \mathbb{R}^{n_y}$ is the measured output. All matrices of system (1) are known and satisfy

$$Z_{k+rT} = Z_k, \quad \forall r \in \mathbb{Z},\ k = 1, \ldots, T \tag{2}$$

For this reason, they are periodic matrices of period $T \in \mathbb{Z}^+$.

Before presenting the main results, some preliminaries about stability and the $\mathscr{H}_2$ performance for periodic discrete-time systems are introduced. The following lemma presents LMI conditions to certify the stability of system (1).

Lemma 1. *Consider system (1) with $u_k = 0$ and $w_k = 0$. System (1) is asymptotically stable, if and only if there exist symmetric positive definite periodic matrices $P_k \in \mathbb{R}^{n_x \times n_x}$ such that*

$$A_k^T P_{k+1} A_k - P_k < 0, \quad k = 1, 2, \ldots, T \tag{3}$$

with $P_{T+1} = P_1$ hold.

Following [7] the $\mathscr{H}_2$ norm of the system (1) is the mean of all the responses corresponding to impulsive inputs applied at each time k of the period on each of the n_w channels. The squared $\mathscr{H}_2$ norm for system (1) is represented by $\|\Sigma\|_2^2$. In [7] the controllability Gramian has been used to compute the $\mathscr{H}_2$ performance for periodic discrete-time systems. In the sequel a condition based on the observability Gramian is presented.

Lemma 2. *Consider system (1) with $u_k = 0$. The system is asymptotically stable and the $\mathscr{H}_2$ norm is lower than a scalar μ, if and only if there exist symmetric positive definite periodic matrices $P_k \in \mathbb{R}^{n_x \times n_x}$ and $X_k \in \mathbb{R}^{n_w \times n_w}$ such that*

$$\|\Sigma\|_2^2 = \min \mu \tag{4}$$

being

$$\sum_{k=1}^{T} Tr(X_k) < T\mu \tag{5}$$

subject to

$$\begin{bmatrix} X_k & B_{w_k}^T P_{k+1} & D_{w_k}^T \\ \star & P_{k+1} & 0 \\ \star & \star & I \end{bmatrix} > 0, \quad k = 1, 2, \ldots, T. \tag{6}$$

$$\begin{bmatrix} P_k & A_k^T P_{k+1} & C_k^T \\ \star & P_{k+1} & 0 \\ \star & \star & I \end{bmatrix} > 0, \quad k = 1, 2, \ldots, T, \tag{7}$$

with $P_{T+1} = P_1$, is valid.

Proof. By applying the Schur complement twice in (6) one has

$$X_k - B_{w_k}^T P_{k+1} B_{w_k} - D_{w_k}^T D_{w_k} > 0 \tag{8}$$

The computation of the trace recovers the condition

$$Tr(X_k) - Tr\left(B_{w_k}^T P_{k+1} B_{w_k} + D_{w_k}^T D_{w_k}\right) > 0. \tag{9}$$

From (7), the application of the Schur complement yields

$$\begin{bmatrix} P_k - A_k^T P_{k+1} A_k & C_k^T \\ C_k & I \end{bmatrix} > 0, \tag{10}$$

and applying Schur complement once again, the observability Gramian condition is recovered

$$P_k - A_k^T P_{k+1} A_k - C_k^T C_k > 0. \tag{11}$$

If there exist symmetric definite positive periodic matrices P_k and X_k satisfying (9) and (11), the scalar μ given by (5) is a bound for the $\mathscr{H}_2$ norm of the system (1). □

3. Observer Design

The first problem to be considered in this paper is to design a periodic observer that can provide an asymptotic estimation of x_k considering the $\mathscr{H}_2$ performance. To this end the estimated system is expressed as follows:

$$\begin{aligned} \hat{x}_{k+1} &= A_k \hat{x}_k + B_{u_k} u_k - L_k \left(y_k - \hat{y}_k \right) \\ \hat{y}_k &= C_k \hat{x}_k + D_{u_k} u_k \end{aligned} \tag{12}$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is the estimated state vector and $L_k \in \mathbb{R}^{n_x \times n_y}$ is the observer gain, being a periodic matrix expressed as in (2). The estimation error is given by $e_k = x_k - \hat{x}_k$. By using (1) and (12) one can write the error dynamics system Σ_e

$$\begin{aligned} e_{k+1} &= \overline{A}_k e_k + \overline{B}_k w_k \\ z_k &= \overline{C}_k e_k \end{aligned} \tag{13}$$

with $\overline{A}_k = (A_k + L_k C_k)$, $\overline{B}_k = (B_{w_k} + L_k D_{w_k})$, and $\overline{C}_k = I_{n_x}$, i.e., $z_k = e_k$. Two different conditions are proposed to design a periodic observer for discrete-time periodic systems. The first condition is based on the specialization of Lemma 2 to deal with the observer design problem.

