Research Article

Deviations for Jumping Times of a Branching Process Indexed by a Poisson Process

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Consider a continuous time process $\{Y_t = Z_{N_t}, t \geq 0\}$, where $\{Z_n\}$ is a supercritical Galton–Watson process and $\{N_t\}$ is a Poisson process which is independent of $\{Z_n\}$. Let τ_n be the n -th jumping time of $\{Y_t\}$, we obtain that the typical rate of growth for $\{\tau_n\}$ is n/λ , where λ is the intensity of $\{N_t\}$. Probabilities of deviations $\{|n^{-1}\tau_n - \lambda^{-1}| > \delta\}$ are estimated for three types of positive δ .

1. Statements of the Main Results

The model of Poisson randomly indexed branching process (PRIBP) $\{Y_t = Z_{N_t}, t \geq 0\}$ was introduced by [1] to study the evolution of stock prices and its statistical investigation was done in [2].

In a recent manuscript [3] the authors there consider the asymptotic properties of $\log Y_t$. Let $\{p_k, k \geq 0\}$ be the offspring distribution of the branching process with mean $m = \sum_k k p_k \in (1, \infty)$; we distinguish between the Shröder case and the Böttcher case depending on whether $p_0 + p_1 > 0$ or $p_0 + p_1 = 0$. In Böttcher case, it was proved in [3] that $\log Y_t$ have similar asymptotic results to the Poisson process $\{N_t\}$. But differences appeared in Shröder case; see [4]. For subcritical and critical PRIBP, one can see [5, 6] for details.

In this paper, we deal with the asymptotic theory for the jumping times of PRIBP defined as follows. For any ω , define

$$\tau_\infty(\omega) = \inf \left\{ t : t > 0; \lim_{s \uparrow t} Y_s(\omega) = \infty \right\}, \quad (1)$$

where $\inf \emptyset = \infty$; then

$$\{\tau_\infty < \infty\} = \bigcup_{l=1}^{\infty} \{\tau_\infty < l\}. \quad (2)$$

Note that $\{Z_n\}$ is independent of $\{N_t\}$; one has

$$\begin{aligned} P(\tau_\infty < l) &\leq P(Y_l = \infty) = \sum_n P(Z_n = \infty) P(N_l = n) \\ &= 0, \end{aligned} \quad (3)$$

and thus $P(\tau_\infty = \infty) = 1$. Define $\tau_0 = 0$ and $\{\tau_n, n \geq 1\}$ as the successive times of jump of the PRIBP $\{Y_t, t \geq 0\}$.

In Böttcher case, the jumping times of $\{Y_t\}$ coincide with that of $\{N_t\}$. Let $\{T_n\}$ be the successive times of jump of $\{N_t\}$; then both τ_n and T_n have a gamma distribution with parameters n and λ . But when $p_0 + p_1 > 0$, at the jumping time of N_t , PRIBP can have no jump, since an individual can replicate himself at this time. So the jumping times of $\{Y_t\}$ are likely to be delayed; see Figure 1 for example. In the path of Figure 1, $\tau_1 = T_1$ and $\tau_2 = T_3, \dots$.

Although $\tau_n \geq T_n$ for all n , the growth rate of τ_n is not too fast as that of T_n . In fact, the typical growth rate of T_n is n/λ by the law of large numbers and we can show that the typical growth rate of τ_n is

$$\sum_{k=1}^n \frac{1}{\lambda(1-p_1^k)} \leq \frac{n}{\lambda} + \frac{1}{\lambda(1-p_1)^2}, \quad (4)$$

and see the proof of Theorem 1. Thus, for almost all the path of Shröder case PRIBP, τ_n/n has a limit λ^{-1} when $n \rightarrow \infty$.

FIGURE 1: A path of Schröder case branching process.

In the rest of this paper, we always assume that our branching process belongs to the Schröder case, $p_0 = 0$ and $Z_0 = 1$.

We are interested in the decay rates about the probabilities of

$$\left\{ \omega : \left| \frac{\tau_n(\omega)}{n} - \frac{1}{\lambda} \right| > \delta \right\} \quad (5)$$

for some positive δ . Typically, there are three classes of δ to be chosen.

