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Research Article

Fractional-Order Accumulative Linear Time-Varying Parameters Discrete Grey Forecasting Model

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Traditional discrete grey forecasting model can effectively predict the development trend of the stabilizing system. However, when the system has disturbance information, the prediction result will have larger error, and there will appear significant downward trend in the stability of the model. In the presence of disturbance information, this paper presents a fractional-order linear time-varying parameters discrete grey forecasting model to deal with the system that contains both linear trend and nonlinear trend. The modeling process of the model and calculation method are given. The perturbation bounds of the new model are analyzed by using the least-squares method of perturbation theory. And it is compared with that of the first-order linear time-varying parameters discrete grey forecasting model. Finally, two real cases are given to verify the effectiveness and practicality of the proposed method.

1. Introduction

The grey system theory was proposed by Deng in 1982. It only needs a small amount of data to establish the model for analysis [1, 2]. Grey prediction model is the main part of grey system theory, and its theoretical basis is the grey of accumulation generation. The prediction model is established by using the characteristic of grey exponential. Since the grey prediction model was proposed, it has been widely concerned by scholars. The existing studies mainly focus on the theoretical development from the aspects of background value optimization [3, 4], parameter optimization [5–7], model expansion [8–10], and the application of natural gas [11, 12], electric power [13–16], environment [17–20], economy [21, 22], and transportation [23, 24].

Dang gave the new method to select initial value [25]. Cui proposed the NGM(1,1,k) model and described the modeling mechanism and modeling process of the model [26]. Zhou proposed the generalized GM(1,1) model and used the new model to simulate and predict China's fuel output from 2003 to 2010 [27]. Combined with the concept of Bernoulli differential equation, Chen proposed the NGBM(1,1) model on the basis of GM(1,1) model [28]. Li proposed 3spGM(1, 1) model and applied the new model to the failure data sets of electric product manufacturing systems [29]. These methods further improve the modeling effect of grey prediction model. However, the transformation of difference equation and differential equation is required in the solving process. There is still error in the grey prediction model for the sequences that conform to the exponential features.

Xie proposed the discrete grey model (DGM(1,1) model). The relationship between DGM(1,1) and GM(1,1) was studied deeply, and the reason for the instability of GM(1,1) was found. It only needs to use difference equation to solve the equation and does not need to convert the difference equation to differential equation. Therefore, the modeling accuracy is effectively improved [30]. Wu proposed the discrete grey prediction model based on fractional-order accumulation, discussed the properties of the model, and gave the calculation method of the model [31–33]. Liu proposed the fractional-order reverse accumulation discrete grey prediction model and discussed the properties of the model [34].

The research mentioned above has positive significance for improving the accuracy of grey prediction model. However, for complex systems with disturbance, the model's robustness is insufficient; how to deal with the system disturbance information is particularly important. To solve this problem, this paper presents a fractional-order accumulation linear time-varying parameters discrete grey prediction model (FTDGM(1,1)model). The modeling process and parametric calculation method of the model are given. It is proven that the model has good stability using the theory of matrix perturbation analysis. Finally, two real cases are given. And the calculation results show that FTDGM(1,1) model can effectively reduce the disturbance caused by disturbance information. The model's robustness and prediction accuracy are improved, and the validity and practicality of the model are further verified.

2. The Fractional-Order Accumulative Linear Time-Varying Parameters Discrete Grey Model

Definition 1 (see [30]). Assume that the nonnegative sequence is $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)\}$. $X^{(1)} = \{x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n)\}$ is the first-order accumulative sequence of $X^{(0)}$.

Among them,

$$x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i), \quad k = 1, 2, \cdots, n. \tag{1}$$

The equation

$$x^{(1)}(k+1) = \beta_1 x^{(1)}(k) + \beta_2, \quad k = 1, 2, \cdots, n-1 \tag{2}$$

is called discrete grey prediction model (DGM(1,1) model).

Theorem 2 (see [30]). *The parameters of the DGM(1,1) model can be solved by using the following least-squares estimation:*

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \left(B^T B\right)^{-1} B^T Y, \tag{3}$$

and, among them,

$$B = \begin{bmatrix} x^{(1)}(1) & 1 \\ x^{(1)}(2) & 1 \\ \vdots & \vdots \\ x^{(1)}(n-2) & 1 \\ x^{(1)}(n-1) & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} x^{(1)}(2) \\ x^{(1)}(3) \\ \vdots \\ x^{(1)}(n-1) \\ x^{(1)}(n) \end{bmatrix}. \tag{4}$$

Definition 3 (see [35]). Assume that the nonnegative sequence is $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)\}$. $X^{(1)} = \{x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n)\}$ is the first-order accumulative sequence of $X^{(0)}$.

Among them,

$$x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i), \quad k = 1, 2, \cdots, n. \tag{5}$$

The equation

$$x^{(1)}(k+1) = \left(\beta_1 k + \beta_2\right) x^{(1)}(k) + \beta_3 k + \beta_4, \quad k = 1, 2, \cdots, n-1 \tag{6}$$

is called linear time-varying parameters discrete grey model (TDGM).