Lemma 3. *If there exist symmetric positive definite periodic matrices $P_k \in \mathbb{R}^{n_x \times n_x}$ and $X_k \in \mathbb{R}^{n_w \times n_w}$ and periodic matrices $Z_k \in \mathbb{R}^{n_x \times n_y}$ such that*

$$\left\| \Sigma_e \right\|_2^2 = \min \mu \tag{14}$$

being

$$\sum_{k=1}^{T} Tr\left(X_k\right) < T\mu \tag{15}$$

subject to

$$\begin{bmatrix} X_k & B_{w_k}^T P_{k+1} + D_{w_k}^T Z_k^T \\ \star & P_{k+1} \end{bmatrix} > 0, \quad k = 1, 2, \ldots, T \tag{16}$$

$$\begin{bmatrix} P_k & A_k^T P_{k+1} + C_k^T Z_k^T & \overline{C}_k^T \\ \star & P_{k+1} & 0 \\ \star & \star & I \end{bmatrix} > 0, \quad k = 1, 2, \ldots, T \tag{17}$$

with $P_{T+1} = P_1$ holding, then the periodic observer, whose matrices are given by $L_k = P_{k+1}^{-1} Z_k$, guarantees the asymptotic stability for the augmented system (13) and μ is the squared $\mathscr{H}_2$ norm of the system.

Proof. By replacing $Z_k = P_{k+1} L_k$ in conditions (16) and (17) one can write

$$\begin{gathered} \begin{bmatrix} X_k & \overline{B}_k^T P_{k+1} \\ \star & P_{k+1} \end{bmatrix} > 0 \\ \begin{bmatrix} P_k & \overline{A}^T P_{k+1} & \overline{C}_k^T \\ \star & P_{k+1} & 0 \\ \star & \star & I \end{bmatrix} > 0 \end{gathered} \tag{18}$$

The rest of the proof follows exactly the same steps presented in the proof of Lemma 2. Note that $\overline{D}_k = 0$ in the error dynamics (13). □

It is important to point that the matrices of the periodic observer gain designed by Lemma 2 are obtained from the inverse of the periodic Lyapunov matrix P_k. This fact can be an issue when considering uncertain systems or when the design of decentralized observers is pursued. To overcome this issue, next Lemma presents a condition that includes slack variables used to decouple the Lyapunov matrices from the system matrices.

Lemma 4. *If there exist symmetric positive definite periodic matrices $P_k \in \mathbb{R}^{n_x \times n_x}$ and $X_k \in \mathbb{R}^{n_w \times n_w}$, periodic matrices $Z_k \in \mathbb{R}^{n_x \times n_y}$ and $F_k \in \mathbb{R}^{n_x \times n_x}$, and a given scalar ξ such that*

$$\left\| \Sigma_e \right\|_2^2 = \min \mu \tag{19}$$

being

$$\sum_{k=1}^{T} Tr\left(X_k\right) < T\mu \tag{20}$$

subject to

$$\begin{bmatrix} -X_k & B_{w_k}^T F_k + D_{w_k}^T Z_k^T \\ \star & P_{k+1} - F_k - F_k^T \end{bmatrix} < 0, \quad k = 1, 2, \ldots, T, \tag{21}$$

$$\begin{bmatrix} P_k & -\xi\left(A_k^T F_k^T + C_k^T Z_k^T\right) & \overline{C}_k^T \\ \star & -P_{k+1} + \xi F_k + \xi F_k^T & 0 \\ \star & \star & I \end{bmatrix} > 0, \quad k = 1, 2, \ldots, T, \tag{22}$$

with $P_{T+1} = P_1$ holding, then the periodic observer, whose matrices are given by $L_k = F_k^{-1} Z_k$, guarantees the asymptotic stability for the augmented system (13) and μ is the squared $\mathscr{H}_2$ norm of the system.

