Deviations for Jumping Times of a Branching Process Indexed by a Poisson Process

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Consider a continuous time process \( Y_t = Z_{N_t}, t \geq 0 \), where \( \{Z_n\} \) is a supercritical Galton–Watson process and \( \{N_t\} \) is a Poisson process which is independent of \( \{Z_n\} \). Let \( \tau_n \) be the \( n \)-th jumping time of \( \{Y_t\} \), we obtain that the typical rate of growth for \( \{\tau_n\} \) is \( n/\lambda \), where \( \lambda \) is the intensity of \( \{N_t\} \).

1. Statements of the Main Results

The model of Poisson randomly indexed branching process (PRIBP) \( \{Y_t = Z_{N_t}, t \geq 0\} \) was introduced by [1] to study the evolution of stock prices and its statistical investigation was done in [2].

In a recent manuscript [3] the authors there consider the asymptotic properties of \( \log Y_t \). Let \( \{p_k, k \geq 0\} \) be the offspring distribution of the branching process with mean \( m = \sum_k k p_k \in (1, \infty) \); we distinguish between the Shr"oder case and the B"ottcher case depending on whether \( p_0 + p_1 > 0 \) or \( p_0 + p_1 = 0 \). In B"ottcher case, it was proved in [3] that \( \log Y_t \) have similar asymptotic results to the Poisson process \( \{N_t\} \). But differences appeared in Shr"oder case; see [4]. For subcritical and critical PRIBP one can see [5, 6] for details.

In this paper, we deal with the asymptotic theory for the jumping times of PRIBP defined as follows. For any \( \omega \), define

\[
\tau_{\infty} (\omega) = \inf \left\{ t : t > 0; \lim_{s \uparrow t} Y_s (\omega) = \infty \right\},
\]

\[
\{ \tau_{\infty} < \infty \} = \bigcup_{l=1}^{\infty} \{ \tau_l < \infty \}.
\]

where \( \inf \emptyset = \infty \); then

Note that \( \{Z_n\} \) is independent of \( \{N_t\} \); one has

\[
P (\tau_{\infty} < l) \leq P (Y_l = \infty) = \sum_n P (Z_n = \infty) P (N_l = n)
\]

\[
= 0,
\]

and thus \( P (\tau_{\infty} = \infty) = 1 \). Define \( \tau_0 = 0 \) and \( \{\tau_n, n \geq 1\} \) as the successive times of jump of the PRIBP \( \{Y_t, t \geq 0\} \).

In B"ottcher case, the jumping times of \( \{Y_t\} \) coincide with that of \( \{N_t\} \). Let \( \{T_n\} \) be the successive times of jump of \( \{N_t\} \); then both \( \tau_n \) and \( T_n \) have a gamma distribution with parameters \( n \) and \( \lambda \). But when \( p_0 + p_1 > 0 \), at the jumping time of \( N_t \), PRIBP can have no jump, since an individual can replicate himself at this time. So the jumping times of \( \{Y_t\} \) are likely to be delayed; see Figure 1 for example. In the path of Figure 1, \( \tau_1 = T_1 \) and \( \tau_2 = T_3, \ldots \).

Although \( \tau_n \geq T_n \) for all \( n \), the growth rate of \( \tau_n \) is not too fast as that of \( T_n \). In fact, the typical growth rate of \( T_n \) is \( n/\lambda \) by the law of large numbers and we can show that the typical growth rate of \( \tau_n \) is

\[
\sum_{k=1}^{n} \frac{1}{\lambda (1 - p_k^l)} \leq \frac{n}{\lambda} + \frac{1}{\lambda (1 - p_1^l)},
\]

and see the proof of Theorem 1. Thus, for almost all the path of Shr"oder case PRIBP, \( \tau_n/n \) has a limit \( \lambda^{-1} \) when \( n \rightarrow \infty \).
In the rest of this paper, we always assume that our branching process belongs to the Shröder case, $p_0 = 0$ and $Z_0 = 1$.