The first one is that $\delta = a\sqrt{n}$ for some fixed $a > 0$. In this case, the event in (5) is said to be a normal deviation event. The decay rate of its probability can be characterized by the following central limit theorem.

Theorem 1. $\{\tau_n\}$ satisfies the law of large number and the central limit theorem; that is, $\tau_n/n \xrightarrow{a.e.} \lambda^{-1}$ and $\lambda\sqrt{n}(\tau_n/n - \lambda^{-1}) \xrightarrow{d} N(0, 1)$ when $n \rightarrow \infty$, where $N(0, 1)$ is standard normal distribution.

Next, if $\delta = a$ for some fixed $a > 0$, the event in (5) is said to be a large deviation event whose probability has an exponential convergence rate by the following large deviation principle.

Theorem 2 (LDP). For any measurable subset B of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x \in B^\circ} \Lambda^*(x) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log P\left(\frac{\tau_n}{n} \in B\right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P\left(\frac{\tau_n}{n} \in B\right) \\ &\leq -\inf_{x \in \bar{B}} \Lambda^*(x), \end{aligned} \quad (6)$$

where B° denotes the interior of B , $\bar{B}$ its closure, and

$$\Lambda^*(x) = \begin{cases} \lambda(1 - p_1)x + \log p_1, & x \geq (\lambda p_1)^{-1}; \\ \lambda x - \log(\lambda x) - 1, & (\lambda p_1)^{-1} > x > 0; \\ +\infty, & x \leq 0. \end{cases} \quad (7)$$

FIGURE 2: $\lambda = 1$, $p_1 = 1/2$.

Remark. By Cramér's theorem (see Theorem 2.2.3 of [7]), T_n/n satisfies the large deviation principle with rate function

$$\Psi(x) = \begin{cases} \lambda x - \log(\lambda x) - 1, & x > 0; \\ +\infty, & x \leq 0. \end{cases} \quad (8)$$

By Theorem 2, the rate function of τ_n/n coincides with that of T_n/n for $x \leq (\lambda p_1)^{-1}$, but differences appeared for large x ; see Figure 2 for example.

If $\delta = \delta_n \rightarrow \infty$ and $\delta_n = o(\sqrt{n})$ as $n \rightarrow \infty$, we call the event in (5) a moderate deviation event. Let $\{a_n, n \geq 0\}$ be a family of positive numbers satisfying

$$\begin{aligned} \frac{a_n}{n} &\rightarrow 0 \\ \text{and } \frac{a_n}{\sqrt{n}} &\rightarrow \infty \end{aligned} \quad (9)$$

as $n \rightarrow \infty$.

As in the case of large deviation principle, based on the Gärtner-Ellis theorem (see [7], page 44), we have the following moderate deviation principle.

Theorem 3 (MDP). For any measurable subset B of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x \in B^\circ} \frac{\lambda^2 x^2}{2} &\leq \liminf_{n \rightarrow \infty} \frac{n}{a_n^2} \log P\left(\frac{\tau_n - \lambda^{-1}n}{a_n} \in B\right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{n}{a_n^2} \log P\left(\frac{\tau_n - \lambda^{-1}n}{a_n} \in B\right) \\ &\leq -\inf_{x \in \bar{B}} \frac{\lambda^2 x^2}{2}. \end{aligned} \quad (10)$$

The rest of the paper is organized as follows. In Section 2, we prove the law of the large number and the central limit

theorem. Section 3 is devoted to the proof of the large deviation principle. In Section 4, the moderate deviation principle is obtained. Basic facts on Gärtner-Ellis theorem are given in the Appendix.

2. The Law of Large Number and the Central Limit Theorem

Proposition 4. *PRIBP is a homogenous continuous time Markov chain in which Q matrix satisfies $q_i = \lambda(1 - p_1^i)$ for all $i \geq 1$.*

Proof. For any nonnegative real numbers $t_0 < t_1 < \dots < t_{n-1} < s < t + s$ and nonnegative integers $i_0 \leq i_1 \leq \dots \leq i_{n-1} \leq i \leq j$, define

$$A = \{Y(s) = i, Y(t_{n-1}) = i_{n-1}, \dots, Y(t_0) = i_0\}. \quad (11)$$

For any nonnegative integers $k_0 \leq k_1 \leq \dots \leq k_{n-1} \leq k_s \leq k_{t+s}$, define

$$\begin{aligned} B(k_0, \dots, k_{n-1}, k_s, k_{t+s}) &= \{N(t_0) = k_0, \dots, N(t_{n-1}) \\ &= k_{n-1}, N(s) = k_s, N(t + s) = k_{t+s}\}. \end{aligned} \quad (12)$$