Proof. First, replace $Z_k = L_k F_k$ in conditions (21) and (22). Pre- and post-multiplying (21) by $\mathscr{J}^T$ and $\mathscr{J}$ with

$$\mathscr{J}^T = \begin{bmatrix} I & \overline{B}_k^T \end{bmatrix} \tag{23}$$

recover

$$-X_k + \overline{B}_k^T P_{k+1} \overline{B}_k < 0, \tag{24}$$

which is equivalent to (6). Pre- and post-multiplying (22) by $\mathscr{S}^T$ and $\mathscr{S}$ with

$$\mathscr{S} = \begin{bmatrix} I & 0 \\ \overline{A} & 0 \\ 0 & I \end{bmatrix} \tag{25}$$

recover

$$\begin{bmatrix} P_k - \overline{A}^T P_{k+1} \overline{A} & \overline{C}_k^T \\ \star & I \end{bmatrix} > 0, \tag{26}$$

which is equivalent to condition (7) presented in Lemma 2. In this way, the rest of the proof follows Lemma 2. □

Differently from Lemma 3, the observer gain designed by Lemma 4 is recovered by a change of variables applied in a slack variable that have been included in the problem. In the uncertain case, this choice allows the use of parameter-dependent Lyapunov matrices that provide better results in terms of the $\mathscr{H}_2$ norm. Moreover, Lemma 4 makes use of a scalar parameter ξ that must be searched to minimize the $\mathscr{H}_2$ norm. Thanks to the slack variables introduced in the problem, the observer gains computed by Lemma 4 rely on matrices that appear in the same time-instant, i.e., $L_k = F_k^{-1} Z_k$.

4. State Feedback

Consider that the actual states x_k are not available for the control system (1). In this case the control input can use the estimated states $\widehat{x}_k$ to provide a state-feedback compensation for system (1), with

$$u_k = K_k \widehat{x}_k. \tag{27}$$

Note that $\widehat{x}_{k+1} = x_{k+1} - e_{k+1}$ and the closed loop system Σ_{cl} can be written as

$$\begin{aligned} x_{k+1} &= \widetilde{A}_k x_k + \widetilde{B}_k \widetilde{K}_k \theta_k \\ y_k &= \widetilde{C} x_k + \widetilde{D}_k \widetilde{K}_k \theta_k \end{aligned} \tag{28}$$

where $\theta_k = \begin{bmatrix} e_k^T & w_k^T \end{bmatrix}^T$ and

$$\begin{aligned} \widetilde{A}_k &= A_k + B_{u_k} K_k, \\ \widetilde{B}_k &= \begin{bmatrix} -B_{u_k} & B_{w_k} \end{bmatrix}, \\ \widetilde{C}_k &= C_k + D_{u_k} K_k, \\ \widetilde{D}_k &= \begin{bmatrix} -D_{u_k} & D_{w_k} \end{bmatrix}, \end{aligned} \tag{29}$$

with

$$\widetilde{K}_k = \begin{bmatrix} K_k & 0 \\ 0 & I \end{bmatrix}. \tag{30}$$

The dual representation of the system, given by

$$\begin{aligned} \overline{x}_{k-1} &= \widetilde{A}_k^T \overline{x}_k + \widetilde{C}^T \theta_k \\ y_k &= \widetilde{K}_k^T \widetilde{B}_k^T \overline{x}_k + \widetilde{K}_k^T \widetilde{D}_k^T \theta_k, \end{aligned} \tag{31}$$

is employed to synthesize the $\mathscr{H}_2$ state-feedback controller. The principle of duality implies that time direction in dual system has reverse direction of the primal system [16]. That is the reason why the evolution of the states $\overline{x}_k$ is backwards. Therefore, primal system (28) is asymptotically stable if the dual system (31) is exponentially unstable. The $\mathscr{H}_2$ norm of the primal system (28) is lower than a positive scalar μ if there exists a function $V(\overline{x}_k) > 0$, for the dual system, satisfying the following conditions:

$$-X_k + \widetilde{C} W_{k-1} \widetilde{C}^T + \widetilde{D} \widetilde{K}_k \widetilde{K}_k^T \widetilde{D}^T < 0 \tag{32}$$

$$-W_k + \widetilde{A}_k W_{k-1} \widetilde{A}_k^T + \widetilde{B}_k \widetilde{K}_k \widetilde{K}_k^T \widetilde{B}_k^T < 0 \tag{33}$$

Conditions (32) and (33) are not in the form of LMIs, since there are products between variables that appear in the matrices $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{K}$ and the Lyapunov matrix W_k. The next lemma presents LMI conditions that can be used to design the $\mathscr{H}_2$ state-feedback controller.

