Research Article

Optimal Convergence Rates for the Strong Solutions to the Compressible MHD Equations with Potential Force

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In this paper, the large-time behavior of solutions to the Cauchy problem for the 3D compressible MHD equations is considered with the effect of external force. We construct the global unique solution with the small initial data near the stationary profile. The optimal $L^p - L^2$ ($1 \leq p \leq 2$) time decay rates of the solution to the system are built in multifrequency decompositions.

1. Introduction

Consider the following Cauchy problem of the full compressible MHD equations affected by the external potential force in $\mathbb{R}^3$:

$$\begin{aligned} \rho_t + \nabla \cdot (\rho u) &= 0, \\ \rho [u_t + (u \cdot \nabla) u] + \nabla P(\rho, \theta) \\ &= \mu \Delta u + (\mu + \lambda) \nabla (\nabla \cdot u) + (\nabla \times b) \times b + \rho F, \\ b_t - \nabla \times (u \times b) &= -\nabla \times (\nabla \times (vb)), \\ \rho c_v [\theta_t + (u \cdot \nabla) \theta] + \theta P_\theta(\rho, \theta) \nabla \cdot u &= \kappa \Delta \theta + \Psi[u], \\ \nabla \cdot b &= 0 \end{aligned} \quad (1)$$

where

$$\begin{aligned} (\rho, u, b, \theta)(0, x) &= (\rho_0, u_0, b_0, \theta_0)(x) \longrightarrow \\ &(\rho_\infty, 0, 0, \theta_\infty), \end{aligned} \quad (2)$$

as $|x| \longrightarrow \infty$.

Here $\rho \geq 0$, $u, b, \theta \in \mathbb{R}^3$, and $p(\rho, \theta)$ are density, velocity, magnetic field, temperature, and pressure, respectively. The constants μ, λ are the shear and bulk viscosity coefficients of the flow, satisfying the physical restrictions $\mu > 0$ and $2\mu + 3\lambda \geq 0$. The resistivity coefficient $\nu > 0$ is inversely

proportional to the electrical conductivity constant and acts as the magnetic diffusivity of magnetic fields. By the positive constant κ we denote the heat conduction coefficient and $c_v > 0$ is the specific heat at constant volume. In addition, $F = F(x)$ is an external force and $\Psi = \Psi[u]$ is the dissipation function:

$$\Psi[u] = \frac{\mu}{2} \sum_{i,j=1}^3 (\partial_i u_j + \partial_j u_i)^2 + \lambda \sum_{j=1}^3 (\partial_j u_j)^2. \quad (3)$$

It will be assumed that $P(\rho, \theta)$ is smooth in a neighborhood of $(\rho_\infty, \theta_\infty)$ with $P_\rho(\rho_\infty, \theta_\infty) > 0$ and $P_\theta(\rho_\infty, \theta_\infty) > 0$ throughout this paper. Both ρ_∞ and θ_∞ are positive constants.

We only consider the potential force $F = -\nabla \Phi(x)$ in this work. Reference [1] has proofed the existence of the stationary solution to problem (1) and (2) under aforementioned assumptions. We give the solution $(\rho_*, u_*, b_*, \theta_*)$ in a neighborhood of $(\rho_\infty, 0, 0, \theta_\infty)$ as

$$\begin{aligned} \int_{\rho_\infty}^{\rho_*(x)} \frac{P_\rho(\eta, \theta_\infty)}{\eta} dx + \Phi(x) &= 0, \\ u_*(x) &= 0, \\ b_*(x) &= 0, \\ \theta_*(x) &= \theta_\infty, \end{aligned} \quad (4)$$

and satisfy

$$\|\rho_* - \rho_\infty\|_{H^k(\mathbb{R}^3)} \leq C \|\Phi\|_{H^k(\mathbb{R}^3)}, \quad 0 \leq k \leq 4, \quad (5)$$

$$\begin{aligned} & \sum_{k=1}^4 \|(1 + |x|) \nabla^k (\rho_* - \rho_\infty)\|_{L^2(\mathbb{R}^3)} \\ & \leq C \sum_{k=1}^4 \|(1 + |x|) \nabla^k \Phi\|_{L^2(\mathbb{R}^3)}. \end{aligned} \quad (6)$$

The global unique solution is constructed to (1) near the steady state $(\rho_*, 0, 0, \theta_\infty)$ when the initial perturbation belongs to the Sobolev space $H^2(\mathbb{R}^3)$. The main results are as follows.