Since Poisson process $\{N_t\}$ is independent of the Galton-Watson process $\{Z_n\}$,

$$\begin{aligned} P(Y(t + s) = j \mid A) &= \sum_{k_0, \dots, k_{n-1}, k_s, k_{t+s}} P(B(k_0, \dots, k_{n-1}, k_s, k_{t+s})) \\ &\cdot P(Z_{k_{t+s}} = j \mid Z_{k_s} = i, Z_{k_{n-1}} = i_{n-1}, \dots, Z_{k_0} = i_0). \end{aligned} \quad (13)$$

Note that the Galton-Watson process is a Markov chain with n -step transition probabilities $P_n(i, j)$, and summing $k_0, \dots, k_{n-1}$, one has

$$\begin{aligned} P(Y(t + s) = j \mid A) &= \sum_{k_s, k_{t+s}} P(N(s) = k_s, N(t + s) = k_{t+s}) \\ &\cdot P(Z_{k_{t+s}} = j \mid Z_{k_s} = i) \\ &= \sum_{k_s, k_{t+s}} P(N(s) = k_s, N(t + s) = k_{t+s}) P_{k_{t+s}-k_s}(i, j) \\ &= E(P_{N(t+s)-N(s)}(i, j)) = E(P_{N(t)}(i, j)). \end{aligned} \quad (14)$$

Similarly,

$$\begin{aligned} P(Y(t + s) = j \mid Y(s) = i) &= E(P_{N(t)}(i, j)) \\ &= P(Y(t + s) = j \mid A), \end{aligned} \quad (15)$$

which means that PRIBP is a homogenous continuous time Markov chain.

Next, note that N_t has a Poisson distribution with parameter $\lambda t > 0$; we have

$$\begin{aligned} E(P_{N(t)}(i, i)) &= e^{-\lambda t} + p_1^i \lambda t e^{-\lambda t} \\ &+ \sum_{n \geq 2} p_1^{ni} P(N(t) = n), \end{aligned} \quad (16)$$

which implies $q_i = \lambda(1 - p^i)$. $\square$

Proof of Theorem 1. Define $X_n(\omega) = Y(\tau_n(\omega))(\omega)$; then $\{X_n\}$ is a homogeneous discrete-time Markov chain. Define $\rho_n = \tau_n - \tau_{n-1}$ for $n \geq 1$; then the conditional distribution of ρ_n relative to $X_1, X_2, \dots, X_{n-1}$ equals exponential distribution with parameter $q_{X_{n-1}}$, where $q_i = \lambda(1 - p^i)$, see page 259 of [8] for example. So

$$\begin{aligned} \tau'_n &:= \tau_n - \sum_{k=1}^n \frac{1}{\lambda(1 - p_1^{X_{k-1}})} \\ &= \sum_{k=1}^n \left(\rho_k - \frac{1}{\lambda(1 - p_1^{X_{k-1}})} \right) \end{aligned} \quad (17)$$

is a square-integrable martingale adapted to the σ -fields $\sigma\{X_1, \dots, X_n\}$. Consequently, there exists a random variable Z such that $\tau'_n \xrightarrow{a.e.} Z$, see page 2 of [9]. Note that $X_{n-1} \geq n$; one has

$$\begin{aligned} \frac{n}{\lambda} &\leq \sum_{k=1}^n \frac{1}{\lambda(1 - p_1^{X_{k-1}})} \leq \sum_{k=1}^n \frac{1}{\lambda(1 - p_1^k)} \\ &\leq \frac{n}{\lambda} + \frac{1}{\lambda(1 - p_1)^2}, \end{aligned} \quad (18)$$

which implies that $\tau_n/n \xrightarrow{a.e.} \lambda^{-1}$ as $n \rightarrow \infty$.