Lemma 5. *If there exist symmetric positive definite periodic matrix $W_k \in \mathbb{R}^{n_x \times n_x}$ and $X_k \in \mathbb{R}^{n_w \times n_w}$ and periodic matrices $G_k \in \mathbb{R}^{n_x \times n_x}$, $F_k \in \mathbb{R}^{n_u \times n_x}$, and $H_k \in \mathbb{R}^{n_y \times n_y}$ and a scalar ξ such that*

$$\left\| \Sigma_{cl} \right\|_2^2 = \min \mu \tag{34}$$

being

$$\sum_{k=1}^{T} Tr\left(X_k\right) < T\mu \tag{35}$$

subject to

$$\begin{bmatrix} -X_k & \xi C_k G_k + \xi D_{u_k} F_k & \widetilde{D}_k H_{1_k} \\ \star & W_{k-1} - \xi G_k - \xi G_k^T & 0_{n_x \times (n_x + n_y)} \\ \star & \star & I_{n_x + n_y} + H_{2_k} + H_{2_k}^T \end{bmatrix} < 0, \quad k = 1, 2, \ldots, T, \tag{36}$$

$$\begin{bmatrix} -W_k + A_k G_k + B_{u_k} F_k + G_k^T A_k^T + F_k^T B_{u_k} & \xi A_k G_k + \xi B_{u_k} F_k - G_k^T & \widetilde{B}_k H_{1_k} \\ \star & W_{k-1} - \xi G_k - \xi G_k^T & 0_{n_x \times (n_x + n_y)} \\ \star & \star & I_{n_x + n_y} + H_{2_k} + H_{2_k}^T \end{bmatrix} < 0, \tag{37}$$

$k = 1, 2, \ldots, T$, with

$$H_{1_k} = \begin{bmatrix} F_k & 0_{n_u \times n_y} \\ 0_{n_y \times n_u} & H_k \end{bmatrix} \qquad H_{2_k} = \begin{bmatrix} -G_k & 0_{n_x \times n_y} \\ 0_{n_y \times n_x} & H_k \end{bmatrix} \tag{38}$$

holding with $W_0 = W_T$, then $K_k = F_k G_k^{-1}$ is a periodic state-feedback gain that guarantees the asymptotic stability for system (28) and μ is a bound to the squared $\mathscr{H}_2$ norm of the system.

Proof. First, consider the inequality

$$\begin{gathered}\left(I+\widetilde{H}_{2_k}\right)^T\left(I+\widetilde{H}_{2_k}\right)>0 \longrightarrow \\ -\widetilde{H}_{2_k}^T\widetilde{H}_{2_k}<I+\widetilde{H}_{2_k}+\widetilde{H}_{2_k}^T.\end{gathered} \tag{39}$$

Applying such inequality, replace $F_k = K_k G_k$ and $H_{1_k} = \widetilde{K}_k \widetilde{H}_{2_k}$ so conditions (36) and (37) can be rewritten as

$$\begin{bmatrix} -X_k & \xi \widetilde{C}_k G_k & \widetilde{D}_k \widetilde{K}_k H_{2_k} \\ \star & W_{k-1}-\xi\left(G_k+G_k^T\right) & 0_{n_x\times(n_x+n_y)} \\ \star & \star & -H_{2_k}^T H_{2_k}\end{bmatrix}<0, \tag{40}$$

$$\begin{bmatrix} -W_k+\widetilde{A}G_k+G_k^T\widetilde{A}_k^T & \xi\widetilde{A}_kG_k-G_k^T & \widetilde{B}_k\widetilde{K}_kH_{2_k} \\ \star & W_{k-1}-\xi\left(G_k+G_k^T\right) & 0_{n_x\times(n_x+n_y)} \\ \star & \star & -H_{2_k}^TH_{2_k}\end{bmatrix} \tag{41}$$
$$<0,$$

respectively. Pre- and post-multiplying the condition (40) by τ and τ^T and pre- and post-multiplying the condition (41) by Λ and Λ^T, with

$$\begin{gathered}\tau=\left[I \;\; \widetilde{C}_k \;\; \widetilde{D}_k\widetilde{K}_k\left(H_{2_K}^T\right)^{-1}\right] \\ \Lambda=\left[I \;\; \widetilde{A}_k \;\; \widetilde{B}_k\widetilde{K}_k\left(H_{2_K}^T\right)^{-1}\right]\end{gathered} \tag{42}$$