Theorem 4. *The parameters of the TDGM(1,1) model can be solved by using the following least-squares estimation:*

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \left(C^T C\right)^{-1} C^T Y, \tag{7}$$

and, among them,

$$C = \begin{bmatrix} x^{(1)}(1) & x^{(1)}(1) & 1 & 1 \\ 2x^{(1)}(2) & x^{(1)}(2) & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (n-2)\, x^{(1)}(n-2) & x^{(1)}(n-2) & n-2 & 1 \\ (n-1)\, x^{(1)}(n-1) & x^{(1)}(n-1) & n-1 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} x^{(1)}(2) \\ x^{(1)}(3) \\ \vdots \\ x^{(1)}(n-1) \\ x^{(1)}(n) \end{bmatrix}. \tag{8}$$

Definition 5 (see [18]). Assume that the nonnegative sequence is $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)\}$.

$X^{(r)} = \{x^{(r)}(1), x^{(r)}(2), \cdots, x^{(r)}(n)\}$ is called the fractional-order accumulative sequence of $X^{(0)}$.

Among them,

$$x^{(r)}(k) = \sum_{i=1}^{k} C_{k-i+r-1}^{k-i} x^{(0)}(i), \quad C_{r-1}^{0} = 1,\ C_{k-1}^{k} = 0,\ k = 1, 2, \cdots, n. \tag{9}$$

Definition 6. Assume the nonnegative sequence $X^{(0)}$; $X^{(r)}$is defined as Definition 5. The equation

$$x^{(r)}(k+1)=(\beta_1 k+\beta_2)x^{(r)}(k)+\beta_3 k+\beta_4, \quad k=1,2,\cdots,n-1 \tag{10}$$

is called fractional-order accumulative linear time-varying parameters discrete grey model (FTDGM).

Theorem 7. *The parameters of the FTDGM(1,1) model can be solved by using the following least-squares estimation:*

$$\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{bmatrix}=(D^TD)^{-1}D^TW, \tag{11}$$

and, among them,

$$D=\begin{bmatrix} x^{(r)}(1) & x^{(r)}(1) & 1 & 1\\ 2x^{(r)}(2) & x^{(r)}(2) & 2 & 1\\ \vdots & \vdots & \vdots & \vdots\\ (n-2)x^{(r)}(n-2) & x^{(r)}(n-2) & n-2 & 1\\ (n-1)x^{(r)}(n-1) & x^{(r)}(n-1) & n-1 & 1\end{bmatrix},$$
$$W=\begin{bmatrix} x^{(r)}(2)\\ x^{(r)}(3)\\ \vdots\\ x^{(r)}(n-1)\\ x^{(r)}(n)\end{bmatrix}. \tag{12}$$

The predicted value of FTDGM(1,1)model is as follows:

$$\hat{x}^{(r)}(k+1)=(\beta_1 k+\beta_2)\hat{x}^{(r)}(k)+\beta_3 k+\beta_4 \tag{13}$$

According to the calculation formula of fractional-order accumulation, it is not difficult to calculate the reduced value of the predicted sequence as follows:

$$\hat{x}^{(0)}(k)=\hat{x}^{(r)}(k)-\sum_{i=1}^{k}C_{k-i+r-1}^{k-i}\hat{x}^{(r)}(i), \quad k=1,2,\ldots,n,\ldots \tag{14}$$

3. Disturbance Analysis of TDGM(1,1) Model and FTDGM(1,1) Model

Theorem 8 (see [36, 37]). *Suppose that $A\in C^{m\times n}$, $b\in C^m$, $A^\dagger$ is the generalized inverse matrix of A. When the column vector of A has linear independence, the function $\|Ax-b\|_2=\min$ has a unique solution.*

Theorem 9 (see [36, 37]). *Suppose that $A\in C^{m\times n}$, $b\in C^m$, $A^\dagger$ is the generalized inverse matrix of A. $B=A+E$, $c=b+k\in C^n$. Suppose that the solutions of function $\|Bx-c\|_2=\min$ and $\|Ax-b\|_2=\min$ are $x+h$ and x, respectively. When $\mathrm{rank}(A)=\mathrm{rank}(B)=n$ and $\|A^\dagger\|_2\|E\|_2<1$, we have the following result:*

$$\|h\|_2\le\frac{s_\dagger}{t_\dagger}\left(\frac{\|E\|_2}{\|A\|}\|x\|+\frac{\|k\|}{\|A\|}+\frac{s_\dagger}{t_\dagger}\frac{\|E\|_2}{\|A\|}\frac{\|r_x\|}{\|A\|}\right). \tag{15}$$

Among them,

$$\begin{aligned} s_\dagger&=\left\|A^\dagger\right\|_2\|A\|,\\ t_\dagger&=1-\left\|A^\dagger\right\|_2\|E\|_2,\\ r_x&=b-Ax. \end{aligned} \tag{16}$$

3.1. Disturbance Analysis of TDGM(1,1) Model. The perturbation bounds of TDGM(1,1) model will be analyzed in the following section.