We are interested in the decay rates about the probabilities of

$$
\{ \omega : \left| \frac{\tau_n(\omega)}{n} - \frac{1}{\lambda} \right| > \delta \} \quad \text{(5)}
$$

for some positive $\delta$. Typically, there are three classes of $\delta$ to be chosen.

The first one is that $\delta = a \sqrt{n}$ for some fixed $a > 0$. In this case, the event in (5) is said to be a normal deviation event. The decay rate of its probability can be characterized by the following central limit theorem.

**Theorem 1.**\{\(\tau_n\)\} satisfies the law of large number and the central limit theorem; that is,\(\frac{\tau_n}{n} \overset{a.e.}{\to} \frac{1}{\lambda}\) and $\lambda \sqrt{n}(\frac{\tau_n}{n} - \frac{1}{\lambda}) \overset{d}{\to} N(0, 1)$ when $n \to \infty$, where $N(0, 1)$ is standard normal distribution.

Next, if $\delta = a$ for some fixed $a > 0$, the event in (5) is said to be a large deviation event whose probability has an exponential convergence rate by the following large deviation principle.

**Theorem 2 (LDP).** For any measurable subset $B$ of $\mathbb{R}$,

\[
-\inf_{x \in B^c} \Lambda^* (x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P \left( \frac{\tau_n}{n} \in B \right) \leq \limsup_{n \to \infty} \frac{1}{n} \log P \left( \frac{\tau_n}{n} \in B \right) \leq -\inf_{x \in B} \Lambda^* (x),
\]

where $B^c$ denotes the interior of $B$, $\overline{B}$ its closure, and

\[
\Lambda^* (x) = \begin{cases} 
\lambda (1 - p_1) x + \log p_1, & x \geq (\lambda p_1)^{-1}; \\
\lambda x - \log (\lambda x) - 1, & (\lambda p_1)^{-1} > x > 0; \\
\infty, & x \leq 0.
\end{cases}
\]

Remark. By Cramér’s theorem (see Theorem 2.2.3 of [7]), $T_n/n$ satisfies the large deviation principle with rate function

$$
\Psi (x) = \begin{cases} 
\lambda x - \log (\lambda x) - 1, & x > 0; \\
\infty, & x \leq 0.
\end{cases}
$$

By Theorem 2, the rate function of $\frac{\tau_n}{n}$ coincides with that of $T_n/n$ for $x \leq (\lambda p_1)^{-1}$, but differences appeared for large $x$; see Figure 2 for example.

If $\delta = \delta_n \to \infty$ and $\delta_n = o(\sqrt{n})$ as $n \to \infty$, we call the event in (5) a moderate deviation event. Let $\{a_n, n \geq 0\}$ be a family of positive numbers satisfying

$$
\frac{a_n}{n} \to 0
$$

and

$$
\frac{a_n}{\sqrt{n}} \to \infty
$$

as $n \to \infty$.

As in the case of large deviation principle, based on the Gärtner-Ellis theorem (see [7], page 44), we have the following moderate deviation principle.

**Theorem 3 (MDP).** For any measurable subset $B$ of $R$,

$$
-\inf_{x \in B^c} \lambda^2 x^2 \leq \liminf_{n \to \infty} \frac{n}{a_n^2} \log P \left( \frac{\tau_n - \lambda^{-1} n}{a_n} \in B \right)
\]

$$
\leq \limsup_{n \to \infty} \frac{n}{a_n^2} \log P \left( \frac{\tau_n - \lambda^{-1} n}{a_n} \in B \right)
\]

$$
\leq -\inf_{x \in B^c} \frac{\lambda^2 x^2}{2}.
\]

The rest of the paper is organized as follows. In Section 2, we prove the law of the large number and the central limit
for any nonnegative real numbers $t_0 < t_1 < \cdots < t_{n-1} < s < t + s$ and nonnegative integers $i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i \leq j$, define

$$A = \{ Y(s) = i, Y(t_{n-1}) = i_{n-1}, \ldots, Y(t_0) = i_0 \}. \quad (11)$$

For any nonnegative integers $k_0 \leq k_1 \leq \cdots \leq k_{n-1} \leq k_i \leq k_{i+1}$, define

$$B(k_0, \ldots, k_{n-1}, k_i, k_k) = \{N(t_0) = k_0, \ldots, N(t_{n-1}) = k_{n-1}, N(s) = k_k, N(t + s) = k_{i+1}\}. \quad (12)$$

