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Research Article

Uniform Asymptotics for a Delay-Claims Risk Model with Constant Force of Interest and By-Claims Arriving according to a Counting Process

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The insurance risk model involving main claims and by-claims has been traditionally studied under the assumption that every main claim may be accompanied with a by-claim occurring after a period of delay, but in reality each main claim can cause many by-claims arriving according to a counting process. To this end, we construct a new insurance risk model that is also perturbed by diffusion with constant force of interest. In the presence of heavy tails and dependence structures among modelling components, we obtain some asymptotic results for the finite-time ruin probability and the tail probability of discounted aggregate claims, where the results hold uniformly for all times in a finite or infinite interval.

1. Introduction

Consider the classical delay-claims risk model with constant force of interest (see Li [1]), whose wealth process is expressed as

$$U_r(t) = xe^{rt} + c\int_{0-}^{t} e^{r(t-s)}ds - \sum_{i=1}^{N(t)} X_i e^{r(t-\tau_i)} - \sum_{i=1}^{\infty} Y_i e^{r(t-\tau_i-T_i)} \mathbf{1}_{\{\tau_i+T_i\le t\}}, \quad t \ge 0. \tag{1}$$

Here $x \ge 0$ is the initial capital of an insurance company, $r > 0$ is the constant force of interest, $c \ge 0$ is the constant rate of premium, $\mathbf{1}_A$ is the indicator function of an event A, X_i, $i \ge 1$, are the insurer's main claims such that each main claim may cause a by-claim Y_i, $i \ge 1$, occurring after a delay period T_i, $i \ge 1$, and τ_i, $i \ge 1$, are the main-claim arrival times constituting a renewal counting process,

$$N(t) = \sup\{i \ge 1 : \tau_i \le t\}, \quad t \ge 0, \tag{2}$$

with the renewal function $\lambda(t) = EN(t)$ for any $t \ge 0$.

We know that model (1) without interest was introduced by Yuen et al. [2] and then studied by Meng and Wang [3] and Li and Wu [4]. For its various discrete-time counterparts, the reader is referred to Waters and Papatriandafylou [5], Yuen and Guo [6], Xiao and Guo [7], and Wu and Li [8]. All these references above conducted risk analyses with light-tailed and mutually independent claims. But recently, Li [1] considered the model (1) with heavy-tailed claim sizes satisfying a certain dependence structure and gave an asymptotic formula for the ruin probability; Gao et al. [9] extended Li's result to the case in which the model is perturbed by diffusion and the main-claim inter-arrival times satisfy a certain dependence structure; Gao et al. [10] discussed the uniform asymptotics for the finite-time ruin probability in the delayed-claims risk model with diffusion, dependence structures, and constant force of interest. It is worth mentioning that, in the above-referenced delay-claims risk models, the main claims and the by-claims such that every main claim may be accompanied with only one by-claim occurring after a period of delay were involved.

But in insurance practice, for a natural and man-made disaster such as an earthquake or a traffic accident, it is very probable that there are many other insurance claims occurring after the immediate one; namely, each main claim can induce more than one by-claim which practically arrive according to a counting process after the immediate main claim.

In the present paper, we propose a new insurance risk model, which involves the main claims and the by-claims such that each main claim can cause multiple by-claims arriving according to a counting process, and in which all the modelling components satisfy the following assumptions.

Assumption 1. Denote by $\theta_i = \tau_i - \tau_{i-1}$, $i \geq 1$, the interarrival times of main claims with $\tau_0 = 0$. Assume that the main claims $\{X_i, i \geq 1\}$ and their interarrival times $\{\theta_i, i \geq 1\}$ are two sequences of nonnegative and identically distributed, but not necessarily independent, random variables (r.v.s) with common distributions F and K, respectively.

Assumption 2. For any fixed $i \geq 1$, we denote, by τ_{ij}, $j \geq 1$, the duration from the time τ_i when the i-th main claim arrives to the time when the i-th main claim induces its j-th by-claim and by $\theta_{ij} = \tau_{ij} - \tau_{i(j-1)}$, $j \geq 1$, the interarrival times of the by-claims caused by the i-th main claim with $\tau_{i0} = \tau_i$. Then the arrival process of by-claims caused by the i-th main claim is

$$\widehat{N}_i(t) = \sup\left\{j \geq 1 : \tau_{ij} \leq t - \tau_i\right\}, \quad t \geq 0, \text{ for } i \geq 1; \quad (3)$$

namely, the by-claims caused by the i-th main claim arrive according to the above counting process. Assume that the by-claim sizes $\{Y_{ij}, j \geq 1, i \geq 1\}$ and their interarrival times $\{\theta_{ij}, j \geq 1, i \geq 1\}$ are two sequences of nonnegative and identically distributed, but not necessarily independent, r.v.s. with common distributions G and H, respectively.

Assumption 3. The total amount of premiums accumulated up to time $t \geq 0$, denoted by $\{C(t), t \geq 0\}$, is a nonnegative and nondecreasing stochastic process with $C(0) = 0$ and $C(t) < \infty$ almost surely (a.s.) for all $0 < t < \infty$.

Assumption 4. The diffusion process $\{B(t), t \geq 0\}$, as a perturbed term, is a standard Brownian motion with volatility parameter $\sigma \geq 0$ and independent of the other sources of randomness. In practice, the diffusion-perturbed term can be interpreted as an additional uncertainty of the aggregate claims or the premium income of the insurance company.

Assumption 5. Assume that $\{X_i, Y_{ij}, j \geq 1, i \geq 1\}$ and $\{\theta_i, \theta_{ij}, j \geq 1, i \geq 1\}$ are mutually independent, and so are $\{\theta_i, i \geq 1\}$ and $\{\theta_{ij}, j \geq 1, i \geq 1\}$.

The delay-claims risk model described by (1) is then extended to a new insurance risk model to match the insurance practice, and the insurer's total surplus up to time $t \geq 0$ satisfies the following equation:

$$\begin{aligned} U_r(t) &= xe^{rt} + \int_{0-}^{t} e^{r(t-s)} dC(s) - \sum_{i=1}^{N(t)} X_i e^{r(t-\tau_i)} \\ &\quad - \sum_{i=1}^{N(t)} \sum_{j=1}^{\widehat{N}_i(t)} Y_{ij} e^{r(t-\tau_i-\tau_{ij})} + \sigma \int_{0-}^{t} e^{r(t-s)} dB(s), \end{aligned} \quad (4)$$

and the discounted aggregate claims up to time $t \geq 0$ can be written as

$$\begin{aligned} D_r(t) &= \sum_{i=1}^{N(t)} X_i e^{-r\tau_i} + \sum_{i=1}^{N(t)} \sum_{j=1}^{\widehat{N}_i(t)} Y_{ij} e^{-r(\tau_i+\tau_{ij})} \\ &= \sum_{i=1}^{N(t)} \left(X_i e^{-r\tau_i} + \sum_{j=1}^{\infty} Y_{ij} e^{-r(\tau_i+\tau_{ij})} \mathbf{1}_{\{\tau_{ij} \leq t-\tau_i\}} \right). \end{aligned} \quad (5)$$

By Assumption 3, it is easy to see that, for any $0 < t < \infty$,

$$0 \leq \widetilde{C}(t) = \int_{0-}^{t} e^{-rs} dC(s) < \infty \quad \text{a.s.}, \quad (6)$$

where $\widetilde{C}(t)$ is the discounted value of premiums accumulated up to time $t > 0$.

We define, as usual, the ruin probability within a finite time $t > 0$ by

$$\begin{aligned} &\psi_r(x,t) \\ &\quad = P\left(U_r(s) < 0 \text{ for some } 0 \leq s \leq t \mid U_r(0) = x\right), \end{aligned} \quad (7)$$

and the infinite-time ruin probability by

$$\begin{aligned} &\psi_r(x,\infty) \\ &\quad = P\left(U_r(t) < 0 \text{ for some } t \geq 0 \mid U_r(0) = x\right). \end{aligned} \quad (8)$$

All limit relationships in the paper are for $x \longrightarrow \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) = O(1)b(x)$ if $\limsup a(x)/b(x) \equiv C < \infty$, write $a(x) = o(1)b(x)$ if $C = 0$, write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $C \leq 1$, write $a(x) \sim b(x)$ if $a(x) \lesssim b(x)$ and $b(x) \lesssim a(x)$, and write $a(x) \asymp b(x)$ if $a(x) = O(1)b(x)$ and $b(x) = O(1)a(x)$.

We notice that, in insurance industry, practitioners usually choose r.v.s. with heavy tails to model large claims. For a distribution V and any $y > 0$, we set

$$J_V^+ = -\lim_{y \longrightarrow \infty} \frac{\log \overline{V}_*(y)}{\log y} = \inf\left\{-\frac{\log \overline{V}_*(y)}{\log y} : y > 1\right\}, \quad (9)$$

and

$$\begin{aligned} J_V^- &= -\lim_{y \longrightarrow \infty} \frac{\log \overline{V}^*(y)}{\log y} \\ &= \sup\left\{-\frac{\log \overline{V}^*(y)}{\log y} : y > 1\right\}, \end{aligned} \quad (10)$$

where $\overline{V}_*(y) = \liminf \overline{V}(xy)/\overline{V}(x)$ and $\overline{V}^*(y) = \limsup \overline{V}(xy)/\overline{V}(x)$. By definition, we say that a distribution V on $[0,\infty)$ belongs to the consistently varying-tailed class, denoted by $V \in \mathscr{C}$, if

$$\begin{aligned} &\lim_{y \searrow 1} \overline{V}_*(y) = 1 \\ &\text{or } \lim_{y \nearrow 1} \overline{V}^*(y) = 1; \end{aligned} \quad (11)$$

belongs to the dominatedly varying-tailed class, denoted by $V \in \mathscr{D}$, if, for all $y > 0$,

$$\overline{V}_*(y) > 0 \quad \text{or } \overline{V}^*(y) < \infty; \tag{12}$$

belongs to the long-tailed class, denoted by $V \in \mathscr{L}$, if, for all $y > 0$,

$$\overline{V}(x+y) \sim \overline{V}(x); \tag{13}$$

belongs to the subexponential class, denoted by $F \in \mathscr{S}$, if

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x), \tag{14}$$

where F^{*2} is the 2-fold convolution of F.

Remark 6. Remark that if $V \in \mathscr{C}$, then $J_V^- > 0$. In fact, for any fixed $x > 0$, $\overline{V}(xy)/\overline{V}(x)$ is monotonically decreasing function of y. So, for $y > z > 0$, $\overline{V}^*(y) \le \overline{V}^*(z)$, and then, by $V \in \mathscr{C}$, $\overline{V}^*(y) \le \lim_{z\nearrow 1}\overline{V}^*(z) = 1$. Because $\limsup_{x\longrightarrow\infty}\lim_{y\longrightarrow\infty}\overline{V}(xy)/\overline{V}(x) = 0$, there exists a sufficiently large number $y_0 > 1$ such that $\overline{V}^*(y) < 1$ for all $y > y_0$, and further $\log \overline{V}^*(y)/\log y < 0$, $y > y_0 > 1$. Therefore by the definition of J_V^-, we know that $J_V^- \ge \sup\{-\log\overline{V}^*(y)/\log y : y > y_0\} > 0$.