Theorem 10. *The solution of TDGM model can be given as the following function: $\|Y-Cx\|_2=\min$. Suppose that the solution of the TDGM(1,1) model is x, and $\hat{x}^{(0)}(1)=x^{(0)}(1)+\varepsilon$. Among them, ε is the disturbance information.*

Then

$$\|h\|_2\le|\varepsilon|\frac{s_\dagger}{t_\dagger}\left(\frac{\|x\|\sqrt{\sum_{i=1}^{n-1}i^2}}{\|B\|}+\frac{\sqrt{n-1}}{\|B\|}+\frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=1}^{n-1}i^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right). \tag{17}$$

Proof.

$$\begin{aligned}\widehat{C}&=C+\Delta C\\ &=\begin{bmatrix} x^{(1)}(1)+\varepsilon & x^{(1)}(1)+\varepsilon & 1 & 1\\ 2\left(x^{(1)}(2)+\varepsilon\right) & x^{(1)}(2)+\varepsilon & 2 & 1\\ \vdots & \vdots & \vdots & \vdots\\ (n-2)\left(x^{(1)}(n-2)+\varepsilon\right) & x^{(1)}(n-2)+\varepsilon & n-2 & 1\\ (n-1)\left(x^{(1)}(n-1)+\varepsilon\right) & x^{(1)}(n-1)+\varepsilon & n-1 & 1\end{bmatrix}\\ &=\begin{bmatrix} x^{(1)}(1) & x^{(1)}(1) & 1 & 1\\ 2x^{(1)}(2) & x^{(1)}(2) & 2 & 1\\ \vdots & \vdots & \vdots & \vdots\\ (n-2)x^{(1)}(n-2) & x^{(1)}(n-2) & n-2 & 1\\ (n-1)x^{(1)}(n-1) & x^{(1)}(n-1) & n-1 & 1\end{bmatrix}\end{aligned}$$

$$+\begin{bmatrix} \varepsilon & \varepsilon & 0 & 0 \\ 2\varepsilon & \varepsilon & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (n-2)\varepsilon & \varepsilon & 0 & 0 \\ (n-1)\varepsilon & \varepsilon & 0 & 0 \end{bmatrix}$$

$$\widehat{Y} = Y + \Delta Y = \begin{bmatrix} x^{(1)}(2) \\ x^{(1)}(3) \\ \vdots \\ x^{(1)}(n-1) \\ x^{(1)}(n) \end{bmatrix} + \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \\ \varepsilon \end{bmatrix}. \tag{18}$$

Assume that the solution of the new model $\|\widehat{Y} - \widehat{C}x\|_2 = \min$ is $\widehat{x}$, and the disturbance is h. The equation $\|Y - Cx\|_2 = \min$ has a unique solution $x = Y^{\dagger}C$ due to the linear independence of column vectors of C.

Since

$$\Delta Y = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \\ \varepsilon \end{bmatrix},$$

$$\Delta C^T \Delta C = \begin{bmatrix} \sum_{i=1}^{n-1} i^2\varepsilon^2 & \sum_{i=1}^{n-1} i\varepsilon^2 & 0 & 0 \\ \sum_{i=1}^{n-1} i\varepsilon^2 & (n-1)\varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

$$\|\Delta Y\|_2 = |\varepsilon| \sqrt{n-1},$$

$$\|\Delta C\|_2 = \sqrt{\lambda_{\max}(\Delta C^T \Delta C)},$$

we have

$$\|\Delta C\|_2 = \sqrt{\sum_{i=1}^{n-1} i^2\varepsilon^2} = |\varepsilon| \sqrt{\sum_{i=1}^{n-1} i^2} \tag{20}$$

Then, the following result can be obtained according to Theorem 9:

$$\begin{aligned} \|h\|_2 &\le \frac{s_\dagger}{t_\dagger}\left(\frac{\|\Delta C\|_2}{\|B\|}\|x\| + \frac{\|\Delta Y\|}{\|B\|} + \frac{s_\dagger}{t_\dagger}\frac{\|\Delta C\|_2}{\|B\|}\frac{\|r_x\|}{\|B\|}\right) \\ &= |\varepsilon|\frac{s_\dagger}{t_\dagger}\left(\frac{\|x\|\sqrt{\sum_{i=1}^{n-1} i^2}}{\|B\|} + \frac{\sqrt{n-1}}{\|B\|} \right. \\ &\quad \left. + \frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=1}^{n-1} i^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right) = Q\left(x^{(0)}(1)\right). \end{aligned} \tag{21}$$

□

Theorem 11. *Assume that the conditions of Theorem 9 remain unchanged, and $\widehat{x}^{(0)}(t) = x^{(0)}(t) + \varepsilon$.*

Then the perturbation bound of the solution is as follows:

$$\begin{aligned} \|h\|_2 &\le |\varepsilon|\frac{s_\dagger}{t_\dagger}\left(\frac{\|x\|\sqrt{\sum_{i=t}^{n-1} i^2}}{\|B\|} + \frac{\sqrt{n-t+1}}{\|B\|}\right. \\ &\quad \left. + \frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=t}^{n-1} i^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right). \end{aligned} \tag{22}$$

Proof.