Since Poisson process $\{N_t\}$ is independent of the Galton-Watson process $\{Z_n\}$,

$$P(Y(t + s) = j \mid A) = \sum_{k_0, \ldots, k_{n-1}, k_i, k_{k+1}} P(B(k_0, \ldots, k_{n-1}, k_i, k_{k+1})) \cdot P(Z_{k_i+1} = j \mid Z_{k_i} = i, Z_{k_{i-1}} = i_{n-1}, \ldots, Z_{k_0} = i_0). \quad (13)$$

Note that the Galton-Watson process is a Markov chain with $n$-step transition probabilities $P_n(i, j)$, and summing $k_0, \ldots, k_{n-1}$, one has

$$P(Y(t + s) = j \mid A) = \sum_{k_0, \ldots, k_{n-1}} P(N(s) = k_k, N(t + s) = k_{k+1}) \cdot P(Z_{k_i+1} = j \mid Z_{k_i} = i) \quad (14)$$

$$= \sum_{k_0, \ldots, k_{n-1}} P(N(s) = k_k, N(t + s) = k_{k+1}) P_{k_i+1-k_k}(i, j)$$

$$= E(P_{N(t+s)-N(t)}(i, j)) = E(P_{N(t)}(i, j)).$$

Similarly,

$$P(Y(t + s) = j \mid Y(s) = i) = E(P_{N(t)}(i, j)) = P(Y(t + s) = j \mid A), \quad (15)$$

which means that PRIBP is a homogenous continuous time Markov chain.

Next, note that $N_t$ has a Poisson distribution with parameter $\lambda t > 0$; we have

$$E(P_{N(t)}(i, i)) = e^{-\lambda t} + \sum_{n \geq 2} \lambda^n P(N(t) = n), \quad (16)$$

which implies $q_i = \lambda(1 - p^i)$.

Proof of Theorem 1. Define $X_n(\omega) = Y(\tau_n(\omega))(\omega)$; then $\{X_n\}$ is a homogenous discrete-time Markov chain. Define $\rho_n = \tau_n - \tau_{n-1}$ for $n \geq 1$; then the conditional distribution of $\rho_n$ relative to $X_1, X_2, \ldots, X_{n-1}$ equals exponential distribution with parameter $q_{X_{n-1}}$, where $q_i = \lambda(1 - p^i)$, see page 259 of [8] for example. So

$$\tau_n = \tau_n - \sum_{k=1}^{n} \frac{1}{\lambda (1 - p^k)} \rho_k - \frac{1}{\lambda (1 - p^1)} \frac{1}{\lambda (1 - p^1)}$$

is a square-integrable martingale adapted to the $\sigma$ fields $\sigma(X_1, X_n)$. Consequently, there exists a random variable $Z$ such that $\tau_n^{1/2} \xrightarrow{a.e.} Z$, see page 2 of [9]. Note that $X_{n-1} \geq n$; one has

$$n^{1/2} \leq \sum_{k=1}^{n} \frac{1}{\lambda (1 - p^k)} \leq n \leq \frac{1}{\lambda (1 - p^1)}$$

which implies that $\tau_n^{1/2} \xrightarrow{a.e.} \lambda^{-1}$ as $n \to \infty$.

Next, we prove $\lambda t^{1/2} \sqrt{n} \xrightarrow{d} N(0, 1)$. Let $\eta_{n,i} = \lambda n^{-1/2}[\rho_i - (\lambda(1 - p^i))^{-1}]$, $F_{i-1}$ be the $\sigma$-field generated by $X_1, \ldots, X_{i-1}$; by Hölder’s inequality, one has

$$E(\eta_{n,i}^3 | F_{i-1})$$

$$\leq \left[ E(\eta_{n,i}^3 | F_{i-1}) \right]^{2/3} \left[ E(\eta_{n,i}^3 | F_{i-1}) \right]^{1/3}, \quad (19)$$

where $I_{A}$ is the indicator function of $A$.