Theorem 1. Suppose that $(\rho_0 - \rho_\infty, u_0, b_0, \theta_0 - \theta_\infty) \in H^2(\mathbb{R}^3)$, there exists constant $\varepsilon > 0$ such that if

$$\begin{aligned} & \|(\rho_0 - \rho_\infty, u_0, b_0, \theta_0 - \theta_\infty)\|_{H^2(\mathbb{R}^3)} + \|\Phi\|_{H^4(\mathbb{R}^3)} \\ & + \sum_{k=1}^4 \|(1 + |x|) \nabla^k \Phi\|_{L^2(\mathbb{R}^3)} \leq \varepsilon, \end{aligned} \quad (7)$$

then the initial value problem (1) and (2) admits a unique solution (ρ, u, b, θ) globally in time which satisfies

$$\begin{aligned} \rho - \rho_* & \in C([0, \infty); H^2(\mathbb{R}^3)) \\ & \cap C^1([0, \infty); H^1(\mathbb{R}^3)), \\ u, b, \theta - \theta_\infty & \in C([0, \infty); H^2(\mathbb{R}^3)) \\ & \cap C^1([0, \infty); L^2(\mathbb{R}^3)). \end{aligned} \quad (8)$$

Moreover, if the initial data $(\rho_0 - \rho_\infty, u_0, b_0, \theta_0 - \theta_\infty)$ is bounded in $L^p(\mathbb{R}^3)$ for any given $1 \leq p \leq 2$, the solution (ρ, u, b, θ) satisfies the following decay-in-time estimates:

$$\begin{aligned} & \|\nabla(\rho - \rho_*, u, b, \theta - \theta_\infty)\|_{H^1(\mathbb{R}^3)} \\ & \leq C(1 + t)^{-(3/2)(1/p-1/2)} \quad \forall t \geq 0, \end{aligned} \quad (9)$$

$$\begin{aligned} & \|(\rho - \rho_*, u, b, \theta - \theta_\infty)\|_{L^q(\mathbb{R}^3)} \leq C(1 + t)^{-(3/2)(1/p-1/q)} \\ & \quad \forall t \geq 0, \quad 2 \leq q \leq 6, \end{aligned} \quad (10)$$

$$\begin{aligned} & \|\partial_t(\rho - \rho_*, u, b, \theta - \theta_\infty)\|_{L^2(\mathbb{R}^3)} \\ & \leq C(1 + t)^{-(3/2)(1/p-1/2)} \quad \forall t \geq 0, \end{aligned} \quad (11)$$

for some positive constant C .

Remark 2. In Theorem 1, using the Sobolev imbedding inequalities in Lemma 3, (7) together with (5) and (6) yields

$$\begin{aligned} & \|\rho - \rho_*\|_{H^4(\mathbb{R}^3)} \\ & + \sum_{k=1}^3 \|(1 + |x|) \nabla^k (\rho_* - \rho_\infty)\|_{L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)} \leq C\varepsilon. \end{aligned} \quad (12)$$

Due to its importance in mathematics and physics, there are a lot of literatures devoted to the mathematical theory of the MHD fluid system, including the global existence, unique, and time decay rates of solutions with or without external forces; see [2–20] and references therein. For the initial value problem for three-dimensional isentropic MHD system ((1) with $\theta = \text{const.}$), Li and Yu [13] obtained the global existence and the L^2 decay rate of classical solutions under the small oscillations on small initial data. Chen and Tan [4] obtained the decay estimate of solutions with the initial data in L^p ($p \in [1, 6/5)$). Noting the special construction of (1), the authors in [18] obtained decay estimate of solutions when the initial data belongs to negative Sobolev space. For the initial boundary value problem for three-dimensional isentropic MHD system (1), we refer to [11]. Pu and Guo [17] extended the global existence and decay rate of solutions in [4, 13] to three-dimensional incompressible nonisentropic MHD system (1). For the initial boundary value for three-dimensional nonisentropic MHD system, the global existence of weak solutions has been established in [9, 10].