recover conditions (32) and (33), concluding the proof. □

Remark 6. The proposed Lemmas can be extended for discrete-time periodic uncertain systems described in the following form:

$$\begin{aligned} x_{k+1} &= A_k(\alpha)x_k + B_{u_k}(\alpha)u_k + B_{w_k}(\alpha)w_k \\ y_k &= C_k(\alpha)x_k + D_{u_k}(\alpha)u_k + D_{w_k}(\alpha)w_k \end{aligned} \tag{43}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector, $u_k \in \mathbb{R}^{n_u}$ is the control input, $w_k \in \mathbb{R}^{n_w}$ is the noise input, and $y_k \in \mathbb{R}^{n_y}$ is the measured output. All matrices in (43) satisfy

$$Z_{k+rT}(\alpha) = Z_k(\alpha), \quad \forall r \in \mathbb{Z},\ k = 1,\dots,T. \tag{44}$$

Thus, they are periodic matrices of period $T \in \mathbb{Z}^+$. Moreover, the matrices $Z_k(\alpha)$ belong to a polytopic domain parameterized in terms of time-invariant parameters α:

$$\mathscr{P} = \left\{Z_k(\alpha) = \sum_{i=1}^{R}\alpha_i Z_{k_i},\ \alpha \in \Lambda_R\right\}, \tag{45}$$

where Z_{k_i}, $i = 1,\dots R$, are the vertices, R is the number of vertices of the polytope, and Λ_R is the unit simplex, given by

$$\Lambda_R = \left\{\alpha \in \mathbb{R}^R : \sum_{i=1}^{R}\alpha_i = 1,\ \alpha_i \geq 0,\ i = 1,\dots,R\right\}. \tag{46}$$

By assumption the parameter α is time-invariant. The observer and state-feedback gains, in the uncertain case, can be computed by applying, respectively, Lemmas 4 and 5, considering that the system and the Lyapunov matrices depend on α. A finite set of conditions, capable of generating robust periodic gains that stabilize the system for all $\alpha \in \Lambda_R$, can be obtained by using, for instance, computational packages such as the ROLMIP toolbox [17].

5. Numerical Experiments

The routines were implemented in MATLAB, version 7.0.1 (R14), and the LMIs were solved using the packages Yalmip [18], SeDuMi [19], and ROLMIP [17].

Example 1. Consider the periodic discrete-time system as in (1) with $T = 3$. The following state space matrices are borrowed from [15]:

$$\begin{gathered} A_1 = \begin{bmatrix} -4.5 & -1 \\ 2.5 & 0.5 \end{bmatrix}, \\ A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \\ A_3 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \\ B_{w_1} = B_{w_2} = B_{w_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}, \\ C_2 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \\ C_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\ B_{u_1} = B_{u_2} = B_{u_3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ D_{u_1} = D_{u_2} = D_{u_3} = 0, \end{gathered} \tag{47}$$

$D_{w_1} = D_{w_2} = D_{w_3} = 0.5$. For this system, Lemma 3 yields $\mu = 1.09$ and a periodic observer L_k given by the following matrices:

$$\begin{gathered} L_1 = \begin{bmatrix} 0.4294 \\ -0.4107 \end{bmatrix}, \\ L_2 = \begin{bmatrix} -0.3159 \\ -0.5126 \end{bmatrix}, \\ L_3 = \begin{bmatrix} -1.9998 \\ -0.9753 \end{bmatrix}. \end{gathered} \tag{48}$$

Figure 1 presents the evolution of the actual states x_k and of the estimated states $\hat{x}_k$, considering initial conditions $x_0 =$