$$\begin{aligned} \widehat{C} &= C + \Delta C \\ &= \begin{bmatrix} x^{(1)}(1) & x^{(1)}(1) & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ t\left(x^{(1)}(t)+\varepsilon\right) & x^{(1)}(t)+\varepsilon & t & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (n-2)\left(x^{(1)}(n-2)+\varepsilon\right) & x^{(1)}(n-2)+\varepsilon & n-2 & 1 \\ (n-1)\left(x^{(1)}(n-1)+\varepsilon\right) & x^{(1)}(n-1)+\varepsilon & n-1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} x^{(1)}(1) & x^{(1)}(1) & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ t\left(x^{(1)}(t)\right) & x^{(1)}(t) & t & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (n-2)\left(x^{(1)}(n-2)\right) & x^{(1)}(n-2) & n-2 & 1 \\ (n-1)\left(x^{(1)}(n-1)\right) & x^{(1)}(n-1) & n-1 & 1 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ t\varepsilon & \varepsilon & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (n-2)\varepsilon & \varepsilon & 0 & 0 \\ (n-1)\varepsilon & \varepsilon & 0 & 0 \end{bmatrix} \end{aligned} \tag{23}$$

$$\widehat{Y} = Y + \Delta Y = \begin{bmatrix} x^{(1)}(2) \\ \vdots \\ x^{(1)}(t) \\ \vdots \\ x^{(1)}(n-1) \\ x^{(1)}(n) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \varepsilon \\ \vdots \\ \varepsilon \\ \varepsilon \end{bmatrix}.$$

Assume that the solution of the new model $\|\widehat{Y} - \widehat{C}x\|_2 = \min$ is $\widehat{x}$, and the disturbance is h. The equation

$\|Y - Cx\|_2 = \min$ has a unique solution $x = Y^{\dagger}C$ due to the linear independence of column vectors of C.

Since

$$\Delta Y = \begin{bmatrix} 0 \\ \vdots \\ \varepsilon \\ \vdots \\ \varepsilon \\ \varepsilon \end{bmatrix},$$

$$\Delta C^T \Delta C = \begin{bmatrix} \sum_{i=t}^{n-1} i^2\varepsilon^2 & \sum_{i=t}^{n-1} i\varepsilon^2 & 0 & 0 \\ \sum_{i=t}^{n-1} i\varepsilon^2 & (n-t+1)\varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

$$\|\Delta Y\|_2 = |\varepsilon| \sqrt{n-t+1},$$

$$\|\Delta C\|_2 = \sqrt{\lambda_{\max}(\Delta C^T \Delta C)},$$

we have

$$\|\Delta C\|_2 = \sqrt{\sum_{i=t}^{n-1} i^2\varepsilon^2} = |\varepsilon| \sqrt{\sum_{i=t}^{n-1} i^2} \tag{25}$$

Then, the following result can be obtained according to Theorem 9:

$$\begin{aligned} \|h\|_2 &\le \frac{s_\dagger}{t_\dagger}\left(\frac{\|\Delta C\|_2}{\|B\|}\|x\| + \frac{\|\Delta Y\|}{\|B\|} + \frac{s_\dagger}{t_\dagger}\frac{\|\Delta C\|_2}{\|B\|}\frac{\|r_x\|}{\|B\|}\right) \\ &= |\varepsilon| \frac{s_\dagger}{t_\dagger}\left(\frac{\|x\|\sqrt{\sum_{i=t}^{n-1} i^2}}{\|B\|} + \frac{\sqrt{n-t+1}}{\|B\|} \right. \\ &\quad \left. + \frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=t}^{n-1} i^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right) = Q\left(x^{(1)}(t)\right). \end{aligned} \tag{26}$$

□

3.2. Disturbance Analysis of TDGM(1,1) Model and FTDGM(1,1) Model

Theorem 12. *The solution of TDGM(1,1) model can be given as the following function:* $\|W - Dx\|_2 = \min$. *Suppose that the solution of the TDGM(1,1) model is x, and* $\hat{x}^{(0)}(1) = x^{(0)}(1) + \varepsilon$. *Among them, ε is the disturbance information.*

Then

$$\begin{aligned} \|h\|_2 \le |\varepsilon| \frac{s_\dagger}{t_\dagger}&\left(\frac{\sqrt{\sum_{i=1}^{n-1}\left(iC_{i+r-2}^{i-1}\right)^2}}{\|B\|}\|x\|\right. \\ &\left. + \frac{\sqrt{\sum_{i=1}^{n-1}\left(C_{i+r-2}^{i-1}\right)^2}}{\|B\|} + \frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=1}^{n-1}\left(iC_{i+r-2}^{i-1}\right)^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right). \end{aligned} \tag{27}$$

Proof.

$$\begin{aligned} \widehat{D} &= D + \Delta D \\ &= \begin{bmatrix} x^{(r)}(1) & x^{(r)}(1) & 1 & 1 \\ 2x^{(r)}(2) & x^{(r)}(2) & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (n-2)x^{(r)}(n-2) & x^{(r)}(n-2) & n-2 & 1 \\ (n-1)x^{(r)}(n-1) & x^{(r)}(n-1) & n-1 & 1 \end{bmatrix} \\ &\quad + \begin{bmatrix} \varepsilon & \varepsilon & 0 & 0 \\ 2r\varepsilon & r\varepsilon & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (n-2)C_{n-4+r}^{n-3}\varepsilon & C_{n-4+r}^{n-3}\varepsilon & 0 & 0 \\ (n-1)C_{n-3+r}^{n-2}\varepsilon & C_{n-3+r}^{n-2}\varepsilon & 0 & 0 \end{bmatrix}. \end{aligned} \tag{28}$$

$$\widehat{W} = W + \Delta W = \begin{bmatrix} x^{(r)}(2) \\ x^{(r)}(3) \\ \vdots \\ x^{(r)}(n-1) \\ x^{(r)}(n) \end{bmatrix} + \begin{bmatrix} r\varepsilon \\ C_{1+r}^{2}\varepsilon \\ \vdots \\ C_{n-3+r}^{n-2}\varepsilon \\ C_{n-2+r}^{n-1}\varepsilon \end{bmatrix}.$$