Note that the conditional distribution of $\rho_n$ relative to $\rho_1, \ldots, \rho_{n-1}$ equals exponential distribution with parameter $\lambda(1 - p_{X_{n-1}})$; one has

$$E(\eta_{n,i}^3 | F_{i-1})$$

$$= (\lambda n)^{-3/2} E \left[ \left| \rho_i - (\lambda(1 - p^i))^{-1} \right|^3 | F_{i-1} \right]$$

$$\leq 4n^{-3/2} \left( \lambda^{1/2} (1 - p^i)^{-3} \right) a.s.$$. \quad (20)
According to (22), (23), and Corollary 3.1 of [9], one has
\[ \sum_{i=1}^{n} E \left( \eta_{n,i}^2 \mid \mathcal{F}_{t-i} \right) \leq Cn^{-1/3} \text{ a.e.} \]
(22)

By formulas (19)-(21) we have
\[ \sum_{i=1}^{n} E \left( \eta_{n,i}^2 f(\eta_{n,i}) \mid \mathcal{F}_{t-i} \right) \leq Cn^{-1/3} \text{ a.e.} \] (23)

According to (22), (23), and Corollary 3.1 of [9], one has \( \lambda r_n / \sqrt{n} \to N(0,1) \). Note that
\[ \frac{\lambda r_n}{\sqrt{n}} - \lambda \sqrt{n} \left( \frac{r_n}{n} - \lambda^{-1} \right) \]
(24)
\[ = \sqrt{n} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( 1 - p_i X_i^{(n)} \right)^{-2} \to 1 \] (25)

and the central limit theorem follows from Theorem 6 in page 39 of [10]. \( \square \)

3. Large Deviation Principle

Let us begin with some lemmas to show the conditions of Gärtner-Ellis Theorem (see Appendix) are satisfied. Define \( \Lambda_n(\theta) = \log E[e^{\theta r_n/n}] \).

**Lemma 5.** For any \( \theta \in \mathbb{R} \), \( \Lambda(\theta) = \lim_{n \to \infty} (1/n) \Lambda_n(n\theta) \) exists and satisfies
\[ \Lambda(\theta) = \begin{cases} \log \lambda - \log (\lambda - \theta), & \theta < \lambda (1-p_1) ; \\ +\infty, & \theta \geq \lambda (1-p_1). \end{cases} \] (26)

Particularly, \( 0 \in D_\Lambda = \{ \theta : \Lambda(\theta) < \infty \} \).

**Proof.** Note that \( r_n = p_1 + \cdots + p_n \), where the conditional distribution of \( p_n \) relative to \( p_1, \ldots, p_{n-1} \) is the same as that relative to \( X_0, X_1, \ldots, X_{n-1} \) and equals exponential distribution with parameter \( \lambda(1-p_1^{X_{n-1}}) \) (see page 259 of [8]); one has
\[ E \left[ e^{\theta r_n} \right] = E \left[ e^{\theta \sum_{i=1}^{n} r_i} \right] \]
(27)
\[ = \left. E \left[ e^{\theta \sum_{i=1}^{n} r_i} \mid r_1, \ldots, r_{n-1} \right] \right] \]
(28)
\[ = E \left[ e^{\theta \sum_{i=1}^{n} r_i} \right] E \left[ e^{\theta r_1} \mid r_1, \ldots, r_{n-1} \right] \]
(29)
\[ = E \left[ e^{\theta \sum_{i=1}^{n} r_i} \right] E \left[ e^{\theta r_1} \mid X_0, \ldots, X_{n-1} \right] \]

If \( \theta < \lambda (1-p_1) \), we have
\[ E \left( e^{\theta r_n} \mid X_0, \ldots, X_{n-1} \right) = \frac{\lambda (1-p_1^{X_{n-1}})}{\lambda (1-p_1^{X_{n-1}}) - \theta}. \] (30)