It is well-known that

$$\mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{S} \subset \mathscr{L}. \tag{15}$$

For more details of heavy-tailed distributions and their applications to insurance and finance, we refer the readers to Bingham et al. [11], Embrechts et al. [12], and McNeil et al. [13].

In recent years, a study trend of risk theory is to introduce various dependence structures to risk models, among which the widely lower orthant dependence structure was proposed by Wang et al. [14]. Say that r.v.s. $\{\xi_i,\ i \ge 1\}$ are widely lower orthant dependent (WLOD), if there exist a sequence of real numbers $\{g_L(n),\ n \ge 1\}$ such that, for each $n \ge 1$ and all $x_i \in (-\infty,\infty)$, $1 \le i \le n$,

$$P\left(\bigcap_{i=1}^{n}\{\xi_i \le x_i\}\right) \le g_L(n)\prod_{i=1}^{n}P(\xi_i \le x_i). \tag{16}$$

Clearly, if $\{\xi_i,\ i \ge 1\}$ are WLOD, then, for each $n \ge 1$ and any $s > 0$,

$$E\exp\left\{-s\sum_{i=1}^{n}\xi_i\right\} \le g_L(n)\prod_{i=1}^{n}E\exp\{-s\xi_i\}. \tag{17}$$

Chen and Yuen [15] introduced a more general dependence structure, namely, pairwise quasi-asymptotic independence structure. Say that r.v.s. $\{\xi_i,\ i \ge 1\}$ are pairwise quasi-asymptotically independent (PQAI), if, for any $1 \le i \ne j < \infty$,

$$\lim_{x\longrightarrow\infty}P\left(|\xi_i| \wedge \xi_j > x \mid \xi_i \vee \xi_j > x\right) = 0, \tag{18}$$

where $x \wedge y = \min\{x,y\}$ and $x \vee y = \max\{x,y\}$. Adopting the term of Li [1], r.v.s. $\{\xi_i,\ i \ge 1\}$ are said to be pairwise strongly quasi-asymptotically independent (PSQAI), if, for any $1 \le i \ne j < \infty$,

$$\lim_{x_i\wedge x_j\longrightarrow\infty}P\left(|\xi_i| > x_i \mid \xi_j > x_j\right) = 0. \tag{19}$$

Clearly, if $\{\xi_i,\ i \ge 1\}$ are PSQAI, then they are PQAI. For further study on the above dependence structures and their analogues, we refer to Geluk and Tang [16], Gao and Liu [17], Li [1], Gao et al. [18], and references therein.

Under the above frameworks, in this paper we consider the new insurance risk model (4) with a feature that each main claim can cause multiple by-claims arriving according to a counting process after a period of delay and then investigate the uniform asymptotics for the finite-time ruin probability and the tail probability of the discounted aggregate claims. As generally acknowledged, the finite-time ruin probability is more practical but much harder than the infinite-time ruin probability, and the uniformity considered in the paper often significantly merits the theoretical value of these asymptotic formulas obtained.

The remaining part of this paper is organized as follows: in Section 2 we will state the main results that are proven in Section 4 after giving some lemmas in Section 3.

2. Main Results

In the section, we state the main results of this paper. For notational convenience, we denote by $\widehat{\lambda}(t)$ the generic renewal function of the by-claim arrival process $\{\widehat{N}_i(t),\ t \ge 0\}$, $i \ge 1$, which have the same distribution because of the assumption that $\{\theta_{ij},\ j \ge 1,\ i \ge 1\}$ are identically distributed in Assumption 2 and the fact that $\{\widehat{N}_i(t),\ t \ge 0\}$ depends only on its inter-arrival times $\{\theta_{ij},\ j \ge 1\}$, $i \ge 1$. Besides, for any $t \ge 0$,

$$\widehat{\lambda}(t) = E\widehat{N}_i(t) = E\widehat{N}_i(\tau_i + t) = \sum_{j=1}^{\infty}P\left(\tau_{ij} \le t\right), \quad i \ge 1, \tag{20}$$

where the second step is due to the result that the process $\{\widehat{N}_i(t),\ t \ge 0\}$ after time τ_i has the same distribution as the whole process; i.e., $\widehat{N}_i(\tau_i + t) \overset{d}{=} \widehat{N}_i(t)$, $i \ge 1$. Define $\Lambda = \{t : \lambda(t) > 0\}$ and $\widehat{\Lambda} = \{t : \widehat{\lambda}(t) > 0\}$. Clearly, $\widehat{\Lambda} \subset \Lambda$. With $\underline{t} = \inf\{t : \lambda(t) > 0\} = \inf\{t : P(\tau_1 \le t) > 0\}$, then

$$\Lambda = \begin{cases} [\underline{t},\infty], & \text{if } P(\tau_1 = \underline{t}) > 0, \\ (\underline{t},\infty], & \text{if } P(\tau_1 = \underline{t}) = 0. \end{cases} \tag{21}$$

Firstly, we are concerned with the local uniformity of finite-time ruin probability of the risk model (4) with PSQAI sizes of main claims and by-claims arriving according to a sequence of arbitrary counting processes.

Theorem 7. *Consider the risk model introduced by (4) with $r \geq 0$, where the sizes of main claims and by-claims, $\{X_i, Y_{ij},\ i \geq 1,\ j \geq 1\}$ are a sequence of PSQAI r.v.s. with common distributions $F \in \mathscr{L} \cap \mathscr{D}$ and $G \in \mathscr{L} \cap \mathscr{D}$, respectively, and the arrival processes of main claims and by-claims, $\{N(t),\ t \geq 0\}$ and $\{\widehat{N}_i(t),\ t \geq 0\}$, $i \geq 1$, are a sequence of arbitrary counting processes such that for any fixed $t > 0$, $EN^{p+1}(t) < \infty$, and $E\widehat{N}_i^{p+1}(t) < \infty$, $i \geq 1$, for some $p > J_F^+ \vee J_G^+$. Then, for any fixed $t_0 \in \Lambda \cap (0, \infty)$ and $T \in (t_0, \infty)$, it holds uniformly for all $t \in [t_0, T]$ that*

$$\psi_r(x,t) \sim \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) d\lambda(s) + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s), \tag{22}$$

if the premium process $\{C(t),\ t \geq 0\}$ is independent of the other sources of randomness.

Remark 8. The uniformity of two bivariate functions $a(x,t) \sim b(x,t)$ for all $t \in \Delta \neq \emptyset$ means that

$$\lim_{x\longrightarrow\infty} \sup_{t\in\Delta} \left| \frac{a(x,t)}{b(x,t)} - 1 \right| = 0. \tag{23}$$

In comparison to the first theorem, the second one discusses the case when the sizes of main claims and by-claims are PQAI r.v.s. and the premium process $\{C(t),\ t \geq 0\}$ is not necessarily independent of the other sources of randomness.

Theorem 9. *Consider the risk model introduced by (4) with $r \geq 0$, where the sizes of main claims and by-claims, $\{X_i, Y_{ij},\ i \geq 1,\ j \geq 1\}$, are a sequence of PQAI r.v.s. with common distributions $F \in \mathscr{C}$ and $G \in \mathscr{C}$, respectively, and the arrival processes of main claims and by-claims, $\{N(t),\ t \geq 0\}$ and $\{\widehat{N}_i(t),\ t \geq 0\}$, $i \geq 1$, satisfy the same conditions as those in Theorem 7. Then relation (22) holds uniformly for all $t \in [t_0, T]$, if, for any fixed $t > 0$,*

$$P\left(\widetilde{C}(t) > 0\right) = o(1)\left(\overline{F}(x) + \overline{G}(x)\right). \tag{24}$$

Remark 10. Tang [19] pointed out that condition (24), which does not require the independence between the premium process and the other sources of randomness, allows for a more realistic case when the premium income varies as a deterministic or stochastic function of the insurer's current reserve. The same concerns condition (33) in Theorem 16 below. Indeed, if, for any fixed $t > 0$, $E\widetilde{C}^p(t) < \infty$ for some $p > J_F^+ \wedge J_G^+$, then condition (24) is satisfied naturally. By Markov's inequality,

$$P\left(\widetilde{C}(t) > x\right) \leq x^{-p} E\widetilde{C}^p(t), \tag{25}$$

which, along with Lemma 22(2), leads to (24).

Remark 11. A comparison of the conditions in Theorems 7 and 9 tells us that the dependence structure among the sizes of main claims and by-claims imposed in Theorem 7 is stronger than that in Theorem 9, but the distribution class of main claims and by-claims in Theorem 7 are larger than that in Theorem 9.

According to Theorems 7 and 9, we now put forward a corollary that gives a uniformly asymptotic formula for the finite-time ruin probability of the risk model (4) with a special case when $r = 0$.

Corollary 12. *Consider the risk model introduced by (4) with $r = 0$; if the conditions of Theorem 7 or Theorem 9 are valid, then it holds uniformly for all $t \in [t_0, T]$ that*

$$\psi_0(x,t) \sim \lambda(t)\overline{F}(x) + \widehat{\lambda} * \lambda(t) \cdot \overline{G}(x) \sim \int_{x}^{x+\lambda(t)} \overline{F}(y)\, dy + \int_{x}^{x+\widehat{\lambda}*\lambda(t)} \overline{G}(y)\, dy, \tag{26}$$

where $\widehat{\lambda} * \lambda(\cdot)$ is the convolution of the renewal functions $\widehat{\lambda}(\cdot)$ and $\lambda(\cdot)$.

Remark 13. Formula (26) contains main-claim distribution F and by-claim distribution G. Without loss of generality, we now only consider the first relation of (26) to give more explanations on the interrelationship between F and G as follows: let $\overline{G}(x) \sim c\overline{F}(x)$ for some $c \in [0, \infty]$, and one of the conditions below holds.

(1) If $c = 0$, then $\psi_0(x,t) \sim \lambda(t)\overline{F}(x)$, which means that if F has tail heavier than that of G; then the main claims dominate the asymptotic analysis of the finite-time ruin probability.

(2) If $c \in (0, \infty)$, then $\psi_0(x,t) \sim [\lambda(t) + c\widehat{\lambda} * \lambda(t)]\overline{F}(x)$; particularly, if $c = 1$, then $\psi_0(x,t) \sim [\lambda(t) + \widehat{\lambda} * \lambda(t)]\overline{F}(x)$, where the second summand of formula (26) inserts only some addition to the first summand.

(3) If $c = \infty$, then $\psi_0(x,t) \sim \widehat{\lambda} * \lambda(t)\overline{G}(x)$, which is the opposite of the first case.