Assume that the solution of the new model $\|\widehat{Y} - \widehat{B}x\|_2 = \min$ is $\hat{x}$, and the disturbance is h. The equation $\|Y - Bx\|_2 = \min$ has a unique solution $x = Y^{\dagger}B$ due to the linear independence of column vectors of B.

Since

$$\Delta W = \begin{bmatrix} r\varepsilon \\ C_{1+r}^{2}\varepsilon \\ \vdots \\ C_{n-3+r}^{n-2}\varepsilon \\ C_{n-2+r}^{n-1}\varepsilon \end{bmatrix},$$

$$\Delta D^T \Delta D = \begin{bmatrix} \sum_{i=1}^{n-1}\left(iC_{i+r-2}^{i-1}\varepsilon\right)^2 & \sum_{i=1}^{n-1} i\left(C_{i+r-2}^{i-1}\varepsilon\right)^2 & 0 & 0 \\ \sum_{i=1}^{n-1} i\left(C_{i+r-2}^{i-1}\varepsilon\right)^2 & \sum_{i=1}^{n-1}\left(C_{i+r-2}^{i-1}\varepsilon\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} \|\Delta W\|_2 &= |\varepsilon| \sqrt{\sum_{i=2}^{n} \left(C_{i+r-2}^{i-1}\right)^2}, \\ \|\Delta D\|_2 &= \sqrt{\lambda_{\max}\left(\Delta D^T \Delta D\right)} \end{aligned} \tag{29}$$

we have

$$\begin{aligned} \|\Delta W\|_2 &= |\varepsilon| \sqrt{\sum_{i=2}^{n} \left(C_{i+r-2}^{i-1}\right)^2}, \\ \|\Delta D\|_2 &= \sqrt{\sum_{i=1}^{n-1} \left(iC_{i+r-2}^{i-1}\varepsilon\right)^2} = |\varepsilon| \sqrt{\sum_{i=1}^{n-1} i^2 \left(C_{i+r-2}^{i-1}\right)^2} \end{aligned} \tag{30}$$

Then, the following result can be obtained according to Theorem 9:

$$\begin{aligned} \|h\|_2 &\le \frac{s_\dagger}{t_\dagger}\left(\frac{\|\Delta D\|_2}{\|B\|}\|x\| + \frac{\|\Delta Y\|}{\|B\|} + \frac{s_\dagger}{t_\dagger}\frac{\|\Delta D\|_2}{\|D\|}\frac{\|r_x\|}{\|B\|}\right) \\ &= |\varepsilon| \frac{s_\dagger}{t_\dagger}\left(\frac{\sqrt{\sum_{i=1}^{n-1}\left(iC_{i+r-2}^{i-1}\right)^2}}{\|B\|}\|x\| + \frac{\sqrt{\sum_{i=2}^{n}\left(i_{i+r-2}^{i-1}\right)^2}}{\|B\|} \right. \\ &\quad \left. + \frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=1}^{n-1}\left(iC_{i+r-2}^{i-1}\right)^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right) = L\left(x^{(0)}(1)\right) \end{aligned} \tag{31}$$

$$\begin{aligned} Q\left(x^{(0)}(1)\right) &= |\varepsilon| \frac{s_\dagger}{t_\dagger}\left(\frac{\|x\|\sqrt{\sum_{i=1}^{n-1} i^2}}{\|B\|} + \frac{\sqrt{n-1}}{\|B\|} \right. \\ &\quad \left. + \frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=1}^{n-1} i^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right) \end{aligned}$$

Since

$$C_{i+r-2}^{i-1} = C_{i-1+r-1}^{i-1} < 1 \tag{32}$$

we have

$$\begin{aligned} \sqrt{\sum_{i=1}^{n-1} i^2 \left(C_{i+r-2}^{i-1}\varepsilon\right)^2} &< \sqrt{\sum_{i=1}^{n-1} i^2}, \\ \sqrt{\sum_{i=2}^{n} \left(C_{i+r-2}^{i-1}\right)^2} &< \sqrt{n-1}. \end{aligned} \tag{33}$$

It is not hard to get $L(x^{(0)}(1)) < Q(x^{(0)}(1))$. □

Theorem 13. *The solution of TDGM(1,1) model can be given as the following function:* $\|W - Dx\|_2 = \min$. *Suppose that the solution of the TDGM(1,1) model is x, and* $\widehat{x}^{(0)}(t) = x^{(0)}(t) + \varepsilon$. *Among them, ε is the disturbance information.*