Next, we prove $\lambda \tau'_n / \sqrt{n} \xrightarrow{d} N(0, 1)$. Let $\eta_{n,i} = \lambda n^{-1/2}[\rho_i - (\lambda(1 - p_1^{X_{i-1}}))^{-1}]$, $\mathcal{F}_{i-1}$ be the σ -field generated by $X_1, \dots, X_{i-1}$; by Hölder's inequality, one has

$$\begin{aligned} E(\eta_{n,i}^2 I_{\{|\eta_{n,i}| > \epsilon\}} \mid \mathcal{F}_{i-1}) &\leq [E(|\eta_{n,i}|^3 \mid \mathcal{F}_{i-1})]^{2/3} [E(I_{\{|\eta_{n,i}| > \epsilon\}} \mid \mathcal{F}_{i-1})]^{1/3}, \end{aligned} \quad (19)$$

where I_A is the indicator function of A .

Note that the conditional distribution of ρ_n relative to $\rho_1, \dots, \rho_{n-1}$ equals exponential distribution with parameter $\lambda(1 - p_1^{X_{n-1}})$; one has

$$\begin{aligned} E(|\eta_{n,i}|^3 \mid \mathcal{F}_{i-1}) &= (\lambda n)^{-3/2} E\left(\left|\rho_i - (\lambda(1 - p_1^{X_{i-1}}))^{-1}\right|^3 \mid \mathcal{F}_{i-1}\right) \\ &\leq 4n^{-3/2} (\lambda^{1/2} (1 - p_1^{X_{i-1}}))^{-3} \\ &\leq 4n^{-3/2} (\lambda^{1/2} (1 - p_1))^{-3} \text{ a.s.} \end{aligned} \quad (20)$$

According to Markov's inequality,

$$\begin{aligned} E\left(I_{\{\eta_{n,i} > \epsilon\}} \mid \mathcal{F}_{i-1}\right) &\leq \epsilon^{-2} E\left(\eta_{n,i}^2 \mid \mathcal{F}_{i-1}\right) \\ &= \left(n\epsilon^2 (1 - p_1^{X_{i-1}})^2\right)^{-1} \\ &\leq \left(n\epsilon^2 (1 - p_1)^2\right)^{-1} \text{ a.s.} \end{aligned} \quad (21)$$

By formulas (19)-(21) we have

$$\sum_{i=1}^n E\left(\eta_{n,i}^2 I_{\{\eta_{n,i} > \epsilon\}} \mid \mathcal{F}_{i-1}\right) \leq Cn^{-1/3} \text{ a.e.} \quad (22)$$

for some positive constant C . Similarly,

$$\sum_{i=1}^n E\left(\eta_{n,i}^2 \mid \mathcal{F}_{i-1}\right) = \frac{1}{n} \sum_{i=1}^n (1 - p_1^{X_{i-1}})^{-2} \xrightarrow{\text{a.e.}} 1. \quad (23)$$

According to (22), (23), and Corollary 3.1 of [9], one has $\lambda\tau'_n/\sqrt{n} \xrightarrow{d} N(0, 1)$. Note that

$$\begin{aligned} &\left| \frac{\lambda\tau'_n}{\sqrt{n}} - \lambda\sqrt{n} \left(\frac{\tau_n}{n} - \lambda^{-1} \right) \right| \\ &= \left| \sqrt{n} - \frac{1}{\sqrt{n}} \sum_{i=1}^n (1 - p_1^{X_{i-1}})^{-1} \right| \leq \left[(1 - p_1)^2 \sqrt{n} \right]^{-1} \rightarrow 0, \end{aligned} \quad (24)$$

0,

and the central limit theorem follows from Theorem 6 in page 39 of [10]. $\square$

3. Large Deviation Principle

Let us begin with some lemmas to show the conditions of Gärtner-Ellis Theorem (see Appendix) are satisfied. Define $\Lambda_n(\theta) = \log E[e^{\theta\tau_n/n}]$.

Lemma 5. For any $\theta \in \mathbb{R}$, $\Lambda(\theta) := \lim_{n \rightarrow \infty} (1/n)\Lambda_n(n\theta)$ exists and satisfies

$$\Lambda(\theta) = \begin{cases} \log \lambda - \log(\lambda - \theta), & \theta < \lambda(1 - p_1); \\ +\infty, & \theta \geq \lambda(1 - p_1). \end{cases} \quad (25)$$

Particularly, $0 \in D_\Lambda := \{\theta : \Lambda(\theta) < \infty\}$.