Remark 14. Considering the conditions on the counting processes $\{N(t),\ t \geq 0\}$ and $\{\widehat{N}_i(t),\ t \geq 0\}$, $i \geq 1$, in Theorems 7 and 9, we observe that the renewal functions $\widehat{\lambda}(t)$ and $\lambda(t)$ are monotonically nondecreasing and finite for any fixed $t > 0$. Then, it follows that, for all $t \in [t_0, T]$,

$$\widehat{\lambda} * \lambda(t) = \int_{0-}^{t} \widehat{\lambda}(t-s)\, d\lambda(s) \leq \widehat{\lambda}(T) \cdot \lambda(T) < \infty, \tag{27}$$

which ensures that the second and third expressions of (26) are well-defined. Especially, for the case when the interarrival times of main claims and by-claims, $\{\theta_i,\ i \geq 1\}$ and $\{\theta_{ij},\ j \geq 1,\ i \geq 1\}$ are two sequences of WLOD r.v.s. such that inequality (32) holds for every $\epsilon > 0$, then relation (27) holds for all $t \in [t_0, T]$. In fact, by Markov's inequality and (17), we get that, for all $t \in [t_0, T]$,

$$\begin{aligned}\widehat{\lambda} * \lambda(t) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{0-}^{t} P\left(\tau_{1j} \le t-s\right) dP\left(\tau_i \le s\right)\\ &\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\left(\tau_{1j} \le T\right)\cdot P\left(\tau_i \le T\right)\\ &\le e^{2T}\sum_{i=1}^{\infty} Ee^{-\tau_i}\cdot \sum_{j=1}^{\infty} Ee^{-\tau_{1j}}\\ &\le e^{2T}\sum_{i=1}^{\infty} g_L(i)\left(Ee^{-\tau_1}\right)^i \cdot \sum_{j=1}^{\infty} g_L(j)\left(Ee^{-\tau_{11}}\right)^j,\end{aligned} \tag{28}$$

Applying (32) and taking $\epsilon = -\log(Ee^{-\tau_1}) - c$ for some constant $c > 0$, we see that for some integer $i_0 > 0$ such that for all $i > i_0$, $g_L(i) \le e^{-ci}(Ee^{-\tau_1})^{-i}$, which implies that

$$\sum_{i=1}^{\infty} g_L(i)\left(Ee^{-\tau_1}\right)^i \le \sum_{i=1}^{i_0} g_L(i)\left(Ee^{-\tau_1}\right)^i + \sum_{i=i_0+1}^{\infty} e^{-ci} < \infty. \tag{29}$$

Similarly, it also holds that

$$\sum_{j=1}^{\infty} g_L(j)\left(Ee^{-\tau_{11}}\right)^j < \infty. \tag{30}$$

Hence, we obtain relation (27) for all $t \in [t_0, T]$.

Remark 15. In Theorems 7 and 9 and Corollary 12, we cannot get the uniformity for all $t \in (\underline{t}, t_0)$. But in practice, the case for $t \in (\underline{t}, t_0)$ is hardly significant since t_0 can be arbitrarily close to $\underline{t}$.

In the third theorem below, we extend the set over which relation (22) holds uniformly to the maximal set Λ, where the total discounted amount of premiums is assumed to be finite; namely,

$$0 \le \widetilde{C} = \int_{0-}^{\infty} e^{-rs} dC(s) < \infty \quad \text{a.s.} \tag{31}$$

Theorem 16. *Under the conditions of Theorem 9 with $r > 0$, we further assume that the interarrival times of main claims and by-claims, $\{\theta_i,\ i \ge 1\}$ and $\{\theta_{ij},\ j \ge 1,\ i \ge 1\}$, are two sequences of WLOD r.v.s. such that*

$$\lim_{n\to\infty} g_L(n)\, e^{-\epsilon n} = 0 \tag{32}$$

holds for every $\epsilon > 0$, depending on F, G, K, and H. Then relation (22) holds uniformly for all $t \in \Lambda$, if one of the following conditions holds.

(1) The premium process $\{C(t),\ t \ge 0\}$ is independent of the other sources of randomness.

(2) The total discounted amount of premiums satisfies

$$P\left(\widetilde{C} > x\right) = o(1)\left(\overline{F}(x) + \overline{G}(x)\right). \tag{33}$$

Remark 17. Remark that relation (32) as a condition in Theorem 16 must hold for every $\epsilon > 0$. See the constants $\epsilon = -\log(Ee^{-rp\tau_1}) - c$, $\epsilon = -\log(Ee^{-r\widehat{p}\tau_1}) - c$ and their analogous in Section 4, where r can be close to 0, p and $\widehat{p}$ depend on the distributions F and G, and τ_1 and τ_{11} are the arrival times of main claims and by-claims, respectively. Thus, relation (32) must hold for every $\epsilon > 0$ in order for Theorem 16 to be valid for such arbitrarily given modelling components, and also the constant ϵ in (32) depends on the distributions F, G, K, and H.

Remark 18. By Lemma 3.3 of Gao et al. [18], the conditions on $\{\theta_i,\ i \ge 1\}$ and $\{\theta_{ij},\ j \ge 1,\ i \ge 1\}$ in Theorem 16 indicate that, for any fixed $t > 0$, $E(N(t))^p < \infty$ and $E(\widehat{N}_i(t))^p < \infty$, $i \ge 1$, for any $p > 0$, which asserts that the conditions on $\{N(t),\ t \ge 0\}$ and $\{\widehat{N}_i(t),\ t \ge 0\}$, $i \ge 1$, in Theorems 7 and 9 are more relaxed than those in Theorem 16. In the paper, we consider in Theorems 7 and 9 that the arrival processes of main claims and by-claims are a sequence of arbitrary counting processes, which means that neither independence, nor a special dependence structure, is required among the interarrival times of main claims and by-claims.

By the uniformity of the finite-time ruin probability for all $t \in \Lambda$ in Theorem 16, we derive the corresponding result on the infinite-time ruin probability (8).

Corollary 19. *Under the conditions of Theorem 16, we have*

$$\begin{aligned}\psi_r(x,\infty) \sim{}& \int_{0-}^{\infty} \overline{F}\left(xe^{rt}\right) d\lambda(t)\\ &+ \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{G}\left(xe^{r(t+s)}\right) d\widehat{\lambda}(s)\, d\lambda(t).\end{aligned} \tag{34}$$

Finally, we consider the uniform asymptotics for the tail probability of the discounted aggregate claims, which has the same uniform asymptotics as the finite-time ruin probability in the same time-interval and then can play an important role to prove the first three theorems.

Theorem 20. *Consider the discounted aggregate claims described by (5), if the conditions of Theorem 7 or Theorem 9 are valid; then it holds uniformly for all $t \in [t_0, T]$ that*

$$\begin{aligned}&P\left(D_r(t) > x\right)\\ &\quad\sim \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) d\lambda(s)\\ &\quad+ \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s).\end{aligned} \tag{35}$$

Furthermore, if the conditions of Theorem 16 are valid, relation (35) still holds uniformly for all $t \in \Lambda$.

Remark 21. Clearly, if, for each $i \ge 1$, $\widehat{N}_i(t) \equiv 1$, $t \ge 0$, namely, every main claim may be accompanied with only one by-claim occurring after a delay period T_i, $i \ge 1$, identically distributed by H, then the main results obtained in this paper coincide with those of Gao et al. [10].

3. Some Lemmas

In the section, we prepare some lemmas to prove the main results. The first lemma is due to Proposition 2.2.1 of Bingham et al. [11] and Lemma 3.5 of Tang and Tsitsiashvili [20].

Lemma 22. *If $V \in \mathscr{D}$ with $J_V^- > 0$, then we have the following.*

(1) For any $0 < \widehat{p} < J_V^- \le J_V^+ < p < \infty$, there exist some $C > 1$ and $D > 0$ such that

$$C^{-1}\left(\frac{x}{y}\right)^{\widehat{p}} \le \frac{\overline{V}(y)}{\overline{V}(x)} \le C\left(\frac{x}{y}\right)^{p} \quad \forall x \ge y \ge D. \tag{36}$$

(2) For any $p > J_V^+$, it holds that $x^{-p} = o(1)\overline{V}(x)$.

The second lemma is from Theorem 3.3(iv) of Cline and Samorodnitsky [21] and Lemma 2.5 of Wang et al. [22].

Lemma 23. *Let ξ be a r.v. with distribution V and η be a nonnegative r.v., independent of ξ and satisfying $E\eta^p < \infty$ for some $p > J_V^+$.*

(1) If $V \in \mathscr{D}$, then $P(\xi\eta > x) \asymp \overline{V}(x)$.

(2) If $V \in \mathscr{C}$, then the distribution of $\xi\eta$ still belongs to the class $\mathscr{C}$.

The third lemma is a restatement of Theorem 2.1 of Li [1]. Also, see Lemma 3.3 of Gao and Liu [17] and Lemma 3.2 of Gao et al. [18]. It should be mentioned that the asymptotic formula in this lemma was firstly developed by Tang and Tsitsiashvili [23].

Lemma 24. *If $\{\xi_i,\ 1 \le i \le n\}$ are n PSQAI (or PQAI) and real-valued r.v.s. with distributions $V_i \in \mathscr{L} \cap \mathscr{D}$ (or $V_i \in \mathscr{C}$), $1 \le i \le n$, respectively, then, for any fixed $0 < a \le b < \infty$,*

$$P\left(\sum_{i=1}^{n} c_i\xi_i > x\right) \sim \sum_{i=1}^{n}\overline{V}_i\left(\frac{x}{c_i}\right) \tag{37}$$

holds uniformly for all $(c_1, c_2, \ldots, c_n) \in [a, b]^n$.

In the following, we present a lemma which plays an important role to prove the main results and is also of its own value.

Lemma 25. *Let V be a distribution in the class $\mathscr{D}$. If ξ is a r.v. such that $P(\xi > x) = o(1)\overline{V}(x)$ and η is a nonnegative r.v., independent of ξ and satisfying $E\eta^p < \infty$ for some $p > J_V^+$, then*

$$P(\xi\eta > x) = o(1)\overline{V}(x). \tag{38}$$

Proof. By $P(\xi > x) = o(1)\overline{V}(x)$, it follows that, for any fixed $\varepsilon > 0$, there exists $x_0 > 0$ such that, for all large $x \ge x_0$,

$$P(\xi > x) \le \varepsilon\overline{V}(x). \tag{39}$$

Hence, for $0 < \delta < 1$ such that $(1-\delta)p > J_V^+$, we derive by Markov's inequality and Lemma 22(2) that, for all large $x \ge x_0$,

$$\begin{aligned} P(\xi\eta > x) &= \left(\int_0^{x^{1-\delta}} + \int_{x^{1-\delta}}^{\infty}\right) P\left(\xi > \frac{x}{t}\right) dP(\eta \le t) \\ &\le \varepsilon\int_0^{x^{1-\delta}} \overline{V}\left(\frac{x}{t}\right) dP(\eta \le t) \\ &\quad + P\left(\eta > x^{1-\delta}\right) \\ &\le \varepsilon\int_0^{\infty} \overline{V}\left(\frac{x}{t}\right) dP(\eta \le t) + x^{-(1-\delta)p}E\eta^p \\ &= \varepsilon P(\zeta\eta > x) + o(1)\overline{V}(x), \end{aligned} \tag{40}$$

where ζ is the r.v. distributed by V. So by Lemma 23(1) and the arbitrariness of $\varepsilon > 0$, the lemma holds immediately. □

Finally, we give the asymptotic upper-bound of the total discounted amount of aggregate claims as follows.