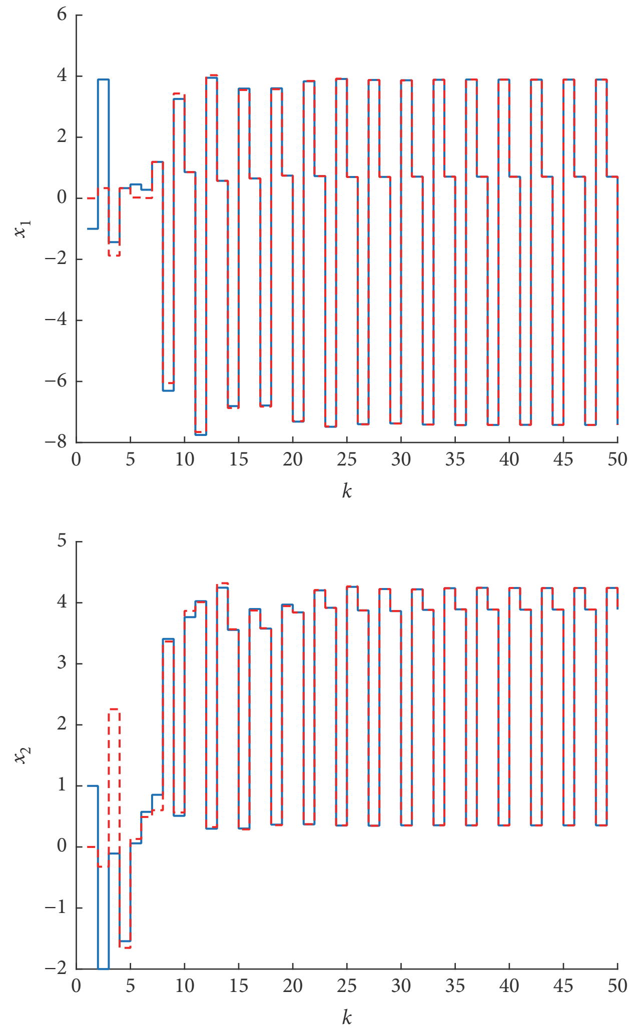

Figure 1: Real states (blue line) and estimated states (red line) for Example 1 by using the observer designed with Lemma 3.

$\begin{bmatrix}-1 & 1\end{bmatrix}^T$, $\hat{x}_0 = \begin{bmatrix}0 & 0\end{bmatrix}^T$, and $u_k = 0$. The system is affected by a disturbance signal w_k given by

$$w_k = \sin(0.5k)\exp(-0.2k). \tag{49}$$

As can be seen in Figure 1 the estimated states converge to the actual states illustrating that the observer designed performs well in the considered scenario, when there is no control input acting in the system.

Figure 2 shows the behavior of μ with variation of the scalar ξ when applying Lemma 4 to design a periodic observer. It can be seen that when $\xi = 1$ the value of the $\mathscr{H}_2$ norm is $\mu = 1.09$ (same value found by Lemma 3); i.e., Lemma 4 is no more conservative than Lemma 3.

Lemma 5 has been employed to compute a periodic state-feedback control. The best performance is achieved when $\xi = 6.25$ which results on $\mu = 9.81$ and a periodic state-feedback controller with matrices given by

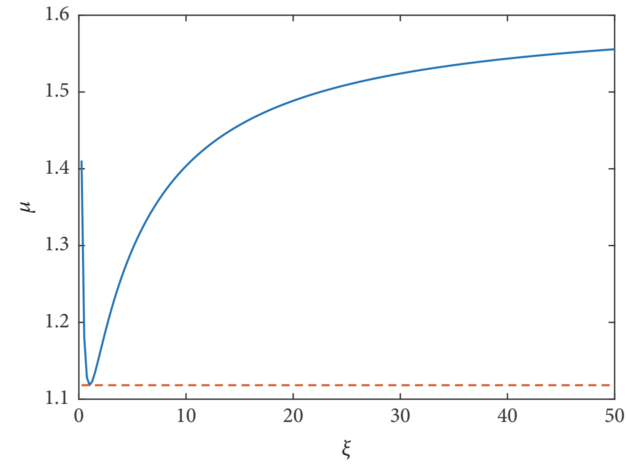

Figure 2: Behavior of μ with variation of the scalar ξ for Example 1 when considering Lemma 4 (solid line) and value of μ obtained by Lemma 3 (dotted line) for the observer design problem.

$$\begin{aligned} K_1 &= \begin{bmatrix}0.7612 & 0.2170\end{bmatrix}, \\ K_2 &= \begin{bmatrix}-0.6450 & -1.5384\end{bmatrix}, \\ K_3 &= \begin{bmatrix}-0.2455 & -1.4631\end{bmatrix}, \end{aligned} \tag{50}$$

Figure 3 shows the trajectory for the real states and for the observed states when applying the observer-based control technique with the periodic observer designed with Lemma 3 and state-feedback controller designed by Lemma 5.