Proof. Note that $\tau_n = \rho_1 + \dots + \rho_n$, where the conditional distribution of ρ_n relative to $\rho_1, \dots, \rho_{n-1}$ is the same as that relative to $X_0, X_1, \dots, X_{n-1}$ and equals exponential distribution with parameter $\lambda(1 - p_1^{X_{n-1}})$ (see page 259 of [8]); one has

$$\begin{aligned} E[e^{\theta\tau_n}] &= E\left[e^{\theta \sum_{i=1}^n \rho_i}\right] \\ &= E\left[E\left[e^{\theta \sum_{i=1}^n \rho_i} \mid \rho_1, \dots, \rho_{n-1}\right]\right] \\ &= E\left[e^{\theta \sum_{i=1}^{n-1} \rho_i} E(e^{\theta\rho_n} \mid \rho_1, \dots, \rho_{n-1})\right] \\ &= E\left[e^{\theta \sum_{i=1}^{n-1} \rho_i} E(e^{\theta\rho_n} \mid X_0, \dots, X_{n-1})\right]. \end{aligned} \quad (26)$$

If $\theta < \lambda(1 - p_1)$, we have

$$E(e^{\theta\rho_n} \mid X_0, \dots, X_{n-1}) = \frac{\lambda(1 - p_1^{X_{n-1}})}{\lambda(1 - p_1^{X_{n-1}}) - \theta}. \quad (27)$$

If $\theta \leq 0$, note that $X_{n-1} \geq n$; by (27) one has

$$\frac{\lambda(1 - p_1^n)}{\lambda(1 - p_1^n) - \theta} \leq E(e^{\theta\rho_n} \mid X_0, \dots, X_{n-1}) \leq \frac{\lambda}{\lambda - \theta}. \quad (28)$$

By (26), (28) and induction, we have

$$\prod_{i=1}^n \frac{\lambda(1 - p_1^i)}{\lambda(1 - p_1^i) - \theta} \leq E[e^{\theta\tau_n}] \leq \frac{\lambda^n}{(\lambda - \theta)^n}, \quad (29)$$

which means $\Lambda(\theta) = \log \lambda - \log(\lambda - \theta)$.

If $0 < \theta < \lambda(1 - p_1)$, note that $X_{n-1} \geq n$; by (27) one has

$$\frac{\lambda}{\lambda - \theta} \leq E(e^{\theta\rho_n} \mid X_0, \dots, X_{n-1}) \leq \frac{\lambda(1 - p_1^n)}{\lambda(1 - p_1^n) - \theta}. \quad (30)$$

We can get $\Lambda(\theta) = \log \lambda - \log(\lambda - \theta)$ similarly.

If $\theta \geq \lambda(1 - p_1)$, note that $X_0 = Z_0 = 1$ and the conditional distribution of ρ_1 relative to X_0 equals exponential distribution with parameter $\lambda(1 - p_1)$; one has

$$E[e^{\theta\tau_n}] \geq E[e^{\theta\rho_1}] = E[E(e^{\theta\rho_1} \mid X_0)] = +\infty, \quad (31)$$

for all $n \geq 1$. Thus $\Lambda(\theta) = +\infty$. $\square$

Lemma 6. Let Λ^* be the Fenchel-Legendre transform of Λ ; then

$$\begin{aligned} \Lambda^*(x) &= \sup_{\theta \in \mathbb{R}} \{\theta x - \Lambda(\theta)\} \\ &= \begin{cases} \lambda(1 - p_1)x + \log p_1, & x \geq (\lambda p_1)^{-1}; \\ \lambda x - \log(\lambda x) - 1, & (\lambda p_1)^{-1} > x > 0; \\ +\infty, & x \leq 0. \end{cases} \end{aligned} \quad (32)$$

In addition, the set of exposed points (see Appendix) of Λ^* is $\mathcal{E} \supset (0, +\infty)$.