Lemma 26. *Under the conditions of Theorem 16, it holds that*

$$\begin{aligned} &P(D_r(\infty) > x) \\ &\quad\lesssim \int_{0-}^{\infty} \overline{F}(xe^{rs})\, d\lambda(s) \\ &\qquad + \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s). \end{aligned} \tag{41}$$

Proof. Following the proof of Lemma 3.5 of Gao and Liu [17], we show that there exists a positive integer n_1 such that

$$P\left(\sum_{i=n_1+1}^{\infty} X_i e^{-r\tau_i} > x\right) = o(1)\overline{F}(x). \tag{42}$$

For any fixed positive integer m, we take $Y_i = \sum_{j=1}^{m} Y_{ij}e^{-r\tau_{ij}}$, $i \ge 1$, identically distributed by Q. Theorem 3.2 of Chen and Yuen [15] gives that

$$\overline{Q}(x) \sim \sum_{j=1}^{m} P\left(Y_{1j}e^{-r\tau_{1j}} > x\right), \tag{43}$$

which, along with $G \in \mathscr{C}$ and Lemma 23(2), leads to $Q \in \mathscr{C}$. By Lemma 3.1 of Chen and Yuen [15] (or Theorem 2.2 of Li [1]) and Theorem 2.5 of Li [1], we obtain that $\{Y_i,\ i \ge 1\}$ are still PQAI. Then by similar derivation of (42), there exists a positive integer n_2 such that

$$\begin{aligned} &P\left(\sum_{i=n_2+1}^{\infty}\sum_{j=1}^{m} Y_{ij}e^{-r(\tau_i+\tau_{ij})} > x\right) \\ &\quad = P\left(\sum_{i=n_2+1}^{\infty} Y_i e^{-r\tau_i} > x\right) = o(1)\overline{Q}(x) \\ &\quad = o(1)\overline{G}(x), \end{aligned} \tag{44}$$

where the last step is due to (43) and Lemma 23(1). Similarly to (42), we still show that there exists a positive integer m_0 such that

$$P\left(\sum_{j=m_0+1}^{\infty} Y_{1j}e^{-r\tau_{1j}} > x\right) = o(1)\overline{G}(x), \quad (45)$$

And from Lemma 25, we can derive that

$$\begin{aligned} &P\left(\sum_{i=1}^{\infty}\sum_{j=m_0+1}^{\infty} Y_{ij}e^{-r(\tau_i+\tau_{ij})} > x\right) \\ &\quad = P\left(\sum_{i=1}^{\infty} e^{-r\tau_i}\cdot\sum_{j=m_0+1}^{\infty} Y_{1j}e^{-r\tau_{1j}} > x\right) = o(1)\overline{G}(x), \end{aligned} \quad (46)$$

If, for some $p > J_G^+$,

$$E\left(\sum_{i=1}^{\infty} e^{-r\tau_i}\right)^p < \infty. \quad (47)$$

In fact, we arbitrarily choose some positive integer k, and set

$$E_k = E\left(\sum_{i=1}^{k} e^{-r\tau_i}\right)^p, \quad k \ge 1. \quad (48)$$

When $0 < J_G^+ < 1$, we prove by the Cr-inequality and (17) that, for $J_G^+ < p \le 1$,

$$\begin{aligned} E_k &\le \sum_{i=1}^{k} Ee^{-rp\tau_i} \le \sum_{i=1}^{k} g_L(i)\left(Ee^{-rp\tau_1}\right)^i \\ &< \sum_{i=1}^{\infty} g_L(i)\left(Ee^{-rp\tau_1}\right)^i. \end{aligned} \quad (49)$$

Arguing as (29) and setting $\epsilon = -\log(Ee^{-rp\tau_1}) - c$ in (32) for some constant $c > 0$, one easily sees that there exists a positive integer i_0 such that, for all $i > i_0$, $g_L(i) \le e^{-ci}(Ee^{-rp\tau_1})^{-i}$. Hence by (49), we have

$$\begin{aligned} E_k &< \sum_{i=1}^{i_0} g_L(i)\left(Ee^{-rp\tau_1}\right)^i + \sum_{i=i_0+1}^{\infty} e^{-ci} \\ &< \sum_{i=1}^{i_0} g_L(i)\left(Ee^{-rp\tau_1}\right)^i + \frac{1}{e^c - 1} < \infty, \end{aligned} \quad (50)$$

which means that the sequence $\{E_k,\ k \ge 1\}$ is bounded from above. Noting that this sequence is monotonically increasing, we apply the monotone bounded convergence theorem to conclude that as $k \longrightarrow \infty$, $\{E_k,\ k \ge 1\}$ has a finite limit, denoted by E_∞, which leads to (47) for $J_G^+ < p \le 1$. When $J_G^+ > 1$, by Minkowski's inequality and along with the similar lines of the proof of the case when $0 < J_G^+ < 1$, we also have that, for $p > J_G^+ > 1$ and for any fixed integer $k \ge 1$,

$$\begin{aligned} E_k &\le \left(\sum_{i=1}^{k}\left(Ee^{-rp\tau_i}\right)^{1/p}\right)^p \\ &\le \left(\sum_{i=1}^{k}\left(g_L(i)\left(Ee^{-rp\tau_1}\right)^i\right)^{1/p}\right)^p \\ &< \left(\sum_{i=1}^{\infty}\left(g_L(i)\left(Ee^{-rp\tau_1}\right)^i\right)^{1/p}\right)^p < \infty. \end{aligned} \quad (51)$$

This, along with the monotone bounded convergence theorem, proves that $\{E_k,\ k \ge 1\}$ has a finite limit as $k \longrightarrow \infty$, and then (47) still holds for $p > J_G^+ > 1$.

Let $n_0 = n_1 \vee n_2$ and m_0 be fixed as above. Notice that, for any $0 < \nu < 1$,

$$\begin{aligned} &P(D_r(\infty) > x) \\ &\quad \le P\left(\sum_{i=1}^{n_0} X_i e^{-r\tau_i} + \sum_{i=1}^{n_0}\sum_{j=1}^{m_0} Y_{ij}e^{-r(\tau_i+\tau_{ij})} > (1-\nu)x\right) \\ &\qquad + P\left(\sum_{i=n_0+1}^{\infty} X_i e^{-r\tau_i} > \frac{\nu x}{3}\right) \\ &\qquad + P\left(\sum_{i=n_0+1}^{\infty}\sum_{j=1}^{m_0} Y_{ij}e^{-r(\tau_i+\tau_{ij})} > \frac{\nu x}{3}\right) \\ &\qquad + P\left(\sum_{i=1}^{\infty}\sum_{j=m_0+1}^{\infty} Y_{ij}e^{-r(\tau_i+\tau_{ij})} > \frac{\nu x}{3}\right) \\ &\quad = \sum_{k=1}^{4} H_k(x,\nu). \end{aligned} \quad (52)$$

For $H_1(x,\nu)$, by Theorem 3.2 of Chen and Yuen [15], we get

$$\begin{aligned} &H_1(x,\nu) \\ &\quad \sim \sum_{i=1}^{n_0} P\left(X_i e^{-r\tau_i} > (1-\nu)x\right) \\ &\qquad + \sum_{i=1}^{n_0}\sum_{j=1}^{m_0} P\left(Y_{ij}e^{-r(\tau_i+\tau_{ij})} > (1-\nu)x\right) \\ &\quad = \sum_{i=1}^{n_0}\int_0^1 \overline{F}\left(\frac{(1-\nu)x}{y}\right) dP\left(e^{-r\tau_i} \le y\right) \\ &\qquad + \sum_{i=1}^{n_0}\sum_{j=1}^{m_0}\int_0^1 \overline{G}\left(\frac{(1-\nu)x}{y}\right) dP\left(e^{-r(\tau_i+\tau_{ij})} \le y\right) \\ &\quad = H_{11}(x,\nu) + H_{12}(x,\nu). \end{aligned} \quad (53)$$

Clearly,

$$H_{11}(x,v)=\sum_{i=1}^{n_0}\int_0^1 \frac{\overline{F}((1-v)x/y)}{\overline{F}(x/y)}\cdot\overline{F}\left(\frac{x}{y}\right)dP\left(e^{-r\tau_i}\le y\right)\le\sup_{0<y\le 1}\frac{\overline{F}((1-v)x/y)}{\overline{F}(x/y)}\sum_{i=1}^{n_0}P\left(X_ie^{-r\tau_i}>x\right), \tag{54}$$

and

$$H_{12}(x,v)=\sum_{i=1}^{n_0}\sum_{j=1}^{m_0}\int_0^1\frac{\overline{G}((1-v)x/y)}{\overline{G}(x/y)}\cdot\overline{G}\left(\frac{x}{y}\right)dP\left(e^{-r(\tau_i+\tau_{ij})}\le y\right)\le\sup_{0<y\le 1}\frac{\overline{G}((1-v)x/y)}{\overline{G}(x/y)}\sum_{i=1}^{n_0}\sum_{j=1}^{m_0}P\left(Y_{ij}e^{-r(\tau_i+\tau_{ij})}>x\right). \tag{55}$$

Thus,

$$\limsup_{x\to\infty}\frac{H_{11}(x,v)}{\sum_{i=1}^{n_0}P\left(X_ie^{-r\tau_i}>x\right)}\le\limsup_{x\to\infty}\sup_{0<y\le1}\frac{\overline{F}((1-v)x/y)}{\overline{F}(x/y)}=\limsup_{x\to\infty}\frac{\overline{F}((1-v)x)}{\overline{F}(x)}, \tag{56}$$

and

$$\limsup_{x\to\infty}\frac{H_{12}(x,v)}{\sum_{i=1}^{n_0}\sum_{j=1}^{m_0}P\left(Y_{ij}e^{-r(\tau_i+\tau_{ij})}>x\right)}\le\limsup_{x\to\infty}\sup_{0<y\le1}\frac{\overline{G}((1-v)x/y)}{\overline{G}(x/y)}=\limsup_{x\to\infty}\frac{\overline{G}((1-v)x)}{\overline{G}(x)}. \tag{57}$$