Example 2. The periodic matrices A_k in Example 1 are modified to illustrate the applicability of the proposed technique under the presence of uncertainties. To this end the following state space matrices are considered:

$$\begin{aligned} A_1 &= \begin{bmatrix}-4.5-\eta & -1 \\ 2.5 & 0.5\end{bmatrix}, \\ A_2 &= \begin{bmatrix}-\eta & 1 \\ 1 & 2\end{bmatrix}, \\ A_3 &= \begin{bmatrix}-\eta & 2 \\ 1 & 1\end{bmatrix}, \end{aligned} \tag{51}$$

with $|\eta| \leq 0.4$. The system can be written as in (43) and each matrix A_k, $k = 1, 2, 3$, has two vertices, i.e., $A_k(\alpha) = \alpha_1 A_{k_1} + \alpha_2 A_{k_2}$. For the system described above, it is possible to find a periodic observer by applying Lemma 3 for uncertain system. In this case $\mu = 1.94$ and the periodic observer is described by the following matrices:

$$L_1 = \begin{bmatrix}1.0011 \\ -0.6390\end{bmatrix},$$

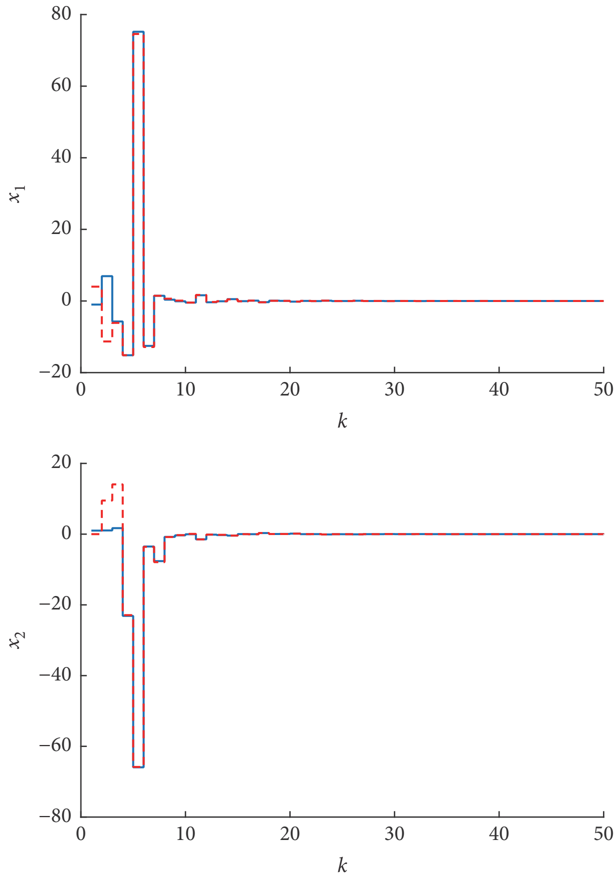

Figure 3: Real states (solid blue line) and observed states (dotted red line) using the observer-based control technique considering the periodic observer designed with Lemma 3 and the state-feedback controller designed with Lemma 5 for Example 1.

$$L_2 = \begin{bmatrix} -0.2794 \\ 0.1695 \end{bmatrix},$$
$$L_3 = \begin{bmatrix} -1.9890 \\ -1.8279 \end{bmatrix} \tag{52}$$

It is possible to reduce the conservatism by considering Lemma 4 with $\xi = 1$, which yields $\mu = 1.64$ and a periodic observer given by the following matrices:

$$L_1 = \begin{bmatrix} 1.1424 \\ -0.7211 \end{bmatrix},$$
$$L_2 = \begin{bmatrix} -0.3625 \\ 0.2343 \end{bmatrix}, \tag{53}$$
$$L_3 = \begin{bmatrix} -1.9915 \\ -1.4081 \end{bmatrix}$$

Figure 4 shows the behavior of μ with variation of the scalar ξ when applying Lemma 4 to design a periodic observer for the uncertain system presented in Example 2. Differently

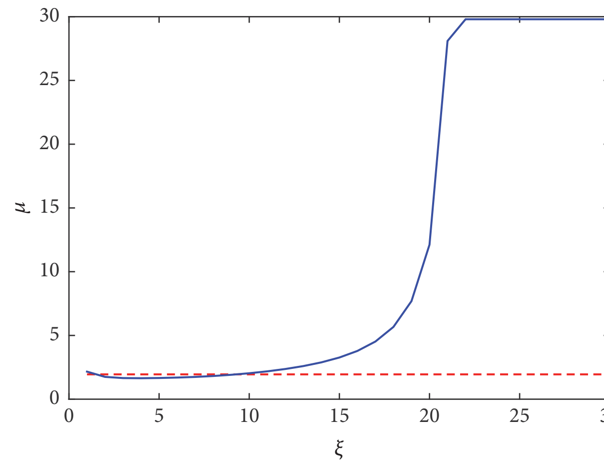

Figure 4: Behavior of μ with variation of the scalar ξ for Example 2 when considering Lemma 4 adapted for uncertain systems (solid line) and value of μ obtained by Lemma 3 adapted for uncertain systems (dotted line).

