Research Article

On the Application of Modified Finite Sine Transform to Structural Mechanics

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The method of modified finite sine transform (MFST) was introduced to solve fourth-order boundary value problems in structural mechanics. The analytical features and computational aspects of MFST were presented, including integral transforms of the derivatives of functions and the term-by-term differentiation of inverse MFST. For the simplicity of demonstrating the solution process, the cantilever beam was used in the discussion. Numerical results for transverse vibration based on the present method were compared and validated with existing results for natural frequencies. Although the present method is not superior to conventional finite sine transform method in one-dimensional structures, it could be significantly efficient for two-dimensional structures with asymmetric boundary conditions, especially including cantilever plates and other plates with only one free edge or two adjacent free edges.

1. Introduction

The modified Fourier series has recently been introduced in [1] as an effective supplement of classical Fourier series for the approximation of nonperiodic functions, whereby in $[-1, 1]$ the $\sin n\pi x$ functions are adjusted by $\sin(n - 1/2)\pi x$, $n \geq 1$. The main immediate advantage of the modified Fourier series is the faster convergence rate [2], which can also be further accelerated [3]. On the other hand, dating back to the late 1940s, the modified finite Fourier sine transform (MFST) was defined by Roettinger [4] to solve the second-order differential equations with mixed/asymmetric boundary conditions (e.g., $y(0) = 0$ and $y'(\pi) = 0$ in $[0, \pi]$), where the integral kernel is of $\sin(n - 1/2)x$, $n = 1, 2, \dots$ in $[0, \pi]$. Such transformation complements the finite Fourier sine transform (FST) which involves the Dirichlet boundary conditions, and the finite Fourier cosine transform (FCT) which is compatible with Neumann boundary conditions of second-order differential equations [5].

In the field of structural mechanics, Fourier sine/cosine series equipped with Stokes's transformation have been extensively applied to higher-order boundary value problems

for beams [6–10], plates [11], and shells [12, 13]. Similarly, FST and FCT have recently been adopted to obtain the exact series solutions of bending of plates with complex boundary conditions [14–18]. However, although MFST was introduced at the same time as FST and FCT [4, 19], very little attention has been paid to the application of MFST to higher-order boundary value problems in structural mechanics. Furthermore, it can be found that the applications of FST and FCT to bending and vibration of plates are mostly reported for symmetry boundary conditions. Plates with asymmetric boundary conditions such as cantilever plates and other plates with only one free edge or two adjacent free edges are rarely addressed by using FST and FCT. One possible reason is that these two methods are not applicable. Instead, MFST can provide efficient solutions.

In the present study, MFST is introduced to solve the fourth-order boundary value problems with asymmetric boundary conditions. At first, MFST was defined and MFST of the function's derivatives were derived. Then, the Stokes's transformation was applied to obtain the term-by-term differentiations of inverse MFST (i.e., modified sine series). A simple example of free vibration of cantilever beam was

analyzed by using MFST to present the solving procedure. At last, more detailed discussions involving MFST and conventional Fourier series as well as FST and FCT are presented. It can be concluded that MFST presents genuine advantages over classical FST and FCT in specific situations of asymmetric boundary conditions.

2. Modified Finite Sine Transform

MFST defined on the interval $(0, l)$ can be given by [4]

$$S^*[f(x)] = \int_0^l f(x) \sin \frac{(m-1/2)\pi x}{l} dx = F_s^*(m), \quad (1)$$

$$m = 1, 2, 3, \dots$$

The inverse transformation is expressed as

$$f(x) = \frac{2}{l} \sum_{m=1}^{\infty} F_s^*(m) \sin \frac{(m-1/2)\pi x}{l} \quad \text{in } (0, l) \quad (2)$$

Similarly, the pair of modified finite cosine transform (MFCT) is defined by

$$C^*[f(x)] = \int_0^l f(x) \cos \frac{(n-1/2)\pi x}{l} dx = F_c^*(n), \quad (3a)$$

$$n = 1, 2, 3, \dots$$

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_c^*(n) \cos \frac{(n-1/2)\pi x}{l} \quad \text{in } (0, l) \quad (3b)$$

It can be found that MFCT and MFST are essentially the same [19] because

$$\begin{aligned} C^*[f(x)] &= \int_0^l f(x) \cos \frac{(n-1/2)\pi x}{l} dx \\ &= -(-1)^{n+1} \int_0^l f(l-x) \sin \frac{(n-1/2)\pi x}{l} dx \quad (4) \\ &= -(-1)^{n+1} S^*[f(l-x)] \end{aligned}$$

Using integration by parts, MFST of derivatives of functions can be obtained, which is presented in Appendix A.