Then

$$\begin{aligned} \|h\|_2 &\le |\varepsilon| \frac{s_\dagger}{t_\dagger}\left(\frac{\sqrt{\sum_{i=t}^{n-1}\left(iC_{i+r-t-1}^{i-t}\right)^2}}{\|B\|}\|x\| \right. \\ &\quad + \frac{\sqrt{\sum_{i=t}^{n}\left(C_{i-t+r-1}^{i-t}\right)^2}}{\|B\|} \\ &\quad \left. + \frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=t}^{n-1}\left(iC_{i+r-t-1}^{i-t}\right)^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right) = L\left(x^{(0)}(t)\right). \end{aligned} \tag{34}$$

Proof.

$$\begin{aligned} \widehat{D} &= D + \Delta D \\ &= \begin{bmatrix} x^{(r)}(1) & x^{(r)}(1) & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ t\left(x^{(r)}(t)\right) & x^{(r)}(t) & t & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (n-2)\left(x^{(r)}(n-2)\right) & x^{(r)}(n-2) & n-2 & 1 \\ (n-1)\left(x^{(r)}(n-1)\right) & x^{(r)}(n-1) & n-1 & 1 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ t\varepsilon & \varepsilon & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (n-2)C_{n-t+r-3}^{n-t-2}\varepsilon & C_{n-t+r-3}^{n-t-2}\varepsilon & 0 & 0 \\ (n-1)C_{n-t+r-2}^{n-t-1}\varepsilon & C_{n-t+r-2}^{n-t-1}\varepsilon & 0 & 0 \end{bmatrix} \end{aligned} \tag{35}$$

$$\widehat{W} = W + \Delta W = \begin{bmatrix} x^{(r)}(2) \\ \vdots \\ x^{(r)}(t) \\ \vdots \\ x^{(r)}(n-1) \\ x^{(r)}(n) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \varepsilon \\ \vdots \\ C_{n-t+r-3}^{n-t-2}\varepsilon \\ C_{n-t+r-2}^{n-t-1}\varepsilon \end{bmatrix}$$

We have

$$\Delta W = \begin{bmatrix} 0 \\ \vdots \\ \varepsilon \\ \vdots \\ C_{n-t+r-3}^{n-t-2}\varepsilon \\ C_{n-t+r-2}^{n-t-1}\varepsilon \end{bmatrix},$$

$$\Delta D^T \Delta D = \begin{bmatrix} \sum_{i=t}^{n-1}\left(iC_{i-t+r-1}^{i-t}\varepsilon\right)^2 & \sum_{i=t}^{n-1} i\left(C_{i-t+r-1}^{i-t}\varepsilon\right)^2 & 0 & 0 \\ \sum_{i=t}^{n-1} i\left(C_{i-t+r-1}^{i-t}\varepsilon\right)^2 & \sum_{i=t}^{n-1}\left(C_{i-t+r-1}^{i-t}\varepsilon\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (36)$$

$$\|\Delta W\|_2 = |\varepsilon| \sqrt{\sum_{i=t}^{n}\left(C_{i-t+r-1}^{i-t}\right)^2},$$

$$\|\Delta D\|_2 = \sqrt{\lambda_{\max}\left(\Delta D^T \Delta D\right)} = |\varepsilon| \sqrt{\sum_{i=t}^{n-1}\left(iC_{i-t+r-1}^{i-t}\right)^2}$$

Then, the following result can be obtained according to Theorem 9.

$$\begin{aligned} \|h\|_2 &\le \frac{s_\dagger}{t_\dagger}\left(\frac{\|\Delta D\|_2}{\|B\|}\|x\| + \frac{\|\Delta W\|}{\|B\|} + \frac{s_\dagger}{t_\dagger}\frac{\|\Delta D\|_2}{\|D\|}\frac{\|r_x\|}{\|B\|}\right) \\ &= |\varepsilon| \frac{s_\dagger}{t_\dagger}\left(\frac{\sqrt{\sum_{i=t}^{n-1}\left(iC_{i-t+r-1}^{i-t}\right)^2}}{\|B\|}\|x\| \right. \\ &\quad + \frac{\sqrt{\sum_{i=t}^{n}\left(C_{i-t+r-1}^{i-t}\right)^2}}{\|B\|} \\ &\quad \left. + \frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=t}^{n-1}\left(iC_{i-t+r-1}^{i-t}\right)^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right) = L\left(x^{(0)}(t)\right) \end{aligned} \quad (37)$$

We have

$$\begin{aligned} Q\left(x^{(0)}(t)\right) &= |\varepsilon| \frac{s_\dagger}{t_\dagger}\left(\frac{\|x\|\sqrt{\sum_{i=t}^{n-1} i^2}}{\|B\|} + \frac{\sqrt{n-t+1}}{\|B\|}\right. \\ &\quad \left. + \frac{s_\dagger}{t_\dagger}\frac{\sqrt{\sum_{i=t}^{n-1} i^2}}{\|B\|}\frac{\|r_x\|}{\|B\|}\right) \end{aligned}$$

$$\begin{aligned} &\sum_{i=1}^{n-t}\left(iC_{k+r-2}^{i-1}\right)^2 < \sum_{i=1}^{n-t} i^2 < \sum_{i=t}^{n-1} i^2, \\ &\sum_{i=1}^{n-t+1}\left(C_{k+r-2}^{i-1}\right)^2 < n-t+1. \end{aligned} \quad (38)$$

Then $L(x^{(0)}(t)) < Q(x^{(0)}(t))$. □

It can be seen from the conclusion of Theorems 12 and 13 that when $r < 1$, the disturbance bound of FTDGM(1,1) model is smaller than that of TDGM(1,1) model. Generally speaking, compared with the TDGM(1,1) model, the FTDGM(1,1) model has better robustness. The proposed FTDGM(1,1) model can effectively reduce the prediction error caused by system disturbance and improve the prediction accuracy of the grey forecasting model. And the solution of optimal parameter r can be given by genetic algorithms.