If the magnetic field disappear ($b = 0$), system (1) deduces to Navier-Stokes equations. We only refer to time decay rates of solutions to the compressible Navier-Stokes equations. If there is no external force, Matsumura and Nishida [21] obtained the convergence rate for the compressible viscous and heat-conductive fluid in $\mathbb{R}^3$:

$$\|[\rho - \rho_\infty, u, \theta - \theta_\infty](t)\|_2 \leq C(1 + t)^{-3/4}, \quad t \geq 0. \quad (13)$$

For the same system, Ponce [22] gave the optimal L^p convergence rate

$$\begin{aligned} & \|\nabla^l[\rho - \rho_\infty, u, \theta - \theta_\infty](t)\|_{L^p} \\ & \leq C(1 + t)^{-(n/2)(1-1/p)-l/2}, \quad l = 0, 1, 2, \quad t \geq 0, \end{aligned} \quad (14)$$

for the space dimension $n = 2$ or 3 . When there is an external potential force $F = -\nabla\Phi(x)$, the first work to give explicit estimates for the decay rates for solutions to problem (1) and (2) was represented by Deckelnick [23]. In [23], when the initial perturbation only belongs to the Sobolev space $H^3(\Omega)$, the following decay estimates were established in an unbounded domain $\Omega \subset \mathbb{R}^3$ (the half space $\mathbb{R}_+^3 = \{x \in \mathbb{R}^3 | x_3 > 0\}$ or the exterior of a bounded domain with smooth boundary):

$$\begin{aligned} & \|\nabla[\rho - \rho_*, u, \theta - \theta_\infty]\|_{L^2(\Omega)} \leq C(1 + t)^{-1/4}, \\ & \|\partial_t[\rho - \rho_*, u, \theta - \theta_\infty]\|_{L^2(\Omega)} \leq C(1 + t)^{-1/2}. \end{aligned} \quad (15)$$

Duan *et al.* [24] obtained a similar result for the nonisentropic case

$$\begin{aligned} & \|\nabla^k[\rho - \rho_*, u, \theta - \theta_\infty](t)\|_{L^p} \leq C(1 + t)^{-(3/2)(1-1/p)}, \\ & \quad 2 \leq p \leq 6, \\ & \|[\rho - \rho_*, u, \theta - \theta_\infty](t)\|_{L^\infty} \leq C(1 + t)^{-5/4}, \end{aligned}$$

$$\begin{aligned}\tilde{S}_1 &= -\nabla \cdot (\tilde{\rho} \tilde{u}) - \nabla \cdot (\tilde{\rho} \tilde{u}), \\ \tilde{S}_2 &= -(\tilde{u} \cdot \nabla) \tilde{u} + \left[\frac{\mu}{\tilde{\rho} + \rho_*} - \frac{\mu}{\rho_\infty} \right] \Delta \tilde{u} + \left[\frac{\mu + \lambda}{\tilde{\rho} + \rho_*} \right. \\ &\quad \left. - \frac{\mu + \lambda}{\rho_\infty} \right] \nabla \nabla \cdot \tilde{u} - \left[\frac{P_p(\tilde{\rho} + \rho_*, \tilde{\theta} + \theta_\infty)}{\tilde{\rho} + \rho_*} \right. \\ &\quad \left. - \frac{P_p(\rho_\infty, \theta_\infty)}{\rho_\infty} \right] \nabla \tilde{\rho} - \left[\frac{P_\theta(\tilde{\rho} + \rho_*, \tilde{\theta} + \theta_\infty)}{\tilde{\rho} + \rho_*} \right. \\ &\quad \left. - \frac{P_\theta(\rho_\infty, \theta_\infty)}{\rho_\infty} \right] \nabla \tilde{\theta} - \left[\frac{P_p(\tilde{\rho} + \rho_*, \tilde{\theta} + \theta_\infty)}{\tilde{\rho} + \rho_*} \right.\end{aligned}\tag{22}$$