3. Term-By-Term Differentiations of Inverse MFST

The inverse MFST will be a modified Fourier sine series, which can be expressed by

$$f(x) = \sum_{m=1}^{\infty} C_m \sin \gamma_m x, \quad 0 < x < l \quad (5)$$

where $\gamma_m = (m-1/2)\pi/l$ and the Fourier coefficient C_m is given by

$$C_m = \frac{2}{l} \int_0^l f(x) \sin \gamma_m x dx \quad (6)$$

Note that the end value $f(0)$ will be forced to be zero but not for $f(l)$ if they are included in the modified sine series in (5). Therefore, $f(0)$ on the interval $[0, l]$ will be defined separately. In this manner, the function of solution $f(x)$ is defined in two separate regions, one is $x = 0$ and the other is $0 < x \leq l$. Thus, Stokes's transformation [20, pp. 375] should be employed to differentiate these series properly. Care must be taken when finding the correct modified cosine series corresponding to $f'(x)$, which should not include the constant term as that of Fourier cosine series. Thus, the first-order derivative $f'(x)$ can be formulated by

$$f'(x) = \sum_{m=1}^{\infty} D_m \cos \gamma_m x, \quad 0 \leq x < l \quad (7)$$

in which

$$D_m = \frac{2}{l} \int_0^l f'(x) \cos \gamma_m x dx \quad (8)$$

The expressions of first four derivatives of $f(x)$ obtained by using Stokes' transformation can be found in Appendix B.

4. Example: Transverse Vibration of Cantilever Beams

In this section, MFST is applied to obtain the series solution of free vibration of a cantilever beam as shown in Figure 1. Such simple example is presented solely for illustrating the procedure and validating the method.

The governing equation of free vibration of Euler beam can be expressed as [21]

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad (9)$$

where $w(x, t)$ is the displacement function, EI is the flexural rigidity, ρ is the density, and A is the cross-sectional area of the beam. Using the method of separation of variables, the solution can be written as

$$w(x, t) = W(x) \cos \omega t \quad (10)$$

Substituting (10) into (9) yields

$$W_{xxxx} - \beta^4 W = 0 \quad (11)$$

where $W_{xxxx} = d^4 W / dx^4$ and $\beta^4 = \rho A \omega^2 / (EI)$. The boundary conditions for the cantilever beam as shown in Figure 1 can be expressed as

$$\begin{aligned} W &= 0, \\ W_{,x} &= 0 \end{aligned} \quad (12a)$$

$$\text{at } x = 0$$

$$\begin{aligned} W_{,xx} &= 0, \\ W_{,xxx} &= 0 \end{aligned} \quad (12b)$$

$$\text{at } x = l$$

FIGURE 1: A cantilever beam.

FIGURE 2: A rotationally restrained beam with a free edge.

Taking MFST to both sides of (11) and substituting (A.3d) and boundary conditions result in

$$\begin{aligned} \gamma_m^4 W_s^*(m) + (-1)^m \gamma_m^2 W_{,x}(l) + \gamma_m W_{,xx}(0) \\ - \beta^4 W_s^*(m) = 0 \end{aligned} \quad (13)$$

where

$$W_s^*(m) = \int_0^l W(x) \sin \gamma_m x \, dx \quad (14a)$$

$$W(x) = \frac{2}{l} \sum_{m=1}^{\infty} W_s^*(m) \sin \gamma_m x \quad (14b)$$

If $W_{,x}(l)$ and $W_{,xx}(0)$ in (13) can be expressed by $W_s^*(m)$, (13) will become the characteristic equations for the frequency parameter β , which can easily be solved as eigenvalue problems. First, $W_{,x}(l)$ can be expressed by $W_s^*(m)$. Based on (B.3), the second derivative of $W(x)$ can be obtained by

$$\begin{aligned} W_{,xx} = \frac{2}{l} \sum_{m=1}^{\infty} \left\{ \frac{2}{l} [-(-1)^m W_{,x}(l) + \gamma_m W(0)] \right. \\ \left. - \gamma_m^2 \frac{2}{l} W_s^*(m) \right\} \sin \gamma_m x \end{aligned} \quad (15)$$