If \( \theta \leq 0 \), note that \( X_{n-1} \geq n \); by (27) one has
\[ \frac{\lambda (1-p_1^{X_{n-1}})}{\lambda (1-p_1^{X_{n-1}}) - \theta} \leq E \left( e^{\theta r_n} \mid X_0, \ldots, X_{n-1} \right) \leq \frac{\lambda}{\lambda - \theta}. \] (31)

By (26), (28) and induction, we have
\[ \frac{\lambda (1-p_1^{X_{n-1}})}{\lambda (1-p_1^{X_{n-1}}) - \theta} \leq E \left( e^{\theta r_n} \mid X_0, \ldots, X_{n-1} \right) \leq \frac{\lambda}{\lambda - \theta} \] (32)

which means \( \Lambda(\theta) = \log \lambda - \log (\lambda - \theta) \).

If \( 0 < \theta < \lambda (1-p_1) \), note that \( X_{n-1} \geq n \); by (27) one has
\[ \frac{\lambda}{\lambda - \theta} \leq E \left( e^{\theta r_n} \mid X_0, \ldots, X_{n-1} \right) \leq \frac{\lambda (1-p_1^{X_{n-1}})}{\lambda (1-p_1^{X_{n-1}}) - \theta}. \]

We can get \( \Lambda(\theta) = \log \lambda - \log (\lambda - \theta) \) similarly.

If \( \theta \geq \lambda (1-p_1) \), note that \( X_0 = Z_0 = 1 \) and the conditional distribution of \( p_1 \) relative to \( X_0 \) equals exponential distribution with parameter \( \lambda(1-p_1) \); one has
\[ E \left[ e^{\theta r_n} \right] \geq E \left[ e^{\theta r_1} \right] = E \left[ E \left( e^{\theta r_1} \mid X_0 \right) \right] = +\infty, \] (33)

for all \( n \geq 1 \). Thus \( \Lambda(\theta) = +\infty \). \( \square \)

**Lemma 6.** Let \( \Lambda^* \) be the Fenchel-Legendre transform of \( \Lambda \); then
\[ \Lambda^* (x) = \sup_{\theta \in \mathbb{R}} \{ \theta x - \Lambda (\theta) \} \]
(34)
\[ = \begin{cases} \lambda (1-p_1) x + \log p_1, & x \geq (\lambda p_1)^{-1} ; \\ \lambda x - \log (\lambda x) - 1, & \lambda p_1^{-1} > x > 0 ; \\ +\infty, & x \leq 0. \end{cases} \] (35)

In addition, the set of exposed points (see Appendix) of \( \Lambda^* \) is \( \mathcal{E} > (0,+\infty) \).

**Proof.** By Lemma 5, if \( x \leq 0 \),
\[ \Lambda^* (x) = \sup_{\theta \in \Lambda(1-p_1)} \{ \theta x - \log \Lambda + \log (\Lambda - \theta) \} \]
(36)
\[ = \lim_{\theta \to +\infty} \left( \theta x - \log \Lambda + \log (\Lambda - \theta) \right) = +\infty. \]

Next, if \( \lambda p_1^{-1} > x > 0 \), then \( \lambda^{-1} < \lambda (1-p_1) \) and
\[ \Lambda^* (x) = \sup_{\theta \in \Lambda(1-p_1)} \{ \theta x - \log \Lambda + \log (\Lambda - \theta) \} \]
(37)
\[ = \theta_0 x - \log \lambda + \log (\lambda - \theta_0) \]
(38)
\[ = \lambda x - \log (\lambda x) - 1. \]
Finally, if \( x \geq (\lambda p_1)^{-1} \), then \( \lambda - x^{-1} \geq \lambda (1 - p_1) \). Note that
\[
\frac{d (\theta x - \log \lambda + \log (\lambda - \theta))}{d \theta} = x - \frac{1}{\lambda - \theta} > 0
\]
for all \( \theta < \lambda (1 - p_1) \); we have
\[
\Lambda^* (x) = \sup_{\theta : \lambda (1 - p_1)} \{ \theta x - \log \lambda + \log (\lambda - \theta) \}
\]
\[
= \lambda (1 - p_1) x + \log p_1.
\]
Equation (32) follows from (33), (34), and (36).