$$\begin{aligned}
& - \frac{P_\rho(\rho_*, \theta_\infty)}{\rho_*} \Big] \nabla \bar{\rho} + \left[\frac{1}{\bar{\rho} + \rho_*} - \frac{1}{\rho_\infty} \right] \left[b \cdot \nabla b \right. \\
& \left. - \frac{1}{2} \nabla |b|^2 \right] + \frac{1}{\rho_\infty} \left[b \cdot \nabla b - \frac{1}{2} \nabla |b|^2 \right], \quad (23)
\end{aligned}$$

$$\tilde{S}_3 = \tilde{b} \cdot \nabla \tilde{u} - \tilde{b} \nabla \cdot \tilde{u} - \tilde{u} \cdot \nabla \tilde{b}, \quad (24)$$

$$\begin{aligned}
\tilde{S}_4 = & -(\tilde{u} \cdot \nabla) \tilde{\theta} - \left[\frac{(\tilde{\theta} + \theta_\infty) P_\theta(\bar{\rho} + \rho_*, \tilde{\theta} + \theta_\infty)}{c_v(\bar{\rho} + \rho_*)} \right. \\
& \left. - \frac{\theta_\infty P_\infty(\rho_\infty, \theta_\infty)}{c_v \rho_\infty} \right] \nabla \cdot \tilde{u} + \left[\frac{\kappa}{c_v(\bar{\rho} + \rho_*)} - \frac{\kappa}{c_v \rho_\infty} \right] \\
& \cdot \Delta \tilde{\theta} + \left[\frac{1}{c_v(\bar{\rho} + \rho_*)} - \frac{1}{c_v \rho_\infty} \right] \Psi[\tilde{u}] + \frac{1}{c_v \rho_\infty} \Psi[\tilde{u}]. \quad (25)
\end{aligned}$$

For obtaining a symmetric system, we denote

$$\begin{aligned}
\sigma(x, t) &= \bar{\rho}(x, t), \\
\omega(x, t) &= \sqrt{\frac{P_1}{\rho_\infty}} \tilde{u}(x, t), \\
m(x, t) &= \sqrt{\frac{P_1}{\rho_\infty}} \tilde{b}(x, t), \\
z(x, t) &= \sqrt{\frac{P_2 \rho_\infty}{P_1 P_3}} \tilde{\theta}(x, t), \\
\mu_1 &= \frac{\mu}{\rho_\infty}, \\
\mu_2 &= \frac{\mu + \lambda}{\rho_\infty}, \\
\gamma_1 &= \sqrt{P_1 \rho_\infty}, \\
\gamma_2 &= \sqrt{P_2 P_3}, \\
\kappa_1 &= \frac{\kappa}{c_v} \sqrt{\frac{P_2}{P_1 P_3 \rho_\infty}}, \quad (26)
\end{aligned}$$

with

$$\begin{aligned}
P_1 &= \frac{P_\rho(\rho_\infty, \theta_\infty)}{\rho_\infty}, \\
P_2 &= \frac{P_\theta(\rho_\infty, \theta_\infty)}{\rho_\infty}, \\
P_3 &= \frac{\theta_\infty P_\theta(\rho_\infty, \theta_\infty)}{c_v \rho_\infty}. \quad (27)
\end{aligned}$$

Then (20) and (21) can be reformulated as

$$\begin{aligned}
\sigma_t + \gamma_1 \nabla \cdot \omega &= S_1, \\
\omega_t - \mu_1 \Delta \omega - \mu_2 \nabla \nabla \cdot \omega + \gamma_1 \nabla \sigma + \gamma_2 \nabla z &= S_2, \\
m_t - \nu \Delta m &= S_3, \quad (28)
\end{aligned}$$

$$z_t - \kappa_1 \Delta z + \gamma_2 \nabla \cdot \omega = S_4,$$

$$(\sigma, \omega, m, z)(x, t)|_{t=0} = (\sigma_0, \omega_0, m_0, z_0)(x),$$

where $S_1 = \tilde{S}_1$, $S_2 = \sqrt{P_1/\rho_\infty} \tilde{S}_2$, $S_3 = \sqrt{P_1/\rho_\infty} \tilde{S}_3$, $S_4 = \sqrt{P_2 \rho_\infty / P_1 P_3} \tilde{S}_4$, and