Substituting the boundary conditions $W(0) = W_{,xx}(l) = 0$ into (15) results in

$$\sum_{m=1}^{\infty} [W_{,x}(l) + (-1)^m \gamma_m^2 W_s^*(m)] = 0 \quad (16)$$

By truncating the infinite series to finite terms of M , $W_{,x}(l)$ can be obtained by

$$W_{,x}(l) = -\frac{1}{M} \sum_{m=1}^M (-1)^m \gamma_m^2 W_s^*(m) \quad (17)$$

However, $W_{,xx}(0)$ can not be directly expressed by $W_s^*(m)$. A slight change needs to be made on the boundary conditions. The cantilever beam in Figure 1 is modified to a rotationally

restrained beam with a free edge as shown in Figure 2. Thus, the boundary condition at $x = 0$ will be

$$W_{,xx}(0) = \frac{R}{EI} W_{,x}(0) \quad (18)$$

The rotational stiffness R can be replaced by the end-fixity factor [15] as follows:

$$\frac{R}{EI} = \frac{1}{L} \frac{3r}{1-r} \quad (19)$$

For cantilever beam, $R = \infty$ and $r = 1$. By using (19), the cantilever condition can be approximated by a value of r close to 1, i.e., $r = 0.9999$. Based on the Stokes' transformation for $W_{,x}(x)$ in (B.2), $W_{,x}(0)$ can be obtained by

$$W_{,x}(0) = \sum_{m=1}^{\infty} \frac{2}{l} \gamma_m W_s^*(m) \quad (20)$$

Substituting (19) and (20) into (18) yields

$$W_{,xx}(0) = \frac{1}{l^2} \frac{6r}{1-r} \sum_{m=1}^{\infty} \gamma_m W_s^*(m) \quad (21)$$

At last, by substituting $W_{,x}(l)$ in (17) and $W_{,xx}(0)$ in (21) into (13), a characteristic equation for the frequency parameter β can be obtained as

$$\begin{aligned} \gamma_m^4 W_s^*(m) - (-1)^m \frac{\gamma_m^2}{M} \sum_{i=1}^M (-1)^i \gamma_i^2 W_s^*(i) \\ + \frac{1}{l^2} \frac{6r}{1-r} \gamma_m \sum_{i=1}^M \gamma_i W_s^*(i) = \beta^4 W_s^*(m) \end{aligned} \quad (22)$$

Substituting $\gamma_m = (m - 1/2)\pi/l$ into (22) results in

$$\begin{aligned} \left(m - \frac{1}{2}\right)^4 W_s^*(m) + \sum_{i=1}^M \left[\frac{2}{\pi^2} \frac{3r}{1-r} \left(m - \frac{1}{2}\right) \left(i - \frac{1}{2}\right) \right. \\ \left. - \frac{(-1)^{m+i}}{M} \left(m - \frac{1}{2}\right)^2 \left(i - \frac{1}{2}\right)^2 \right] W_s^*(i) \\ = \lambda^4 W_s^*(m) \end{aligned} \quad (23)$$

FIGURE 3: Convergence of the fundamental frequency parameter λ with $r = 0.9999$.

where

$$\lambda^4 = \frac{\rho A l^4}{EI \pi^4} \omega^2 \quad (24)$$

Equation (23) can be expressed in the following matrix form:

$$\mathbf{A}\mathbf{W} = \lambda^4 \mathbf{W} \quad (25)$$

where $\mathbf{W} = [W_s^*(1), W_s^*(2), \dots, W_s^*(M)]$ and $\mathbf{A}$ is the corresponding coefficient matrix of size $M \times M$ whose elements can be expressed as

$$A_{ii} = \left(i - \frac{1}{2}\right)^4 + \frac{2}{\pi^2} \frac{3r}{1-r} \left(i - \frac{1}{2}\right)^2 - \frac{1}{M} \left(i - \frac{1}{2}\right)^4 \quad (26a)$$

$$A_{ij} = \frac{2}{\pi^2} \frac{3r}{1-r} \left(i - \frac{1}{2}\right) \left(j - \frac{1}{2}\right) - \frac{(-)^{i+j}}{M} \left(i - \frac{1}{2}\right)^2 \left(j - \frac{1}{2}\right)^2 \quad (26b)$$

in which $i = 1, 2, \dots, M$, $j = 1, 2, \dots, M$.