Because $F\in\mathscr{C}$ and $G\in\mathscr{C}$, letting $v\searrow 0$ yields that

$$H_{11}(x,v)\lesssim\sum_{i=1}^{n_0}P\left(X_ie^{-r\tau_i}>x\right), \tag{58}$$

and

$$H_{12}(x,v)\lesssim\sum_{i=1}^{n_0}\sum_{j=1}^{m_0}P\left(Y_{ij}e^{-r(\tau_i+\tau_{ij})}>x\right). \tag{59}$$

Hence, substituting (58) and (59) into (53) leads to

$$H_1(x,v)\lesssim\sum_{i=1}^{n_0}P\left(X_ie^{-r\tau_i}>x\right)+\sum_{i=1}^{n_0}\sum_{j=1}^{m_0}P\left(Y_{ij}e^{-r(\tau_i+\tau_{ij})}>x\right). \tag{60}$$

For $H_k(x,v)$, $k=2,3,4$, combining (42), (44), and (46) and using $F\in\mathscr{C}\subset\mathscr{D}$, $G\in\mathscr{C}\subset\mathscr{D}$ and Lemma 23(1), we derive that

$$H_2(x,v)=o(1)P\left(X_1e^{-r\tau_1}>x\right) \tag{61}$$

and

$$H_k(x,v)=o(1)P\left(Y_{11}e^{-r(\tau_1+\tau_{11})}>x\right),\quad i=3,4. \tag{62}$$

Therefore, we substitute the above results of $H_k(x,v)$, $i=1,2,3,4$, into (52) to obtain that

$$\begin{aligned}P(D_r(\infty)>x)&\lesssim\sum_{i=1}^{\infty}P\left(X_ie^{-r\tau_i}>x\right)+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P\left(Y_{ij}e^{-r(\tau_i+\tau_{ij})}>x\right)\\&=\int_{0-}^{\infty}\overline{F}\left(xe^{rt}\right)d\lambda(t)+\int_{0-}^{\infty}\int_{0-}^{\infty}\overline{G}\left(xe^{r(t+s)}\right)d\widehat{\lambda}(s)\,d\lambda(t),\end{aligned} \tag{63}$$

where the second term in the last step is due to (20). □

4. Proofs of Main Results

In this section, we proceed to prove the main results of this paper. First of all, we should give the proof of Theorem 20 that is helpful to prove Theorems 7–16.

Proof of Theorem 20. In the first half of this proof, we deal with the uniformity of (35) for all $t\in[t_0,T]$ under the conditions of Theorem 7 or Theorem 9, arbitrarily choosing some positive integer N. Note that, for all $t\in[t_0,T]$,

$$P(D_r(t)>x)=\left(\sum_{n=1}^{N}\sum_{m=1}^{N}+\sum_{n=1}^{N}\sum_{m=N+1}^{\infty}+\sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty}\right)\cdot P\left(\sum_{i=1}^{n}X_ie^{-r\tau_i}+\sum_{i=1}^{n}\sum_{j=1}^{m}Y_{ij}e^{-r(\tau_i+\tau_{1j})}>x,\ N(t)=n,\ \widehat{N}_1(t)=m\right)=\sum_{k=1}^{3}I_k(x,N). \tag{64}$$

Firstly, we consider $I_1(x,N)$. For $m,n\ge 1$, we denote by $A(\overrightarrow{y})$ and $B(\overrightarrow{z})$ the joint distributions of random vectors $\{\tau_1,\tau_2,\ldots,\tau_{n+1}\}$ and $\{\tau_{11},\tau_{12},\ldots,\tau_{1(m+1)}\}$, respectively, where $\overrightarrow{y}=(y_1,y_2,\ldots,y_{n+1})$ and $\overrightarrow{z}=(z_{11},z_{12},\ldots,z_{1(m+1)})$, and write $\Omega_n=\{0\le y_1\le\cdots\le y_n\le t<y_{n+1}\}$ and $\Omega_m=\{0\le z_{11}\le\cdots\le z_{1m}\le t-y_1<z_{1(m+1)}\}$. So by Lemma 24, it holds uniformly for all $t\in[t_0,T]$ and $1\le m,n\le N$ that

$$
\begin{aligned}
I_1(x,N) &= \sum_{n=1}^{N}\sum_{m=1}^{N}\int_{\Omega_n}\int_{\Omega_m} P\left(\sum_{i=1}^{n} X_i e^{-ry_i} + \sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij} e^{-r(y_i+z_{1j})} > x\right) dB\left(\overrightarrow{z}\right) dA\left(\overrightarrow{y}\right) \\
&\sim \sum_{n=1}^{N}\sum_{m=1}^{N}\int_{\Omega_n}\int_{\Omega_m}\left(\sum_{i=1}^{n} P\left(X_i e^{-ry_i} > x\right) + \sum_{i=1}^{n}\sum_{j=1}^{m} P\left(Y_{ij} e^{-r(y_i+z_{1j})} > x\right)\right) dB\left(\overrightarrow{z}\right) dA\left(\overrightarrow{y}\right) \\
&= \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{i=1}^{n} P\left(X_i e^{-r\tau_i} > x,\ N(t)=n\right) P\left(\widehat{N}_1(t)=m\right) \\
&\quad + \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{i=1}^{n}\sum_{j=1}^{m} P\left(Y_{ij} e^{-r(\tau_i+\tau_{1j})} > x,\ N(t)=n,\ \widehat{N}_1(t)=m\right) = I_{11}(x,N) + I_{12}(x,N).
\end{aligned}
\tag{65}
$$

For $I_{11}(x,N)$, it follows that

$$
\begin{aligned}
I_{11}(x,N) &= \left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} - \sum_{n=1}^{N}\sum_{m=N+1}^{\infty} - \sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty}\right) \\
&\quad \cdot \sum_{i=1}^{n} P\left(X_i e^{-r\tau_i} > x,\ N(t)=n\right) P\left(\widehat{N}_1(t)=m\right) \\
&= I_{111}(x) - I_{112}(x,N) - I_{113}(x,N).
\end{aligned}
\tag{66}
$$

Clearly, for all $t \in [t_0, T]$,

$$
\begin{aligned}
I_{111}(x) &= \sum_{i=1}^{\infty} P\left(X_i e^{-r\tau_i}\mathbf{1}_{\{\tau_i \le t\}} > x\right) \\
&= \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) d\lambda(s).
\end{aligned}
\tag{67}
$$

By the condition on $\{N(t),\ t \ge 0\}$, we prove that uniformly for all $t \in [t_0, T]$, as $N \longrightarrow \infty$,

$$
\begin{aligned}
I_{112}(x,N) &\le \overline{F}(x) \cdot EN(T) \cdot P\left(\widehat{N}_1(T) > N\right) \\
&= o(1)\overline{F}(x),
\end{aligned}
\tag{68}
$$

and

$$
I_{113}(x,N) \le \overline{F}(x) \cdot EN(T)\mathbf{1}_{\{N(T)>N\}} = o(1)\overline{F}(x). \tag{69}
$$

Further, by $F \in \mathscr{L} \cap \mathscr{D} \subset \mathscr{D}$, we get that uniformly for all $t \in [t_0, T]$, as $N \longrightarrow \infty$,

$$
\begin{aligned}
I_{112}(x,N) &= o(1)\overline{F}\left(xe^{rt_0}\right) \\
&= o(1)P\left(X_1 e^{-r\tau_1}\mathbf{1}_{\{\tau_1 \le t_0\}} > x\right) \\
&= o(1)\sum_{i=1}^{\infty} P\left(X_i e^{-r\tau_i}\mathbf{1}_{\{\tau_i \le t\}} > x\right),
\end{aligned}
\tag{70}
$$

and, similarly,

$$
I_{113}(x,N) = o(1)\sum_{i=1}^{\infty} P\left(X_i e^{-r\tau_i}\mathbf{1}_{\{\tau_i \le t\}} > x\right). \tag{71}
$$

Hence, it holds uniformly for all $t \in [t_0, T]$ that

$$
I_{11}(x,N) \sim \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) d\lambda(s). \tag{72}
$$

For $I_{12}(x,N)$, which is divided as

$$
\begin{aligned}
I_{12}(x,N) &= \left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} - \sum_{n=1}^{N}\sum_{m=N+1}^{\infty} - \sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty}\right) \\
&\quad \cdot \sum_{i=1}^{n}\sum_{j=1}^{m} P\left(Y_{ij} e^{-r(\tau_i+\tau_{1j})} > x,\ N(t)=n,\ \widehat{N}_1(t)=m\right) \\
&= I_{121}(x) - I_{122}(x,N) - I_{123}(x,N),
\end{aligned}
\tag{73}
$$

by (20), we obtain that, uniformly, for all $t \in [t_0, T]$,

$$
\begin{aligned}
&I_{121}(x) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} P\left(Y_{ij} e^{-r(\tau_i+\tau_{ij})} > x,\ N(t)=n,\ \widehat{N}_i(t)=m\right) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\left(Y_{ij} e^{-r(\tau_i+\tau_{ij})}\mathbf{1}_{\{\tau_i+\tau_{ij} \le t\}} > x\right) \\
&= \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s).
\end{aligned}
\tag{74}
$$

Considering that the counting processes $\{N(t),\ t \ge 0\}$ and $\{\widehat{N}_i(t),\ t \ge 0\}$, $i \ge 1$, depend only on their inter-arrival times $\{\theta_i,\ i \ge 1\}$ and $\{\theta_{ij},\ j \ge 1\}$, $i \ge 1$, respectively, and in Assumption 5, $\{\theta_i,\ i \ge 1\}$ and $\{\theta_{ij},\ j \ge 1,\ i \ge 1\}$ are mutually independent, we know that $\{N(t),\ t \ge 0\}$ is independent of $\{\widehat{N}_i(t),\ t \ge 0\}$, $i \ge 1$. Thus, by $G \in \mathscr{C} \subset \mathscr{D}$ and similar derivation of $I_{112}(x)$, we obtain that uniformly for all $t \in [t_0, T]$, respectively, as $N \longrightarrow \infty$,

$$
\begin{aligned}
I_{122}(x,N) &\le \overline{G}(x) \cdot EN(T) \cdot E\widehat{N}_1(T)\mathbf{1}_{\{\widehat{N}_1(T)>N\}} \\
&= o(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\left(Y_{ij} e^{-r(\tau_i+\tau_{ij})}\mathbf{1}_{\{\tau_i+\tau_{ij} \le t\}} > x\right),
\end{aligned}
\tag{75}
$$

and

$$I_{123}(x,N) \le \overline{G}(x) \cdot E\widehat{N}_1(T) \cdot EN(T)\, \mathbf{1}_{\{N(T)>N\}} = o(1) \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\left(Y_{ij} e^{-r(\tau_i+\tau_{ij})} \mathbf{1}_{\{\tau_i+\tau_{ij}\le t\}} > x\right). \quad (76)$$