$$\begin{aligned}
(\sigma_0, \omega_0, m_0, z_0)(x) &= \left(\rho_0 \right. \\
& \left. - \rho_*, \sqrt{\frac{\rho_\infty}{P_1}} u_0, \sqrt{\frac{\rho_\infty}{P_1}} b_0, \sqrt{\frac{P_2 \rho_\infty}{P_1 P_3}} (\theta_0 - \theta_\infty) \right)(x) \longrightarrow \quad (29)
\end{aligned}$$

$$(0, 0, 0, 0),$$

$$\text{as } |x| \longrightarrow \infty.$$

Then, we will focus on considering the global existence and time decay rates of the solution (ρ, u, b, θ) to the steady state $(\rho_*, 0, 0, \theta_\infty)$, that is, the existence and decay rates of the perturbed solution (σ, ω, m, z) to problem (28). First of all, some inequalities are listed as follows for late use.

Lemma 3 (see [28]). *Let $f \in H^2(\mathbb{R}^3)$. Then*

$$\begin{aligned}
\text{(i)} \quad & \|f\|_{L^\infty} \leq C \|\nabla f\|^{1/2} \|\nabla f\|_{H^1}^{1/2} \leq C \|\nabla f\|_{H^1}; \\
\text{(ii)} \quad & \|f\|_{L^6} \leq C \|\nabla f\|; \\
\text{(iii)} \quad & \|f\|_{L^q} \leq C \|f\|_{H^1}, \quad 2 \leq q \leq 6.
\end{aligned} \quad (30)$$

Lemma 4 (see [24]). *Let $r_1, r_2 > 0$, then it holds that*

$$\begin{aligned}
& \int_0^t (1+t-\tau)^{-r_1} (1+\tau)^{-r_2} d\tau \\
& \leq C(r_1, r_2) (1+t)^{-\min\{r_1, r_2, r_1+r_2-1-\eta\}}, \quad (31)
\end{aligned}$$

for an arbitrarily small $\eta > 0$.

3. Global Existence

We will use the energy method to establish the global existence of solutions to the problem (28) in H^2 -framework in this section. Let us define the solution space and the solution norm of the initial value problem (25) by

$$\begin{aligned}
X(0, T) &= \{(\sigma, \omega, m, z) \mid \sigma \in C^0(0, T; H^2(\mathbb{R}^3)) \\
& \cap C^1(0, T; H^1(\mathbb{R}^3)), \omega, m, z \\
& \in C^0(0, T; H^2(\mathbb{R}^3)) \cap C^1(0, T; L^2(\mathbb{R}^3)), \nabla \sigma \\
& \in L^2(0, T; H^1(\mathbb{R}^3)), \nabla(\omega, m, z) \\
& \in L^2(0, T; H^2(\mathbb{R}^3))\},
\end{aligned}$$

$$N(0, T)^2 = \sup_{0 \leq t \leq T} \|(\sigma, \omega, m, z)(t)\|_2^2 + \int_0^T (\|\nabla \sigma(t)\|_1^2 + \|\nabla(\omega, m, z)(t)\|_2^2) dt, \quad (32)$$

where $T \in [0, +\infty]$. By the standard continuity argument, the global existence of solutions to (25) will be obtained from the combination of the local existence result with some a priori estimates.

Proposition 5 (local existence). *Suppose that the initial data satisfy $(\sigma_0, \omega_0, m_0, z_0) \in H^2(\mathbb{R}^3)$ and (7). Then there exists a positive constant $T_1 > 0$ depending on $\|(\sigma_0, \omega_0, m_0, z_0)\|_2$, such that the initial value problem (25) has a unique solution $(\sigma, \omega, m, z) \in X(0, T_1)$ which satisfies $N(0, T_1) \leq 2N(0, 0)$.*

Proof. The proof can be stated by using the standard iteration arguments. Refer to [2], for instance. $\square$