Proof. By Lemma 5, if $x \leq 0$,

$$\begin{aligned} \Lambda^*(x) &= \sup_{\theta < \lambda(1 - p_1)} \{\theta x - \log \lambda + \log(\lambda - \theta)\} \\ &= \lim_{\theta \rightarrow -\infty} (\theta x - \log \lambda + \log(\lambda - \theta)) = +\infty. \end{aligned} \quad (33)$$

Next, if $(\lambda p_1)^{-1} > x > 0$, then $\lambda - x^{-1} < \lambda(1 - p_1)$ and

$$\begin{aligned} \Lambda^*(x) &= \sup_{\theta < \lambda(1 - p_1)} \{\theta x - \log \lambda + \log(\lambda - \theta)\} \\ &\stackrel{\theta_0 = \lambda - x^{-1}}{=} \theta_0 x - \log \lambda + \log(\lambda - \theta_0) \\ &= \lambda x - \log(\lambda x) - 1. \end{aligned} \quad (34)$$

Finally, if $x \geq (\lambda p_1)^{-1}$, then $\lambda - x^{-1} \geq \lambda(1 - p_1)$. Note that

$$\frac{d(\theta x - \log \lambda + \log(\lambda - \theta))}{d\theta} = x - \frac{1}{\lambda - \theta} > 0 \quad (35)$$

for all $\theta < \lambda(1 - p_1)$; we have

$$\begin{aligned} \Lambda^*(x) &= \sup_{\theta < \lambda(1-p_1)} \{\theta x - \log \lambda + \log(\lambda - \theta)\} \\ &\stackrel{\theta_0 = \lambda(1-p_1)}{=} \theta_0 x - \log \lambda + \log(\lambda - \theta_0) \\ &= \lambda(1 - p_1)x + \log p_1. \end{aligned} \quad (36)$$

Equation (32) follows from (33), (34), and (36).

In addition, for any $\theta < \lambda$, $\Lambda'(\theta) = (\lambda - \theta)^{-1}$, so the range of $\Lambda'(\theta)$ for $\theta < \lambda$ is $(0, +\infty)$, which is a subset of $\mathcal{E}$, the set of exposed points of Λ^* ; see Lemma 2.3.9 of [7]. $\square$

Proof of Theorem 2. Note that for any $x \leq 0$, $\Lambda^*(x) = +\infty$ and the set of exposed points $\mathcal{E} \supset (0, +\infty)$, then for any open set G ,

$$\inf_{x \in G \cap \mathcal{E}} \Lambda^*(x) = \inf_{x \in G \cap (0, +\infty)} \Lambda^*(x) = \inf_{x \in G} \Lambda^*(x). \quad (37)$$

Consequently, Theorem 2 follows from Lemma 5, Lemma 6, and the Gärtner-Ellis theorem (see Appendix). $\square$

4. Moderate Deviation Principle

In this section, we deal with the proof of Theorem 3. Define

$$\Lambda_n(\theta) = \log E \left[\exp \left\{ \theta \cdot \frac{\tau_n - \lambda^{-1}n}{a_n} \right\} \right]. \quad (38)$$

Lemma 7. For each $\theta \in R$, one has,

$$\Delta(\theta) = \lim_{n \rightarrow \infty} \frac{n}{a_n^2} \Lambda_n \left(\frac{a_n^2 \theta}{n} \right) = \frac{\theta^2}{2\lambda^2}. \quad (39)$$

Particularly, $0 \in D_\Delta := \{\theta : \Delta(\theta) < \infty\}$. In addition, let Δ^* be the Fenchel-Legendre transform of Δ ; then

$$\Delta^*(x) = \frac{\lambda^2 x^2}{2}, \quad (40)$$

and the set of exposed points of Δ^* is $\mathcal{F} = R$.