Then, it holds uniformly for all $t \in [t_0, T]$ that

$$I_{12}(x,N) \sim \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s), \quad (77)$$

which, along with (65) and (72), shows that, uniformly, for all $t \in [t_0, T]$,

$$I_1(x,N) \sim \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) d\lambda(s) + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s). \quad (78)$$

Subsequently, we consider $I_2(x,N)$. Clearly, for all $t \in [t_0, T]$,

$$\begin{aligned} I_2(x,N) &\le \sum_{n=1}^{N} P\left(\sum_{i=1}^{n} X_i > \frac{x}{2}\right) P(N(t)=n) \\ &\cdot P\left(\widehat{N}_1(T) > N\right) + \sum_{n=1}^{N}\sum_{m=N+1}^{\infty} P\left(\sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij} > \frac{x}{2}\right) \\ &\cdot P\left(\widehat{N}_1(t) = m\right) P(N(t)=n) = I_{21}(x,N) \\ &+ I_{22}(x,N). \end{aligned} \quad (79)$$

For $I_{21}(x,N)$, it follows from Lemma 24 with $c_1 = c_2 = \cdots = c_n \equiv 1$ that, uniformly for all $t \in [t_0, T]$,

$$\begin{aligned} I_{21}(x,N) &\sim \overline{F}\left(\frac{x}{2}\right) \sum_{n=1}^{N} nP(N(t)=n) \\ &\cdot P\left(\widehat{N}_1(T) > N\right) \\ &\le \overline{F}\left(\frac{x}{2}\right) EN(T) \cdot P\left(\widehat{N}_1(T) > N\right). \end{aligned} \quad (80)$$

Thus, by $F \in \mathscr{D}$ and similar derivation of $I_{112}(x,N)$, it holds uniformly for all $t \in [t_0, T]$ that

$$I_{21}(x,N) = o(1) \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) d\lambda(s), \quad \text{as } N \longrightarrow \infty. \quad (81)$$

For $I_{22}(x,N)$, by $G \in \mathscr{D}$ and Lemma 22(1), there exist two positive constants C and D such that, for any $p > J_G^+$ and $N+1 < x/2D$, we prove that, uniformly for all $t \in [t_0, T]$,

$$\begin{aligned} I_{22}(x,N) &\le \left(\sum\sum_{\{N+1\le mn\le x/2D\}} + \sum\sum_{\{mn>x/2D\}}\right) P\left(\sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij} \right. \\ &\left. > \frac{x}{2}\right) P\left(\widehat{N}_1(t) = m\right) P(N(t)=n) \\ &\le \sum\sum_{\{N+1\le mn\le x/2D\}} mn\overline{G}\left(\frac{x}{2mn}\right) P\left(\widehat{N}_1(t) = m\right) \\ &\cdot P(N(t)=n) + P\left(N(T)\widehat{N}_1(T) > \frac{x}{2D}\right) \\ &\le C\overline{G}\left(\frac{x}{2}\right) \sum\sum_{\{N+1\le mn\le x/2D\}} (mn)^{p+1} P\left(\widehat{N}_1(t) = m\right) \\ &\cdot P(N(t)=n) + \left(\frac{x}{2D}\right)^{-p} E\left(N(T)\widehat{N}_1(T)\right)^p \\ &\lesssim C\overline{G}\left(\frac{x}{2}\right) E\left[N(T)\widehat{N}_1(T)\right]^{p+1} \mathbf{1}_{\{N(T)\widehat{N}_1(T)>N\}}, \end{aligned} \quad (82)$$

where in the last step we used Lemma 22(2). So by the similar derivation of $I_{122}(x,N)$ or $I_{123}(x,N)$, it holds uniformly for all $t \in [t_0, T]$ that

$$I_{22}(x) = o(1) \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s), \quad (83)$$

which, along with (79) and (81), implies that uniformly for all $t \in [t_0, T]$,

$$I_2(x) = o(1)\left(\int_{0-}^{t} \overline{F}\left(xe^{rs}\right) d\lambda(s) + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s)\right). \quad (84)$$

Finally, we turn to $I_3(x)$. By similar derivation of $I_2(x)$, we have that, uniformly for all $t \in [t_0, T]$,

$$\begin{aligned} I_3(x) &\le \sum_{n=N+1}^{\infty} P\left(\sum_{i=1}^{n} X_i > \frac{x}{2}\right) P(N(t)=n) \\ &+ \sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty} P\left(\sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij} > \frac{x}{2}\right) P\left(\widehat{N}_1(t) = m\right) \\ &\cdot P(N(t)=n) = o(1)\left(\int_{0-}^{t} \overline{F}\left(xe^{rs}\right) d\lambda(s)\right. \\ &\left. + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s)\right). \end{aligned} \quad (85)$$

Consequently, we substitute (78), (84), and (85) into (64) to show that relation (35) holds uniformly for all $t \in [t_0, T]$.

In the second half of this proof, we extend the uniformity of (35) to an infinite set Λ under the conditions of Theorem 16.

Clearly, the uniformity of (35) for all $t \in \Lambda \cap [0, T_0]$ follows by copying the proof in the first half with the only modification that we use Lemma 2.1 of Wang et al. [14] to show the uniformity of (70) for all $t \in \Lambda \cap [0, T_0]$ under the conditions of Theorem 16, where $T_0 \in \Lambda$ is an arbitrarily fixed number. So it suffices to show that relation (35) holds uniformly for all $t \in [T_0, \infty]$. By Lemma 22(1) and $F \in \mathscr{C} \subset \mathscr{D}$, it holds that, for all $x \geq D$ and all $t \in \Lambda$,

$$\frac{\int_t^\infty \overline{F}(xe^{rs})\, d\lambda(s)}{\int_{0-}^t \overline{F}(xe^{rs})\, d\lambda(s)} = \frac{\int_t^\infty \overline{F}(xe^{rs}) / \overline{F}(x)\, d\lambda(s)}{\int_{0-}^t \overline{F}(xe^{rs}) / \overline{F}(x)\, d\lambda(s)} \leq C^2 \frac{\int_t^\infty e^{-r\widehat{p}s} d\lambda(s)}{\int_{0-}^t e^{-rps} d\lambda(s)}. \tag{86}$$

From (17), it follows that

$$\begin{aligned}\int_{0-}^\infty e^{-r\widehat{p}s} d\lambda(s) &= \sum_{n=1}^\infty \int_{0-}^\infty e^{-r\widehat{p}s} dP(\tau_n \leq s) \\ &= \sum_{n=1}^\infty E\left(e^{-r\widehat{p}\tau_n}\right) \\ &\leq \sum_{n=1}^\infty g_L(n) \left(Ee^{-r\widehat{p}\tau_1}\right)^n.\end{aligned} \tag{87}$$

By the similar derivation of (29), we set $\epsilon = -\log(Ee^{-r\widehat{p}\tau_1}) - c$ for some constant $c > 0$ in (32); there exists a positive integer n_3 such that, for all $n > n_3$,

$$g_L(n) \leq e^{-cn} \left(Ee^{-r\widehat{p}\tau_1}\right)^{-n}. \tag{88}$$

Thus,

$$\int_{0-}^\infty e^{-r\widehat{p}s} d\lambda(s) \leq \sum_{n=1}^{n_3} g_L(n) \left(Ee^{-r\widehat{p}\tau_1}\right)^n + \sum_{n=n_3+1}^\infty e^{-cn} < \infty. \tag{89}$$

Similarly,

$$\int_{0-}^\infty e^{-rps} d\lambda(s) < \infty. \tag{90}$$

Hence, the third term of (86) tends to 0 as $t \longrightarrow \infty$. Then, for an arbitrarily fixed $\varepsilon > 0$, there exists a large number $T_1 \in \Lambda$ such that, for all $x \geq D$,

$$\int_{T_1}^\infty \overline{F}(xe^{rs})\, d\lambda(s) \leq \varepsilon \int_{0-}^{T_1} \overline{F}(xe^{rs})\, d\lambda(s). \tag{91}$$

Again by Lemma 22(1) and $G \in \mathscr{C} \subset \mathscr{D}$, it holds that for, all $x \geq D$ and all $t \in \Lambda$,

$$\begin{aligned}&\frac{\int_t^\infty \int_{0-}^\infty \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s)}{\int_{0-}^t \int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s)} \\ &\quad = \frac{\int_t^\infty \int_{0-}^\infty \overline{G}\left(xe^{r(s+u)}\right) / \overline{G}(x)\, d\widehat{\lambda}(u)\, d\lambda(s)}{\int_{0-}^t \int_{0-}^{t-s} \overline{G}\left(xe^{r(s+u)}\right) / \overline{G}(x)\, d\widehat{\lambda}(u)\, d\lambda(s)} \\ &\quad \leq C^2 \frac{\int_t^\infty \int_{0-}^\infty e^{-r\widehat{p}(s+u)} d\widehat{\lambda}(u)\, d\lambda(s)}{\int_{0-}^t \int_{0-}^{t-s} e^{-rp(s+u)} d\widehat{\lambda}(u)\, d\lambda(s)}.\end{aligned} \tag{92}$$

By similar derivation of (89), we still get that

$$\begin{aligned}&\int_{0-}^\infty e^{-r\widehat{p}u} d\widehat{\lambda}(u) < \infty, \\ &\int_{0-}^\infty e^{-rpu} d\widehat{\lambda}(u) < \infty.\end{aligned} \tag{93}$$

which, along with (89) and (90), implies that, respectively,

$$\begin{aligned}&\int_{0-}^\infty \int_{0-}^\infty e^{-r\widehat{p}(s+u)} d\widehat{\lambda}(u)\, d\lambda(s) \\ &\quad = \int_{0-}^\infty e^{-r\widehat{p}u} d\widehat{\lambda}(u) \cdot \int_{0-}^\infty e^{-r\widehat{p}s} d\lambda(s) < \infty,\end{aligned} \tag{94}$$

and

$$\begin{aligned}&\int_{0-}^\infty \int_{0-}^\infty e^{-rp(s+u)} d\widehat{\lambda}(u)\, d\lambda(s) \\ &\quad = \int_{0-}^\infty e^{-rpu} d\widehat{\lambda}(u) \cdot \int_{0-}^\infty e^{-rps} d\lambda(s) < \infty.\end{aligned} \tag{95}$$

Thus, the third term of (92) tends to 0 as $t \longrightarrow \infty$, which yields that there exists a large number $T_2 \in \Lambda$ such that, for all $x \geq D$,

$$\begin{aligned}&\int_{T_2}^\infty \int_{0-}^\infty \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s) \\ &\quad \leq \varepsilon \int_{0-}^{T_2} \int_{0-}^{T_2-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s).\end{aligned} \tag{96}$$

Similarly, there also exists a large number $T_3 \in \Lambda$ such that, for all $x \geq D$,

$$\begin{aligned}&\int_{0-}^{T_3} \int_{T_3-s}^\infty \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s) \\ &\quad \leq \varepsilon \int_{0-}^{T_3} \int_{0-}^{T_3-s} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s).\end{aligned} \tag{97}$$