Proposition 6 (a priori estimates). *Suppose that the initial value problem (28) has a solution $(\sigma, \omega, m, z) \in X(0, T)$, where T is a positive constant. Under the assumptions of Theorem 1, there exist a positive constant C_0 and a small positive constant δ which are independent of T , such that if*

$$\sup_{0 \leq t \leq T} \|(\sigma, \omega, m, z)(t)\|_2 \leq \delta, \quad (33)$$

where $\delta > \max\{\varepsilon, \sqrt{C_0}\varepsilon\}$ and ε is given by (7), then it holds that, for any $t \in [0, T]$,

$$\begin{aligned} & \|(\sigma, \omega, m, z)(t)\|_2^2 \\ & + \int_0^t (\|\nabla \sigma(\tau)\|_1^2 + \|\nabla(\omega, m, z)(\tau)\|_2^2) d\tau \\ & \leq C_0 \|(\sigma_0, \omega_0, m_0, z_0)\|_2^2 < \delta^2, \end{aligned} \quad (34)$$

where C_0 is independent of δ .

Remark 7. The global existence and uniqueness of the solutions stated in Theorem 1 follow from Propositions 5 and 6. By (12), (33), and the Sobolev inequality, we have

$$\frac{1}{2}\rho_\infty \leq \sigma + \rho_* \leq 2\rho_\infty. \quad (35)$$

This will be used in following statement. Before proving Proposition 6, we supply Lemmas 8 and 9.

Lemma 8. *For $0 \leq k \leq 2$, it holds that*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla^k(\sigma, \omega, m, z)(t)\|^2 + \frac{\mu_1}{2} \|\nabla^k \nabla \omega(t)\|^2 \\ & + \frac{\mu_2}{2} \|\nabla^k \nabla \cdot \omega(t)\|^2 + \frac{\nu}{2} \|\nabla^k \nabla m(t)\|^2 \\ & + \frac{k_1}{2} \|\nabla^{k+1} z(t)\|^2 \\ & \leq C\delta \{ \|\nabla \sigma(t)\|_1^2 + \|\nabla(\omega, m, z)(t)\|_2^2 \}, \end{aligned} \quad (36)$$

for any $0 \leq t \leq T$.

Proof. For each multi-index α with $|\alpha| = k$, operating ∂_x^α to the first 4 equations of (28), then multiplying the resulting equations by $\partial_x^\alpha \sigma$, $\partial_x^\alpha \omega$, $\partial_x^\alpha m$, and $\partial_x^\alpha z$, respectively, summing up, and then integrating the result over $\mathbb{R}^3$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha(\sigma, \omega, m, z)(t)\|^2 + \mu_1 \|\nabla \partial_x^\alpha \omega(t)\|^2 \\ & + \mu_2 \|\nabla \cdot \partial_x^\alpha \omega(t)\|^2 + \nu \|\nabla \partial_x^\alpha m(t)\|^2 \\ & + \kappa_1 \|\nabla \partial_x^\alpha z(t)\|^2 = \langle \partial_x^\alpha \sigma(t), \partial_x^\alpha S_1(t) \rangle \\ & + \langle \partial_x^\alpha \omega(t), \partial_x^\alpha S_2(t) \rangle + \langle \partial_x^\alpha m(t), \partial_x^\alpha S_3(t) \rangle \\ & + \langle \partial_x^\alpha z(t), \partial_x^\alpha S_4(t) \rangle \triangleq I_1^k(t) + I_2^k(t) + I_3^k(t) \\ & + I_4^k(t), \end{aligned} \quad (37)$$

where the nonlinear terms in (22)-(25) have the following equivalence properties:

$$\begin{aligned} S_1 & \sim \partial_i \sigma \omega_i + \sigma \partial_i \omega_i + \partial_i \bar{\rho} \omega_i + \bar{\rho} \partial_i \omega_i, \\ S_2 & \sim \omega_i \partial_i \omega_j + \sigma \partial_i \partial_j \omega_j + \sigma \partial_j \partial_i \omega_i + \sigma \partial_j \sigma + z \partial_j \sigma \\ & + \sigma \partial_j z + z \partial_j z + \bar{\rho} \partial_i \partial_j \omega_j + \bar{\rho} \partial_j \partial_i \omega_i + \bar{\rho} \partial_j \sigma \\ & + \sigma \partial_j \bar{\rho} + z \partial_j \bar{\rho} + \bar{\rho} \partial_j z \\ & + (\sigma + \bar{\rho} + 1)(m_i \partial_i m_j + m_j \partial_i m_j), \\ S_3 & \sim \omega_i \partial_i m_j + m_i \partial_i \omega_j + m_j \partial_i \omega_i, \\ S_4 & \sim \omega_i \partial_i z + \sigma \partial_i \partial_j z + \sigma \partial_j \omega_i + z \partial_i \omega_i + \sigma \Psi[\omega] + \Psi[\omega] \\ & + \bar{\rho} \partial_i \partial_j z + \bar{\rho} \partial_j \omega_i + \bar{\rho} \Psi[\omega]. \end{aligned} \quad (38)$$