In addition, for any \( \theta < \lambda, \Lambda' (\theta) = (\lambda - \theta)^{-1} \), so the range of \( \Lambda' (\theta) \) for \( \theta < \lambda \) is \((0, +\infty)\), the set of exposed points of \( \Lambda^* \); see Lemma 2.3.9 of [7].

**Proof of Theorem 2.** Note that for any \( x \leq 0, \Lambda^* (x) = +\infty \) and the set of exposed points \( \mathcal{E} > (0, +\infty) \), then for any open set \( G \),
\[
\inf_{x \in \mathcal{G} \cap \mathcal{E}} \Lambda^* (x) = \inf_{x \in \mathcal{G}} \Lambda^* (x) = \inf_{x \in \mathcal{G}} \Lambda^* (x) .
\]
Consequently, Theorem 2 follows from Lemma 5, Lemma 6, and the Gärtner-Ellis theorem (see Appendix).

**4. Moderate Deviation Principle**

In this section, we deal with the proof of Theorem 3. Define
\[
\Lambda_n (\theta) = \log E \left[ \exp \left\{ \theta \frac{\tau_n - \lambda^{-1} n}{a_n} \right\} \right].
\]

**Lemma 7.** For each \( \theta \in \mathbb{R} \), one has,
\[
\Delta (\theta) = \lim_{n \to \infty} \frac{n}{a_n^2} \Lambda_n (\theta) = \frac{\theta^2}{2 \lambda^2}.
\]

Particularly, \( 0 \in D_\lambda = \{ \theta : \Delta (\theta) < \infty \} \). In addition, let \( \Lambda^* \) be the Fenchel-Legendre transform of \( \Delta \); then
\[
\Lambda^* (x) = \frac{x^2 \lambda^2}{2} ,
\]
and the set of exposed points of \( \Lambda^* \) is \( \mathcal{F} = R \).

**Proof.** For any \( \theta \in \mathbb{R} \), we have
\[
\Lambda_n (\frac{a_n^2 \theta}{n}) = \log E \left[ \exp \left\{ \frac{a_n^2 \theta}{n} \left( \frac{\tau_n - \lambda^{-1} n}{a_n} \right) \right\} \right]
\]
\[
= \log E \left[ \exp \left\{ \frac{a_n \theta}{n} \left( \frac{\tau_n - \sum_{i=1}^{n} \frac{1}{\lambda (1 - p_1^{X_{i-1}})} }{1} \right) \right\} \right] \ldots (41)
\]
\[
\ldots (41)
\]
\[
\leq E \left[ \exp \left\{ \frac{-\theta a_n}{n \lambda (1 - p_1)} \right\} \right] \leq \frac{\lambda}{-\theta \lambda (1 - p_1)} - a_n \theta / n
\]
\[
\leq E \left[ \exp \left\{ \frac{-\theta a_n}{n \lambda (1 - p_1)} \right\} \right] \leq E \left[ \exp \left\{ \frac{-\theta a_n}{n \lambda} \right\} \right] \leq \frac{\lambda}{-\theta \lambda (1 - p_1)} - a_n \theta / n
\]
\[
\leq \frac{\lambda}{-\theta \lambda (1 - p_1)} - a_n \theta / n
\]
For any \( \theta \geq 0 \),
According to (45)-(47) and induction, we obtain

\[ I_n(\theta) = \frac{n}{\lambda} \exp \left\{ -\theta a_n \right\} \frac{\lambda}{-\theta a_n} \]

\[ \leq E \left[ \exp \left\{ \frac{\lambda}{n} \right\} \right] \]

\[ \leq \prod_{i=1}^{n} \exp \left\{ -\theta a_n \frac{\lambda}{n} \right\} \frac{\lambda}{-\theta a_n} \]

\[ = H_n(\theta). \]