Let $T_0 = T_1 \vee T_2 \vee T_3$. Clearly, $T_0 \in \Lambda$. On the one hand, by Lemma 26 we get that, uniformly for all $t \in (T_0, \infty]$,

$$\begin{aligned} P(D_r(t) > x) &\le P(D_r(\infty) > x) \\ &\lesssim \int_{0-}^{\infty} \overline{F}(xe^{rs})\, d\lambda(s) \\ &\quad + \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{G}\left(xe^{r(s+u)}\right) d\widehat{\lambda}(u)\, d\lambda(s) \\ &= J_1(x) + J_2(x). \end{aligned} \tag{98}$$

For $J_1(x)$, by (91), we have that, for all $t \in (T_0, \infty]$,

$$\begin{aligned} J_1(x) &\le \left(\int_0^t + \int_{T_0}^{\infty}\right) \overline{F}(xe^{rs})\, d\lambda(s) \\ &\le (1+\varepsilon) \int_0^t \overline{F}(xe^{rs})\, d\lambda(s). \end{aligned} \tag{99}$$

For $J_2(x)$, by (96) and (97), we also have that, for all $t \in (T_0, \infty]$,

$$\begin{aligned} J_2(x) &\le \left(\int_{0-}^{t}\int_{0-}^{t-s} + \int_{0-}^{T_0}\int_{T_0-s}^{\infty} + \int_{T_0}^{\infty}\int_{0-}^{\infty}\right) \\ &\quad \cdot \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s) \le (1+2\varepsilon) \\ &\quad \cdot \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s). \end{aligned} \tag{100}$$

Hence, substituting (99) and (100) into (98) and considering the arbitrariness of $\varepsilon > 0$, it holds uniformly for all $t \in (T_0, \infty]$ that

$$\begin{aligned} &P(D_r(t) > x) \\ &\quad \lesssim \int_0^t \overline{F}(xe^{rs})\, d\lambda(s) \\ &\quad + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s). \end{aligned} \tag{101}$$

On the other hand, by (35) with $t = T_0$, we attain that, uniformly for all $t \in (T_0, \infty]$,

$$\begin{aligned} P(D_r(t) > x) &\ge (D_r(T_0) > x) \\ &\sim \int_{0-}^{T_0} \overline{F}(xe^{rs})\, d\lambda(s) \\ &\quad + \int_{0-}^{T_0}\int_{0-}^{T_0-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s) \\ &= J_3(x) + J_4(x). \end{aligned} \tag{102}$$

For $J_3(x)$, by (91), we get that, for all $t \in (T_0, \infty]$,

$$\begin{aligned} J_3(x) &\ge \frac{1}{1+\varepsilon} \int_0^{\infty} \overline{F}(xe^{rs})\, d\lambda(s) \\ &\ge \frac{1}{1+\varepsilon} \int_0^{t} \overline{F}(xe^{rs})\, d\lambda(s). \end{aligned} \tag{103}$$

For $J_4(x)$, by (96) and (97), we also get that, for all $t \in (T_0, \infty]$,

$$\begin{aligned} J_4(x) &\ge \frac{1}{1+2\varepsilon} \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s) \\ &\ge \frac{1}{1+2\varepsilon} \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s). \end{aligned} \tag{104}$$

Thus, by (102)-(104) and the arbitrariness of $\varepsilon > 0$, it holds uniformly for all $t \in (T_0, \infty]$ that

$$\begin{aligned} &P(D_r(t) > x) \\ &\quad \gtrsim \int_0^t \overline{F}(xe^{rs})\, d\lambda(s) \\ &\quad + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s), \end{aligned} \tag{105}$$

which, along with (101), shows that relation (35) holds uniformly for all $t \in (T_0, \infty]$. □

Proof of Theorem 7. By the surplus process (4), we obtain its discounted value as

$$\widetilde{U}_r(t) = x + \widetilde{C}(t) - D_r(t) + \sigma\widetilde{B}(t), \quad t \ge 0, \tag{106}$$

where $D_r(t)$ and $\widetilde{C}(t)$ are defined by (5) and (6), respectively, and $\widetilde{B}(t) = \int_{0-}^{t} e^{-rs} dB(s)$. Hence by (7), we see that, for all $t \in \Lambda$,

$$\begin{aligned} \psi_r(x,t) = P\Big(&D_r(s) - \sigma\widetilde{B}(s) > x \\ &+ \widetilde{C}(s) \text{ for some } 0 < s \le t \mid \widetilde{U}_r(0) = x\Big). \end{aligned} \tag{107}$$

Noting that the stochastic integral $\widetilde{B}(t)$, $0 < t \le \infty$, has a normal distribution with mean 0 and variance $(1 - e^{-2rt})/2r$, then by the classic martingale inequalities, $\widehat{B}(T) \equiv \sigma \sup_{t\in[0,T]} |\widetilde{B}(t)|$, $0 < T \le \infty$, has finite moments of arbitrary orders; and further for a distribution $V \in \mathscr{D}$, it holds that

$$P\left(\widehat{B}(T) > x\right) = o(1)\,\overline{V}(x). \tag{108}$$

From (107), it follows that, for all $t \in [t_0, T]$,

$$\psi_r(x,t) \le P\left(D_r(t) + \widehat{B}(T) > x\right), \tag{109}$$

and

$$\psi_r(x,t) \ge P\left(D_r(t) - \widehat{B}(T) > x + \widetilde{C}(T)\right). \tag{110}$$

On the one hand, by $F \in \mathscr{L} \cap \mathscr{D}$, $G \in \mathscr{L} \cap \mathscr{D}$, and the uniformity of (35) for all $t \in [t_0, T]$, we know that, for all $t \in [t_0, T]$, the distributions of $D_r(t)$ belong to $\mathscr{L} \cap \mathscr{D} \subset \mathscr{S}$. Also by (35) and (108), it holds uniformly for all $t \in [t_0, T]$ that

$$\begin{aligned} P\left(\widehat{B}(T) > x\right) &= o(1)\, P\left(D_r(t_0) > x\right) \\ &= o(1)\, P\left(D_r(t) > x\right), \end{aligned} \tag{111}$$

where the second step is from the fact that $D_r(t)$ is monotonically increasing in $t \in [t_0, T]$. So by mimicking the proof of Lemma 4.5 of Tang [24] and considering the independence between $\widehat{B}(T)$ and $D_r(t), t \in [t_0, T]$, we prove that, uniformly for all $t \in [t_0, T]$,

$$P\left(D_r(t) + \widehat{B}(T) > x\right) \sim P\left(D_r(t) > x\right). \tag{112}$$

Therefore by (109) and the uniformity of (35) for all $t \in [t_0, T]$, it holds uniformly for all $t \in [t_0, T]$ that

$$\begin{aligned}\psi_r(x,t) &\lesssim P\left(D_r(t) > x\right) \\ &\sim \int_0^t \overline{F}\left(xe^{rs}\right) d\lambda(s) \\ &\quad + \int_{0-}^t \int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s).\end{aligned} \tag{113}$$

On the other hand, letting $\widetilde{Z}(T) = \widetilde{C}(T) + \widehat{B}(T)$, and using the dominated convergence theorem and the independence between $\widetilde{Z}(T)$ and $D_r(t), t \in [t_0, T]$, we have that, for all $t \in [t_0, T]$,

$$\begin{aligned}&\lim \frac{P\left(D_r(t) > x + \widetilde{Z}(T)\right)}{P\left(D_r(t) > x\right)} \\ &= \int_0^\infty \lim \frac{P\left(D_r(t) > x + y\right)}{P\left(D_r(t) > x\right)} dP\left(\widetilde{Z}(T) \le y\right) = 1,\end{aligned} \tag{114}$$

where the second step is due to the fact that, for all $t \in [t_0, T]$, the distributions of $D_r(t)$ are long-tailed. Then by (110) and the uniformity of (35) for all $t \in [t_0, T]$, it holds uniformly for all $t \in [t_0, T]$ that

$$\begin{aligned}\psi_r(x,t) &\gtrsim P\left(D_r(t) > x\right) \\ &\sim \int_0^t \overline{F}\left(xe^{rs}\right) d\lambda(s) \\ &\quad + \int_{0-}^t \int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s).\end{aligned} \tag{115}$$

Consequently, we derive from (113) and (115) that relation (22) holds uniformly for all $t \in [t_0, T]$. □

Proof of Theorem 9. According to the proof of Theorem 7, it remains to show the uniformly asymptotic lower-bound of $\psi_r(x,t)$. By $F \in \mathscr{C}$ and $G \in \mathscr{C}$, one has that, for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that, for all large x,

$$\overline{F}((1+\delta)x) \ge (1-\varepsilon)\overline{F}(x), \tag{116}$$

and

$$\overline{G}((1+\delta)x) \ge (1-\varepsilon)\overline{G}(x). \tag{117}$$

Then by (110), it holds that, for the fixed $\delta > 0$ as above and all $t \in [t_0, T]$,

$$\begin{aligned}\psi_r(x,t) &\ge P\left(D_r(t) - \widehat{B}(T) > (1+\delta)x\right) \\ &\quad - P\left(\widetilde{C}(T) > \delta x\right) = L_1(x,\delta) - L_2(x,\delta).\end{aligned} \tag{118}$$

For $L_1(x,\delta)$, by similar derivation of (114) and the uniformity of (35) for all $t \in [t_0, T]$, we derive that, uniformly for all $t \in [t_0, T]$,

$$\begin{aligned}L_1(x,\delta) &\sim \int_{0-}^t \overline{F}\left((1+\delta)xe^{rs}\right) d\lambda(s) \\ &\quad + \int_{0-}^t \int_{0-}^{t-s} \overline{G}\left((1+\delta)xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s) \ge (1 \\ &\quad - \varepsilon)\left(\int_{0-}^t \overline{F}\left(xe^{rs}\right) d\lambda(s) \right.\\ &\quad \left. + \int_{0-}^t \int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s)\right),\end{aligned} \tag{119}$$

where in the second step we used (116) and (117). For $L_2(x,\delta)$, by (24), $F \in \mathscr{C} \subset \mathscr{D}$, $G \in \mathscr{C} \subset \mathscr{D}$, and similar argument of (70), we conclude that, uniformly for all $t \in [t_0, T]$,

$$\begin{aligned}L_2(x,\delta) &= o(1)\left(\overline{F}(\delta x) + \overline{G}(\delta x)\right) = o(1) \\ &\quad \cdot \left(\sum_{i=1}^\infty P\left(X_i e^{-r\tau_i} \mathbf{1}_{\{\tau_i \le t} \ \} > x\right)\right. \\ &\quad \left. + \sum_{i=1}^\infty \sum_{j=1}^\infty P\left(Y_{ij} e^{-r(\tau_i + \tau_{ij})} \mathbf{1}_{\{\tau_i + \tau_{ij} \le t\}} > x\right)\right).\end{aligned} \tag{120}$$