Proof. For any $\theta \in R$, we have

$$\begin{aligned} \Lambda_n \left(\frac{a_n^2 \theta}{n} \right) &= \log E \left[\exp \left\{ \frac{a_n^2 \theta}{n} \cdot \frac{\tau_n - \lambda^{-1}n}{a_n} \right\} \right] \\ &= \log E \left[\exp \left\{ \frac{a_n \theta}{n} \left(\tau_n - \sum_{i=1}^n \frac{1}{\lambda(1 - p_1^{X_{i-1}})} \right) \right\} \right] \\ &\quad \cdot \exp \left\{ \frac{a_n \theta}{n} \cdot \sum_{i=1}^n \frac{p_1^{X_{i-1}}}{\lambda(1 - p_1^{X_{i-1}})} \right\}. \end{aligned} \quad (41)$$

For any $\theta \geq 0$, note that $X_{i-1} \geq i$; one has

$$\begin{aligned} 0 &\leq \frac{a_n \theta}{n} \cdot \sum_{i=1}^n \frac{p_1^{X_{i-1}}}{\lambda(1 - p_1^{X_{i-1}})} \leq \frac{a_n \theta}{n} \cdot \sum_{i=1}^n \frac{p_1^i}{\lambda(1 - p_1^i)} \\ &\leq \frac{a_n \theta}{n} \frac{1}{\lambda(1 - p_1)^2}. \end{aligned} \quad (42)$$

Similarly, for $\theta < 0$, the above inequality should be reversed. Thus, by (41), if $\Delta(\theta)$ exists, one has

$$\begin{aligned} \Delta(\theta) &= \lim_{n \rightarrow \infty} \frac{n}{a_n^2} \log E \\ &\quad \cdot \left[\exp \left\{ \frac{a_n \theta}{n} \left(\tau_n - \sum_{i=1}^n \frac{1}{\lambda(1 - p_1^{X_{i-1}})} \right) \right\} \right]. \end{aligned} \quad (43)$$

Define $\beta_i = \rho_i - (\lambda(1 - p_1^{X_{i-1}}))^{-1}$; then

$$\tau_n - \sum_{i=1}^n \frac{1}{\lambda(1 - p_1^{X_{i-1}})} = \sum_{i=1}^n \beta_i. \quad (44)$$

Note that the conditional distribution of ρ_n relative to $\rho_1, \dots, \rho_{n-1}$ is the same as that relative to $X_0, X_1, \dots, X_{n-1}$ and equals exponential distribution with parameter $\lambda(1 - p_1^{X_{n-1}})$; one has

$$\begin{aligned} E \left[\exp \left\{ \frac{a_n \theta}{n} \sum_{i=1}^n \beta_i \right\} \right] \\ = E \left[\exp \left\{ \frac{a_n \theta}{n} \sum_{i=1}^{n-1} \beta_i \right\} E \left[\exp \left\{ \frac{a_n \theta}{n} \beta_n \right\} \mid \mathcal{F}_{n-1} \right] \right], \end{aligned} \quad (45)$$

where $\mathcal{F}_{i-1}$ is the σ -field generated by $X_1, \dots, X_{i-1}$. After a simple calculation,

$$\begin{aligned} E \left[\exp \left\{ \frac{a_n \theta}{n} \beta_i \right\} \mid \mathcal{F}_{i-1} \right] \\ = \exp \left\{ \frac{-\theta a_n}{n\lambda(1 - p_1^{X_{i-1}})} \right\} \cdot \frac{\lambda(1 - p_1^{X_{i-1}})}{\lambda(1 - p_1^{X_{i-1}}) - a_n \theta / n} \end{aligned} \quad (46)$$

For $\theta > 0$,

$$\begin{aligned} \exp \left\{ \frac{-\theta a_n}{n\lambda(1 - p_1^i)} \right\} \cdot \frac{\lambda}{\lambda - a_n \theta / n} \\ \leq E \left[\exp \left\{ \frac{a_n \theta}{n} \beta_i \right\} \mid \mathcal{F}_{i-1} \right] \\ \leq \exp \left\{ \frac{-\theta a_n}{n\lambda} \right\} \cdot \frac{\lambda(1 - p_1^i)}{\lambda(1 - p_1^i) - a_n \theta / n}. \end{aligned} \quad (47)$$

According to (45)-(47) and induction, we obtain

$$\begin{aligned}
 I_n(\theta) &:= \prod_{i=1}^n \left[\exp \left\{ \frac{-\theta a_n}{n\lambda(1-p_1^i)} \right\} \cdot \frac{\lambda}{\lambda - a_n\theta/n} \right] \\
 &\leq E \left[\exp \left\{ \frac{a_n\theta}{n} \sum_{i=1}^n \beta_i \right\} \right] \\
 &\leq \prod_{i=1}^n \left[\exp \left\{ \frac{-\theta a_n}{n\lambda} \right\} \cdot \frac{\lambda(1-p_1^i)}{\lambda(1-p_1^i) - a_n\theta/n} \right] \\
 &=: H_n(\theta).
 \end{aligned} \tag{48}$$