Similarly, for \( \theta < 0 \), the above inequality should be reversed. According to (18) and \( \log(1+x) = x - x^2/2 + o(x^2) \) as \( x \to 0 \), one has

\[ \log I_n(\theta) = -\frac{\theta a_n}{\lambda} \sum_{i=1}^{n} \frac{1}{1-p_i^1} - n \log \left( 1 - \frac{\theta a_n}{n\lambda} \right) \]

\[ = -\frac{\theta a_n}{\lambda} \left( 1 + b_n \right) + \frac{\theta a_n}{\lambda} + \frac{\theta^2 a_n^2}{2n\lambda^2} + o \left( \frac{a_n^2}{n} \right) \]

\[ = \frac{\theta a_n b_n}{\lambda} + \frac{\theta^2 a_n^2}{2n\lambda^2} + o \left( \frac{a_n^2}{n} \right), \]

where \( b_n \) belongs to \([0,1/(n(1-p^1_1))]\). Hence,

\[ \lim_{n \to \infty} \frac{n}{a_n^2} \log I_n(\theta) = \frac{\theta^2}{2\lambda^2}, \]

and, similarly,

\[ \lim_{n \to \infty} \frac{n}{a_n^2} \log H_n(\theta) = \frac{\theta^2}{2\lambda^2}. \]

Equation (39) is followed by (43), (48), (50), and (51). Consequently,

\[ \Delta^*(x) = \sup_{\theta \in R} \{ \theta x - \Delta(\theta) \} = \sup_{\theta \in R} \{ \theta x - \frac{\theta^2}{2\lambda^2} \} \]

\[ = \frac{\lambda^2 x^2}{2}. \]

In addition, for any \( \theta \in R \), \( \Delta'(\theta) = \theta/\lambda^2 \); so the range of \( \Delta'(\theta) \) is \( R \), which means \( \mathcal{F} = R \); see Lemma 2.3.9 of [7]. \( \square \)

**Proof of Theorem 2.** Note that the set of exposed points of \( \Delta^* \) is \( R \); Theorem 3 follows from Lemma 7 and the Gärtner-Ellis theorem. \( \square \)

**Appendix**

**The Gärtner-Ellis Theorem**

Consider a stochastic process \( \{S_t\}_{t \geq 0} \), where \( S_t \) possesses the law \( \nu_n \) and logarithmic moment generating function \( \Lambda_n(\theta) := \log E(e^{\theta S_n}) \).

**Assumption A.** For each \( \theta \in R \) and \( 0 < b_n \to \infty \), the logarithmic moment generating function, defined as the limit

\[ \Lambda(\theta) = \lim_{n \to \infty} \frac{1}{b_n^2} \Lambda_n \left( b_n \theta \right) \]

exists as an extended real number. Further, the origin belongs to the interior of \( \{ \theta : \Lambda(\theta) < \infty \} \).

**Definition.** Let \( \Lambda^* \) be the Fenchel-Legendre transform of \( \Lambda \). \( y \in R \) is an exposed point of \( \Lambda^* \) if for some \( \theta \in R \) and all \( x \neq y \) it is verified that \( \theta y - \Lambda^*(y) > \theta x - \Lambda^*(x) \). \( \theta \) in the previous equation is called an exposing hyperplane.

Let \( \mathcal{E} \) be the set of exposed points of \( \Lambda^* \) whose exposing hyperplane belongs to the interior of \( \{ \theta : \Lambda(\theta) < \infty \} \). The following lemma is the Gärtner-Ellis theorem in large deviation theory; see [7] page 44.

**Lemma A.1.** Let Assumption A holds.

(a) For any closed set \( F \),

\[ \limsup_{n \to \infty} \frac{1}{b_n} \log \nu_n(F) \leq -\inf_{x \in F} \Lambda^*(x). \]

(b) For any open set \( G \),

\[ \liminf_{n \to \infty} \frac{1}{b_n} \log \nu_n(G) \geq -\inf_{x \in G \cap \mathcal{E}} \Lambda^*(x). \]

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there is no conflict of interest regarding the publication of this paper.

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**References**