Thus by the arbitrariness of $\varepsilon > 0$, we obtain the uniformly asymptotic lower-bound of $\psi_r(x,t)$ for all $t \in [t_0, T]$; namely, $\int_{0-}^t \overline{F}(xe^{rs}) d\lambda(s) + \int_{0-}^t \int_{0-}^{t-s} \overline{G}(xe^{r(u+s)}) d\widehat{\lambda}(u) d\lambda(s)$. □

Proof of Corollary 12. Clearly, when $r = 0$, it follows from Theorem 7 or Theorem 9 that, uniformly for all $t \in [t_0, T]$,

$$\psi_0(x,t) \sim \lambda(t)\overline{F}(x) + \widehat{\lambda} * \lambda(t) \cdot \overline{G}(x). \tag{121}$$

By $G \in \mathscr{L}$, we prove that, uniformly for all $t \in [t_0, T]$,

$$\begin{aligned}\widehat{\lambda} * \lambda(t) \cdot \overline{G}(x) &\ge \int_x^{x + \widehat{\lambda} * \lambda(t)} \overline{G}(y)\, dy \\ &\ge \widehat{\lambda} * \lambda(t) \cdot \overline{G}\left(x + \widehat{\lambda} * \lambda(T)\right) \\ &\sim \widehat{\lambda} * \lambda(t) \cdot \overline{G}(x).\end{aligned} \tag{122}$$

Similarly,

$$\lambda(t) \cdot \overline{G}(x) \sim \int_x^{x+\lambda(t)} \overline{F}(y)\, dy. \tag{123}$$

Hence, the claimed formulas are established □

Proof of Theorem 16. By (107), it holds that, for all $t \in \Lambda$,

$$\begin{aligned}P\left(D_r(t) - \widehat{B}(\infty) > x + \widetilde{C}\right) &\le \psi_r(x,t) \\ &\le P\left(D_r(t) + \widehat{B}(\infty) > x\right),\end{aligned} \tag{124}$$

where $\widehat{B}(\infty) = \sigma \sup_{t\in[0,\infty]} |\widetilde{B}(t)|$ and $\widetilde{C}$ is defined in (31). For the uniformly asymptotic upper-bound of $\psi_r(x,t)$ for all $t \in \Lambda$, by (124), the uniformity of (35) for all $t \in \Lambda$, and the similar derivation of (114), it holds uniformly for all $t \in \Lambda$ that

$$\psi_r(x,t) \lesssim \int_{0-}^{t} \overline{F}(xe^{rs})\, d\lambda(s) + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s). \tag{125}$$

Hence, we only need to prove the uniformly asymptotic lower-bound of $\psi_r(x,t)$ for all $t \in \Lambda$; namely,

$$\psi_r(x,t) \gtrsim \int_{0-}^{t} \overline{F}(xe^{rs})\, d\lambda(s) + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s) \tag{126}$$

holds uniformly for all $t \in \Lambda$.

Actually, by going along the same lines of the proofs of relation (115) and Theorem 9 with the only change that we use the uniformity of (35) for all $t \in \Lambda \cap [0, T_0]$ instead of that for all $t \in [t_0, T]$, we obtain the uniformity of (126) for all $t \in \Lambda \cap [0, T_0]$ under conditions 1 and 2 of this theorem, respectively. Therefore, we will achieve the proof if we show that (126) holds uniformly for all $t \in (T_0, \infty]$. In what follows, we formulate the proof into two parts.

In the first part, we consider the case of condition 1. By (124), (35) with $t = T_0$, and similar derivation of (115), we prove that, uniformly for all $t \in (T_0, \infty]$,

$$\begin{aligned}\psi_r(x,t) &\geq P\left(D_r(T_0) > x + \widetilde{C} + \widehat{B}(\infty)\right) \\ &\sim \int_{0}^{T_0} \overline{F}(xe^{rs})\, d\lambda(s) + \int_{0-}^{T_0}\int_{0-}^{T_0-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s) \\ &\gtrsim \int_{0-}^{t} \overline{F}(xe^{rs})\, d\lambda(s) + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s),\end{aligned} \tag{127}$$

where the last step is due to the same derivation as that in (105).

In the second part, we consider the case of condition 2. Let $\delta > 0$ be fixed as that in (116) and (117). Again by (124), it follows that, for all $t \in (T_0, \infty]$,

$$\psi_r(x,t) \geq P\left(D_r(T_0) - \widehat{B}(\infty) > (1+\delta)x\right) - P\left(\widetilde{C} > \delta x\right) = L_1'(x,\delta) - L_2'(x,\delta). \tag{128}$$

For $L_1'(x,\delta)$, by (35) with $t = T_0$ and the similar derivation of (119), we have that, uniformly for all $t \in (T_0, \infty]$,

$$\begin{aligned}L_1'(x,\delta) &\gtrsim (1-\varepsilon)\left(\int_{0-}^{T_0} \overline{F}(xe^{rs})\, d\lambda(s) + \int_{0-}^{T_0}\int_{0-}^{T_0-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s)\right) \gtrsim (1-\varepsilon) \\ &\cdot \left(\int_{0-}^{t} \overline{F}(xe^{rs})\, d\lambda(s) + \int_{0-}^{t}\int_{0-}^{t-s} \overline{G}\left(xe^{r(u+s)}\right) d\widehat{\lambda}(u)\, d\lambda(s)\right),\end{aligned} \tag{129}$$

where the last step is also due to the same derivation as that in (105). For $L_2'(x,\delta)$, using (33) and arguing as (120) can imply that, uniformly for all $t \in (T_0, \infty]$,

$$L_2'(x,\delta) = o(1)\left(\sum_{i=1}^{\infty} P\left(X_i e^{-r\tau_i}\mathbf{1}_{\{\tau_i \le t\}} > x\right) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\left(Y_{ij} e^{-r(\tau_i+\tau_{ij})}\mathbf{1}_{\{\tau_i+\tau_{ij} \le t\}} > x\right)\right). \tag{130}$$

As a result, by the arbitrariness of $\varepsilon > 0$, we obtain the uniformity of relation (126) for all $t \in (T_0, \infty]$ immediately. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] J. Li, "On pairwise quasi-asymptotically independent random variables and their applications," *Statistics and Probability Letters*, vol. 83, pp. 2081–2087, 2013.

[2] K. C. Yuen, J. Guo, and K. W. Ng, "On ultimate ruin in a delayed-claims risk model," *Journal of Applied Probability*, vol. 42, no. 1, pp. 163–174, 2005.

[3] H. Meng and G. Wang, "On the expected discounted penalty function in a delayed-claims risk model," *Acta Mathematicae Applicatae Sinica*, vol. 28, no. 2, pp. 215–224, 2012.

[4] J. Li and R. Wu, "The Gerber-Shiu discounted penalty function for a compound binomial risk model with by-claims," *Acta Mathematicae Applicatae Sinica*, vol. 31, no. 1, pp. 181–190, 2015.

[5] H. R. Waters and A. Papatriandafylou, "Ruin probabilities allowing for delay in claims settlement," *Insurance: Mathematics and Economics*, vol. 4, no. 2, pp. 113–122, 1985.

[6] K. C. Yuen and J. Y. Guo, "Ruin probabilities for time-correlated claims in the compound binomial model," *Insurance: Mathematics and Economics*, vol. 29, no. 1, pp. 47–57, 2001.

[7] Y. Xiao and J. Guo, "The compound binomial risk model with time-correlated claims," *Insurance: Mathematics and Economics*, vol. 41, no. 1, pp. 124–133, 2007.

[8] X. Wu and S. Li, "On a discrete time risk model with time-delayed claims and a constant dividend barrier," *Insurance Markets and Companies: Analyses and Actuarial Computations*, vol. 3, no. 1, pp. 50–57, 2012.

[9] Q. Gao, E. Zhang, and N. Jin, "The ultimate ruin probability of a dependent delayed-claim risk model perturbed by diffusion with constant force of interest," *Bulletin of the Korean Mathematical Society*, vol. 52, no. 3, pp. 895–906, 2015.

[10] Q. Gao, J. Zhuang, and Z. Huang, "Uniform asymptotics for a delay-claim risk model with diffusion, dependence structures and constant force of interest," *Journal of Computational and Applied Mathematics*, vol. 353, pp. 219–231, 2019.

[11] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, UK, 1987.

[12] P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin, Germany, 1997.

[13] A. J. McNeil, R. Frey, and P. Embrechts, *Quantitative Risk Management*, Princeton Series in Finance, Princeton University Press, Princeton, NJ, USA, 2005.

[14] K. Wang, Y. Wang, and Q. Gao, "Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate," *Methodology and Computing in Applied Probability*, vol. 15, no. 1, pp. 109–124, 2013.

[15] Y. Chen and K. C. Yuen, "Sums of pairwise quasi-asymptotically independent random variables with consistent variation," *Stochastic Models*, vol. 25, no. 1, pp. 76–89, 2009.

[16] J. Geluk and Q. Tang, "Asymptotic tail probabilities of sums of dependent subexponential random variables," *Journal of Theoretical Probability*, vol. 22, no. 4, pp. 871–882, 2009.

[17] Q. Gao and X. Liu, "Uniform asymptotics for the finite-time ruin probability with upper tail asymptotically independent claims and constant force of interest," *Statistics and Probability Letters*, vol. 83, no. 6, pp. 1527–1538, 2013.

[18] Q. Gao, N. Jin, and H. Shen, "Asymptotic behavior of the finite-time ruin probability with pairwise quasi-asymptotically independent claims and constant interest force," *Rocky Mountain Journal of Mathematics*, vol. 44, no. 5, pp. 1505–1528, 2014.

[19] Q. Tang, "Asymptotic ruin probabilities of the renewal model with constant interest force and regular variation," *Scandinavian Actuarial Journal*, no. 1, pp. 1–5, 2005.

[20] Q. Tang and G. Tsitsiashvili, "Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks," *Stochastic Processes and Their Applications*, vol. 108, no. 2, pp. 299–325, 2003.

[21] D. B. Cline and G. Samorodnitsky, "Subexponentiality of the product of independent random variables," *Stochastic Processes and Their Applications*, vol. 49, no. 1, pp. 75–98, 1994.

[22] D. Wang, C. Su, and Y. Zeng, "Uniform estimate for maximum of randomly weighted sums with applications to insurance risk theory," *Science China Mathematics*, vol. 48, no. 10, pp. 1379–1394, 2005.

[23] Q. Tang and G. Tsitsiashvili, "Randomly weighted sums of subexponential random variables with application to ruin theory," *Extremes*, vol. 6, no. 3, pp. 171–188, 2003.

[24] Q. Tang, "The ruin probability of a discrete time risk model under constant interest rate with heavy tails," *Scandinavian Actuarial Journal*, no. 3, pp. 229–240, 2004.