When $k = 0$, by using the Hölder inequality, Lemma 3, (12), the a priori assumption (33), and the Young inequality, we obtain

$$\begin{aligned} I_1^0(t) & \leq C \{ |\langle \sigma(t), (\partial_i \sigma \omega_i)(t) \rangle| \\ & + |\langle \sigma(t), (\sigma \partial_i \omega_i)(t) \rangle| + |\langle \sigma(t), (\partial_i \bar{\rho} \omega_i)(t) \rangle| \\ & + |\langle \sigma(t), (\bar{\rho} \partial_i \omega_i)(t) \rangle| \} \leq C \left\{ \|\sigma\|_{L^6} \|\partial_i \sigma\| \|\omega_i\|_{L^3} \right. \\ & + \|\sigma\|_{L^6} \|\sigma\|_{L^3} \|\partial_i \omega\| \\ & + \left\| \frac{\sigma}{1+|x|} \right\| \|(1+|x|) \partial_i \bar{\rho}\|_{L^3} \|\omega_i\|_{L^6} \\ & + \|\sigma\|_{L^6} \|\bar{\rho}\|_{L^3} \|\partial_i \omega_i\| \} \leq C \{ \|\nabla \sigma\|^2 \|\omega\|_1 \\ & + \|\nabla \sigma\| \|\sigma\|_1 \|\nabla \omega\| + \|\nabla \sigma\| \|(1+|x|) \partial_i \bar{\rho}\|_1 \|\nabla \omega\| \\ & + \|\nabla \sigma\| \|\bar{\rho}\|_1 \|\nabla \omega\| \} \leq C\delta \{ \|\nabla \sigma\|^2 + \|\nabla \omega\|^2 \}, \end{aligned} \quad (39)$$

where we have used the Hardy inequality as follows:

$$\left\| \frac{\sigma}{1+|x|} \right\| \leq \|\nabla \sigma\|. \quad (40)$$

In a similar way, we get

$$\begin{aligned} I_2^0(t) &\leq C\delta \left\{ \|\nabla\sigma\|^2 + \|\nabla\omega\|_1^2 + \|\nabla m\|^2 + \|\nabla z\|^2 \right\}, \\ I_3^0(t) &\leq C\delta \left\{ \|\nabla\omega\|^2 + \|\nabla m\|^2 \right\}, \\ I_4^0(t) &\leq C\delta \left\{ \|\nabla z\|_1^2 + \|\nabla\omega\|^2 \right\}. \end{aligned} \quad (41)$$

Putting (39), (41) into (37) and taking δ sufficiently small, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|(\sigma, \omega, m, z)(t)\|^2 + \frac{\mu_1}{2} \|\nabla\omega(t)\|^2 \\ &\quad + \frac{\mu_2}{2} \|\nabla \cdot \omega(t)\|^2 + \frac{\nu}{2} \|\nabla m(t)\|^2 + \frac{\kappa_1}{2} \|\nabla z(t)\|^2 \\ &\leq C\delta \left\{ \|\nabla\sigma(t)\|^2 + \|\nabla^2\omega(t)\|^2 + \|\nabla^2z(t)\|^2 \right\}. \end{aligned} \quad (42)$$

When $1 \leq k \leq 2$, the terms on the RHS of (37) can be estimated as follows. For $I_1^k(t)$, from (38), we get

$$\begin{aligned} I_1^k(t) &\leq C \left\{ |\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha (\partial_i \sigma \omega_i)(t) \rangle| \right. \\ &\quad + |\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha (\sigma \partial_i \omega_i)(t) \rangle| \\ &\quad + |\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha (\partial_i \bar{\rho} \omega_i)(t) \rangle| \\ &\quad \left. + |\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha (\bar{\rho} \partial_i \omega_i)(t) \rangle| \right\}. \end{aligned} \quad (43)$$