The natural frequencies can be obtained by solving the eigenvalue problem of (25) and the corresponding mode shape can be determined in terms of modified Fourier sine series by substituting $W_s^*(m)$ of the eigenvector into (14b). The convergence of the first frequency parameter λ is shown in Figure 3 for $r = 0.9999$. From the comparison with the exact value of [21], it can be found that the present result of $M = 500$ is close to the exact value and the convergence rate is not uniform after $M = 1000$ because overflow problems occurred. Thus, in present research, $M = 500$ is used to obtain the numerical results of λ for validation.

As illustrated in Table 1, numerical results for the first six frequency parameters obtained by using MFST for rotationally restrained beams with $r = 0.9999$ are listed and compared with exact values of a cantilever beam in [21]. Excellent agreements between the two methods can be observed in Table 1.

5. Discussion and Remarks

On the other hand, by using Fourier cosine series reported in [9] (i.e., inverse process of FCT), the frequency equation for cantilever beams can be expressed by

$$R_{11}R_{22} - R_{12}R_{21} = 0 \quad (27)$$

TABLE 1: Comparisons of frequency parameters, λ , between rotationally restrained beam with $r_0 = 0.9999$ and cantilever beam.

Mode	λ		
	Ref. [21]	Present	%
1	0.5969	0.5968	-0.01
2	1.4942	1.4951	0.06
3	2.5002	2.5008	0.02
4	3.5000	3.5017	0.05
5	4.5000	4.5013	0.03
6	5.5000	5.5025	0.05

where

$$R_{11} = -\frac{1}{\lambda^4} + \sum_{n=1}^{\infty} \frac{2}{n^4 - \lambda^4} \quad (28a)$$

$$R_{12} = R_{21} = \sum_{n=1}^{\infty} (-1)^n \frac{2n^2}{n^4 - \lambda^4} \quad (28b)$$

$$R_{22} = -1 + \sum_{n=1}^{\infty} \frac{2\lambda^4}{n^4 - \lambda^4} \quad (28c)$$

The frequency parameter λ can be obtained by solving the nonlinear equation of (27). By comparing with (25); one main disadvantage for solving (27) is that its procedure involves root-finding of nonlinear functions, which inevitably needs initial trial values and is often inconvenient and cumbersome. Furthermore, when extending such solution process to dynamic analysis of two-dimensional structures such as previous attempts made by [22, 23] for completely free plates, the resulting nonlinear functions are highly complex and their root-finding for the frequency parameter is extremely difficult as pointed out by [17].

In addition, the solution process of MFST can be easily extended to two-dimensional structural mechanics problems such as plates or shells with one free edge, two adjacent free edges, and three free edges, because all these two-dimensional structures involve so-called asymmetric boundary conditions for two opposite edges (i.e., one edge free and the opposite edge simply supported or clamped). For instance, the transverse vibration of cantilever plates can be solved by combining MFST and FCT for two-dimensional functions. For plates with two adjacent free edges, taking

MFST on both directions of plates can be adopted. If plates have only one free edge, the joint transforms of MFST and FST would be used. Conversely, the conventional FST and FCT might not be efficiently applied to dynamic analysis of these plates with asymmetric boundary conditions. Therefore, although MFST has not shown significant advantages over conventional Fourier sine/cosine series, FST, or FCT in one-dimensional problems, it exhibits genuine advantages in two-dimensional structures with free edges.

6. Conclusions

This paper is devoted to analytical features, computational aspects, and applications of modified finite sine transform (MFST). Special care must be taken when finding derivatives of inverse MFST by using Stokes's transformation since the constant term associated with cosine series should not be included. Although only free vibration of cantilever beam was treated for demonstration, MFST can be extended for two-dimensional structures with asymmetric boundary conditions. Despite no significant advantages over conventional Fourier sine/cosine series, FST, or FCT in one-dimensional problems, MFST exhibits genuine advantages in two-dimensional structures with asymmetric boundary conditions such as cantilever plates and other plates with only one free edge or two adjacent free edges.