Similarly, for $\theta < 0$, the above inequality should be reversed. According to (18) and $\log(1+x) = x - x^2/2 + o(x^2)$ as $x \rightarrow 0$, one has

$$\begin{aligned}
 \log I_n(\theta) &= \frac{-\theta a_n}{n\lambda} \sum_{i=1}^n \frac{1}{1-p_1^i} - n \log \left(1 - \frac{a_n\theta}{n\lambda} \right) \\
 &= \frac{-\theta a_n}{\lambda} (1 + b_n) + \frac{\theta a_n}{\lambda} + \frac{\theta^2 a_n^2}{2n\lambda^2} + o\left(\frac{a_n^2}{n}\right) \\
 &= \frac{-\theta a_n b_n}{\lambda} + \frac{\theta^2 a_n^2}{2n\lambda^2} + o\left(\frac{a_n^2}{n}\right),
 \end{aligned} \tag{49}$$

where b_n belongs to $[0, 1/(n(1-p_1^2))]$. Hence,

$$\lim_{n \rightarrow \infty} \frac{n}{a_n^2} \log I_n(\theta) = \frac{\theta^2}{2\lambda^2}, \tag{50}$$

and, similarly,

$$\lim_{n \rightarrow \infty} \frac{n}{a_n^2} \log H_n(\theta) = \frac{\theta^2}{2\lambda^2}. \tag{51}$$

Equation (39) is followed by (43), (48), (50), and (51). Consequently,

$$\begin{aligned}
 \Delta^*(x) &= \sup_{\theta \in R} \{\theta x - \Delta(\theta)\} = \sup_{\theta \in R} \left\{ \theta x - \frac{\theta^2}{2\lambda^2} \right\} \\
 &= \frac{\lambda^2 x^2}{2}.
 \end{aligned} \tag{52}$$

In addition, for any $\theta \in R$, $\Delta'(\theta) = \theta/\lambda^2$; so the range of $\Delta'(\theta)$ is R , which means $\mathcal{F} = R$; see Lemma 2.3.9 of [7]. $\square$

Proof of Theorem 2. Note that the set of exposed points of Δ^* is R ; Theorem 3 follows from Lemma 7 and the Gärtner-Ellis theorem. $\square$

Appendix

The Gärtner-Ellis Theorem

Consider a stochastic process $\{S_n\}_{n \geq 0}$, where S_n possesses the law ν_n and logarithmic moment generating function $\Lambda_n(\theta) := \log E(e^{\theta S_n})$.

Assumption A. For each $\theta \in R$ and $0 < b_n \rightarrow \infty$, the logarithmic moment generating function, defined as the limit

$$\Lambda(\theta) = \lim_{n \rightarrow \infty} \frac{1}{b_n} \Lambda_n(b_n \theta) \tag{A.1}$$

exists as an extended real number. Further, the origin belongs to the interior of $\{\theta : \Lambda(\theta) < \infty\}$.

Definition. Let Λ^* be the Fenchel-Legendre transform of Λ . $y \in R$ is an exposed point of Λ^* if for some $\theta \in R$ and all $x \neq y$ it is verified that $\theta y - \Lambda^*(y) > \theta x - \Lambda^*(x)$. θ in the previous equation is called an exposing hyperplane.

Let $\mathcal{E}$ be the set of exposed points of Λ^* whose exposing hyperplane belongs to the interior of $\{\theta : \Lambda(\theta) < \infty\}$. The following lemma is the Gärtner-Ellis theorem in large deviation theory; see [7] page 44.

Lemma A.1. *Let Assumption A holds.*

(a) *For any closed set F ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n} \log \nu_n(F) \leq -\inf_{x \in F} \Lambda^*(x). \tag{A.2}$$

(b) *For any open set G ,*

$$\liminf_{n \rightarrow \infty} \frac{1}{b_n} \log \nu_n(G) \geq -\inf_{x \in G \cap \mathcal{E}} \Lambda^*(x). \tag{A.3}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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