4. Numerical Illustrations

In order to test the modeling effect of the model, two real cases will be given in the following section.

Case 1. The GDP of Guangdong province in 2001-2009 is used to build different grey prediction model. The data is shown in Table 1. Different grey prediction models were established based on the given data. And the advantages of the model were tested by comparing the prediction accuracies of different models.

The highest prediction accuracy in [38] is time-varying parameters grey model (TVGM(1,1) model). This Paper established DGM(1,1) model, TVGM(1,1) model, and FTDGM(1,1) model, respectively. The calculation results are shown in Table 2 and Figure 1. The parameters of FTDGM(1,1) model are as follows: $\beta_1 = 0.0185, \beta_2 = 0.5166, \beta_3 = 3530.48, \beta_4 = 7817.70, r = 0.41$. It can be seen from Table 2 that, because of the nature of exponential function, large errors will appear after the second prediction of the DGM(1,1) model. TVGM(1,1) model can effectively increase the prediction accuracy along with adjusting and optimizing the time-varying parameters. However, the prediction error is still relatively large. The proposed FTDGM(1,1) model in this paper shows good robustness, and the prediction error does not increase with the passage of time. It indicated that the FTDGM(1,1) model has better anti-interference and long-term memory, which can be used to predict medium-term goals.

Case 2. As one of the important symbols of China's transportation modernization, highway reflects the degree and level of a country's modernization. Compared with railway and air or water transportation, highway transportation is the more used mode in passenger and cargo transportation. Highway transportation is point-to-point direct, flexible, and convenient. It is very important to accurately predict the length of highway transportation route. The length of Chinese highway transportation route in 2010-2017 is used to build

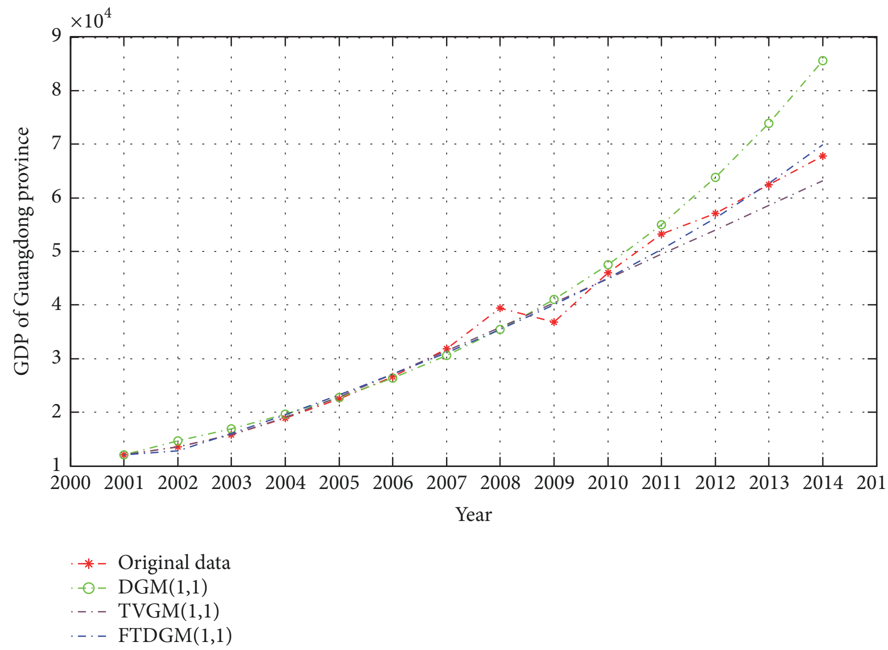

Figure 1: Calculation results of different models of Case 1.

Table 1: Gross domestic product in Guangdong province in 2001-2014 (one hundred million yuan).

Year	2001	2002	2003	2004	2005	2006	2007
GDP	12039.25	13502.42	15844.64	18864.62	22557.37	26587.76	31777.01
Year	2008	2009	2010	2011	2012	2013	2014
GDP	39482.56	36796.71	46013	53210	57068	62475	67792

Source: Guangdong Statistical Yearbook.

Table 2: Predicted values and predicted errors of different grey prediction models.