from the precisely known case discussed in Example 1, in the uncertain case Lemma 4 can provide better results than Lemma 3. This is due to the fact that for uncertain systems Lemma 4 can make use of a periodic uncertain matrix $P_k(\alpha)$, $k = 1, \ldots, T$, since the Lyapunov matrix is not used to recover the periodic observer gains. In Lemma 3 the Lyapunov matrix is used to recover the matrices of the periodic observer and in this way it cannot depend upon the parameter α.

The state-feedback controller has been designed by considering Lemma 5 for uncertain systems. Considering $\xi = 20$, the $\mathscr{H}_2$ performance is given by $\mu = 10.71$ and the matrices of the periodic state-feedback control are given by

$$\begin{aligned} K_1 &= \begin{bmatrix} 1.4096 & 0.1913 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} 0.1993 & -0.9671 \end{bmatrix}, \\ K_3 &= \begin{bmatrix} -0.2581 & -2.0508 \end{bmatrix}. \end{aligned} \tag{54}$$

Temporal simulations for different values of α_1 and α_2 are presented in Figure 5 to certify that the observer-based control can stabilize the system over all the domain of uncertainties. The same input disturbance w_k considered in Example 1 is applied in this case.

6. Conclusion

This paper introduced new conditions for the computation of robust periodic observer gains and robust periodic stabilizing gains for periodic discrete-time systems considering the $\mathscr{H}_2$ performance. Two conditions were proposed for observer design: the first one uses the Lyapunov matrix to recover the observer gains, while the second one makes use of slack variables to compute the periodic observer. The conditions proposed for the periodic state-feedback control design take into account the fact that the feedback

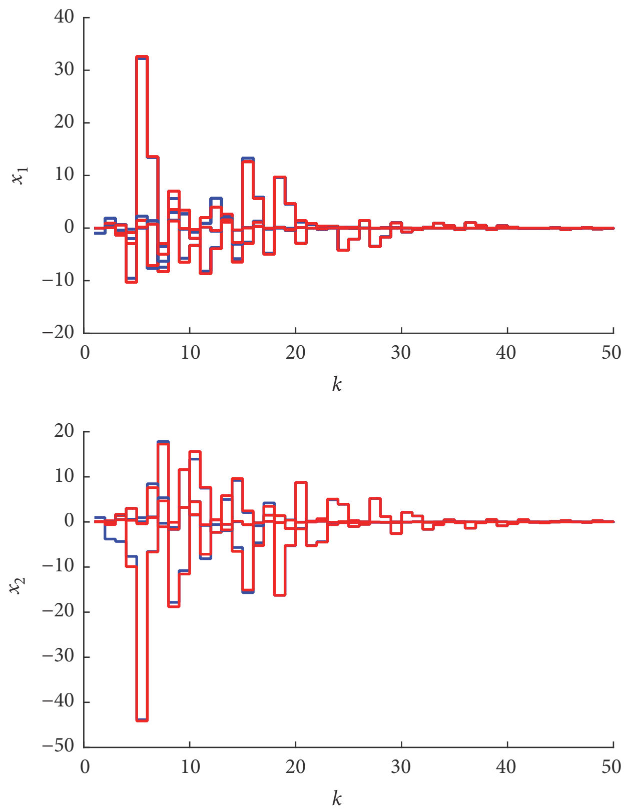

Figure 5: Real states (blue line) and observed states (red line) using the observer-based control technique with periodic observer designed with Lemma 4 and a periodic state-feedback controller designed with Lemma 5 for Example 2.

is performed using the estimated states. The dual representation of the discrete-time periodic system has been employed to tackle with the periodic state-feedback control problem. Numerical experiments illustrated the potential of the proposed observer-based control technique to deal with periodic discrete-time systems even in the presence of uncertainties.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was partially supported by the Brazilian agencies CNPq Grants (402830/2016-4, 425800/2018-0) and FAPEMIG (APQ-00692-17).

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