Appendix

A. MFST of Derivatives of Functions

Define

$$\gamma_m = \frac{(m-1/2)\pi}{l}, \quad m = 1, 2, 3, \dots \quad (\text{A.1})$$

It leads to

$$F_s^*(m) = \int_0^l f(x) \sin \gamma_m x \, dx \quad (\text{A.2a})$$

$$F_c^*(m) = \int_0^l f(x) \cos \gamma_m x \, dx \quad (\text{A.2b})$$

MFST of first four derivatives of functions can be expressed as

$$S^*[f'(x)] = -\gamma_m F_c^*(m) - (-1)^m f(l) \quad (\text{A.3a})$$

$$S^*[f''(x)] = -\gamma_m^2 F_s^*(m) + \gamma_m f(0) - (-1)^m f'(l) \quad (\text{A.3b})$$

$$S^*[f'''(x)] = \gamma_m^3 F_c^*(m) + (-1)^m \gamma_m^2 f(l) + \gamma_m f'(0) - (-1)^m f''(l) \quad (\text{A.3c})$$

$$S^*[f^{(4)}(x)] = \gamma_m^4 F_s^*(m) - \gamma_m^3 f(0) + (-1)^m \gamma_m^2 f'(l) + \gamma_m f''(0) - (-1)^m f'''(l) \quad (\text{A.3d})$$

In general, MFST of even-order derivatives of functions can be given by

$$S^*[f^{(2r)}(x)] = (-1)^r \gamma_m^{2r} F_s^*(m) + \sum_{s=0}^{r-1} (-1)^s \cdot \gamma_m^{2s} [\gamma_m f^{(2r-2s-2)}(0) - (-1)^m f^{(2r-2s-1)}(l)] \quad (\text{A.4})$$

Similarly, MFST of odd-order derivatives can be obtained by

$$S^*[f^{(2r-1)}(x)] = (-1)^r \gamma_m^{2r-1} F_c^*(m) + (-1)^r (-1)^m \cdot \gamma_m^{2r-2} f(l) + \sum_{s=0}^{r-2} (-1)^s \cdot \gamma_m^{2s} [\gamma_m f^{(2r-2s-3)}(0) - (-1)^m f^{(2r-2s-2)}(l)] \quad (\text{A.5})$$

in which $r = 2, 3, 4, \dots$

B. Term-By-Term Differentiations of Inverse MFST

Using integration by parts, D_m in (8) can be expressed by

$$\begin{aligned} D_m &= \frac{2}{l} \int_0^l f'(x) \cos \gamma_m x \, dx \\ &= \frac{2}{l} \left[f(x) \cos \gamma_m x \Big|_0^l + \gamma_m \int_0^l f(x) \sin \gamma_m x \, dx \right] \\ &= -\frac{2}{l} f(0) + \gamma_m C_m \end{aligned} \quad (\text{B.1})$$

Substituting D_m into (7), the first derivative of $f(x)$ on the interval $[0, l]$ can be given by

$$f'(x) = \sum_{m=1}^{\infty} \left[-\frac{2}{l} f(0) + \gamma_m C_m \right] \cos \gamma_m x, \quad (0 \leq x < l) \text{ and } f'(l) \quad (\text{B.2})$$

By using the similar procedure, the second, third, and fourth derivatives of $f(x)$ on the interval $[0, l]$ can be obtained as follows.

$$\begin{aligned} f''(x) &= \sum_{m=1}^{\infty} \left\{ \frac{2}{l} \left[-(-1)^m f'(l) + \gamma_m f(0) \right] \right. \\ &\quad \left. - \gamma_m^2 C_m \right\} \sin \gamma_m x, \quad (0 < x \leq l) \text{ and } f''(0) \end{aligned} \quad (\text{B.3})$$

$$f'''(x) = \sum_{m=1}^{\infty} \left\{ \frac{2}{l} [-f''(0) - (-1)^m \gamma_m f'(l) + \gamma_m^2 f(0)] - \gamma_m^3 C_m \right\} \cos \gamma_m x, \quad (B.4)$$

$$(0 \leq x < l) \text{ and } f'''(l)$$

$$f''''(x) = \sum_{m=1}^{\infty} \left\{ \frac{2}{l} [-(-1)^m f'''(l) + \gamma_m f''(0) + (-1)^m \gamma_m^2 f'(l) - \gamma_m^3 f(0)] + \gamma_m^4 C_m \right\} \sin \gamma_m x, \quad (B.5)$$

$$(0 < x \leq l) \text{ and } f''''(0)$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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