Year	Original value	DGM(1,1) model		TVGM(1,1) model		FTDGM(1,1) model	
		Predicted value	Relative error (%)	Predicted value	Relative error (%)	Predicted value	Relative error (%)
2010	46013	48326.89	5.03	44920.35	2.37	44989.65	2.22
2011	53210	56295.98	5.80	49471.34	7.03	50379.20	5.32
2012	57068	65579.18	14.91	54030.11	5.32	56267.53	1.40
2013	62475	76393.18	22.28	58593.05	6.21	62739.40	0.42
2014	67792	88990.4	31.27	63158.22	6.83	69893.15	3.10
MAPE			15.86		5.55		2.49

MAPE (mean absolute percentage error) = $100\%(1/n)\sum_{k=1}^{n}|(x(k)-\hat{x}(k))/x(k)|$.

different grey prediction model. The unit of highway mileage is ten thousand kilometers. The calculation results of different models are shown in Table 3 and Figure 2. The parameters of FTDGM(1,1) model are as follows: $\beta_1 = 0.06$, $\beta_2 = -1.46$, $\beta_3 = 17.44$, $\beta_4 = 7.76$, $r = 0.85$.

It can be seen from the calculation results that the TDGM(1,1) model is better than the DGM(1,1) model in describing the internal evolution. However, neither the DGM(1,1) model nor the TDGM(1,1) model can accurately describe the development trend of the system. In particular, the prediction error of the second step is relatively large, indicating that the memory of the integer-order model is insufficient for the FTDGM(1,1) model. The prediction error of the FTDGM(1,1) model is only 0.64%, which shows that the model has strong extrapolation ability and good memory.

5. Concluding Remarks

On the basis of the traditional discrete grey prediction model, this paper proposed the FTDGM(1,1) model. The parametric solution method of the model was given. By using the matrix disturbance theory, the disturbance boundary of the model

Table 3: The calculation results of different grey models.

Year	Original value	DGM(1,1) model		TDGM(1, 1) model		FTDGM(1,1) model, r=0.85	
		Simulation value	Relative error(%)	Simulation value	Relative error (%)	Simulation value	Relative error (%)
2010	7.41	7.41	0	7.41	0	7.41	0
2011	8.49	8.65	1.91	8.49	0.05	8.49	0.00
2012	9.62	9.46	1.69	9.65	0.30	9.64	0.23
2013	10.44	10.34	0.98	10.37	0.71	10.33	1.07
2014	11.19	11.30	0.97	11.28	0.80	11.45	2.29
2015	12.35	12.35	0.00	12.30	0.42	11.97	3.06
MAPE			1.11		0.46		1.33
2016	13.1	13.50	3.05	13.46	2.73	13.22	0.92
2017	13.64	14.76	8.18	14.78	8.36	13.69	0.37
MAPE			5.61		5.54		0.64

Source: China Statistical Yearbook.

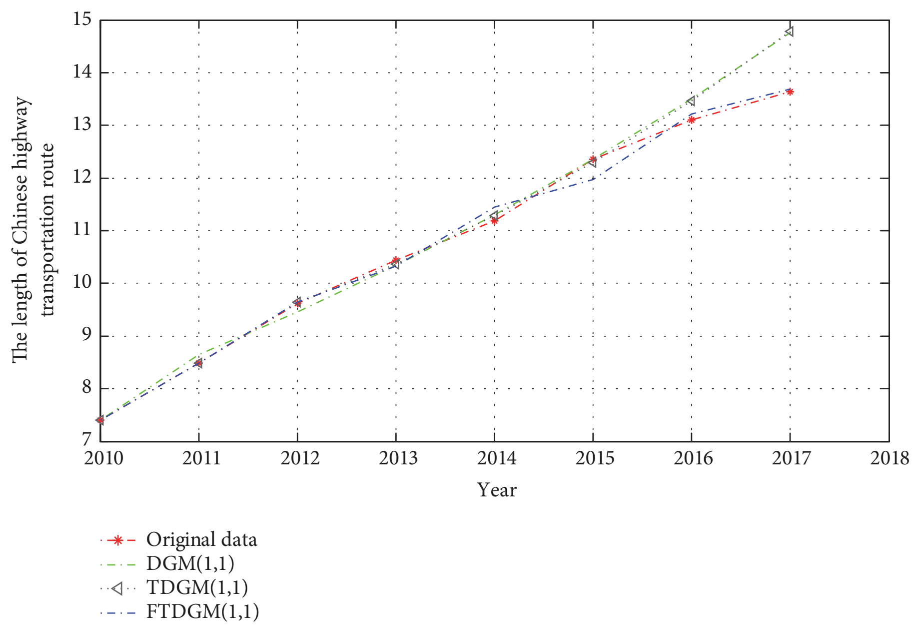

Figure 2: Calculation results of different models of Case 2.

was analyzed, and it was proved that the FTDGM(1,1) model had better robustness than the TDGM(1,1) model.

Two real cases were used to test the effect of the proposed FTDGM(1,1) model. It was found that the prediction accuracy is higher than that of the existing models, which further verifies the superiority of the proposed FTDGM(1,1) model. At the same time, the FTDGM(1,1) model had better memory than other discrete grey prediction models. Research results of this paper further expand the application scope of the grey prediction model. And the reasons for the existence of short-term memory model deserve further discussion.

Data Availability

The data are from China Statistical Yearbook, website of China's national bureau of statistics, http://www.stats.gov.cn/, Guangdong Statistical Yearbook, and Guangdong Statistical Information Network, http://www.gdstats.gov.cn/tjsj/gmjjhs/.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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