For the first term on the RHS of (43), by using integration by parts, the Hölder inequality, the Young inequality, Lemma 3, and the a priori assumption (33), we have

$$\begin{aligned} &|\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha (\partial_i \sigma \omega_i)(t) \rangle| \\ &\leq C |\langle \partial_x^\alpha \sigma, \partial_x^\alpha \partial_i \sigma \omega_i \rangle| \\ &\quad + C \sum_{\beta \leq \alpha, |\beta| \leq |\alpha|-1} C_\alpha^\beta |\langle \partial_x^\alpha \sigma, \partial_x^\beta \partial_i \sigma \partial_x^{\alpha-\beta} \omega_i \rangle| \\ &= \frac{C}{2} |\langle (\partial_x^\alpha \sigma)^2, \partial_i \omega_i \rangle| \\ &\quad + C \left\{ \sum_{|\beta|=0} + \sum_{1 \leq |\beta| \leq |\alpha|-1} \right\} |\langle \partial_x^\alpha \sigma, \partial_x^\beta \partial_i \sigma \partial_x^{\alpha-\beta} \omega_i \rangle| \\ &\leq C \|\partial_i \omega_i\|_{L^\infty} \|\partial_x^\alpha \sigma\|^2 + C \|\partial_x^\alpha \sigma\| \|\partial_i \sigma\|_{L^3} \|\partial_x^\alpha \omega_i\|_{L^6} \\ &\quad + \delta \|\partial_x^\alpha \sigma\|^2 + \frac{C}{\delta} \sum_{1 \leq |\beta| \leq |\alpha|-1} \|\nabla \partial_x^{\alpha-\beta} \omega_i\|^2 \|\partial_x^\beta \partial_i \sigma\|^2 \\ &\leq C\delta \|\partial_x^\alpha \sigma\| \|\nabla\omega\|_2 + C\delta \|\partial_x^\alpha \sigma\| \|\nabla \partial_x^\alpha \omega\| + \delta \|\partial_x^\alpha \sigma\|^2 \\ &\quad + C\delta \|\nabla\omega\|_1^2 \leq C\delta \left\{ \|\partial_x^\alpha \sigma\|^2 + \|\nabla\omega\|_2^2 \right\}, \end{aligned} \quad (44)$$

where the terms including the sum of β with $1 \leq |\beta| \leq |\alpha| - 1$ will be vanished if $|\alpha| = 1$. Similarly, we have

$$\begin{aligned} &|\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha (\sigma \partial_i \omega_i)(t) \rangle| \\ &= \left\{ \sum_{|\beta|=0} + \sum_{1 \leq |\beta| \leq |\alpha|-1} + \sum_{|\beta|=|\alpha|} \right\} C_\alpha^\beta |\langle \partial_x^\alpha \sigma, \\ &\quad \partial_x^{\alpha-\beta} \sigma \partial_x^\beta \partial_i \omega_i \rangle| \leq C \|\partial_i \omega_i\|_{L^\infty} \|\partial_x^\alpha \sigma\|^2 \\ &\quad + C \sum_{1 \leq |\beta| \leq |\alpha|} \|\partial_x^\alpha \sigma\| \|\partial_x^{\alpha-\beta} \sigma\|_{L^6} \|\partial_x^\beta \partial_i \omega_i\|_{L^3} + C \|\partial_x^\alpha \sigma\| \\ &\quad \cdot \|\sigma\|_{L^\infty} \|\partial_x^\alpha \partial_i \omega\| \leq C\delta \|\partial_x^\alpha \sigma\| \|\partial_i \omega_i\|_2 + \delta \|\partial_x^\alpha \sigma\|^2 \\ &\quad + \frac{C}{\delta} \sum_{1 \leq |\beta| \leq |\alpha|-1} \|\partial_x^{\alpha-\beta} \nabla \sigma\|^2 \|\partial_x^\beta \partial_i \omega_i\|_1^2 + C\delta \|\partial_x^\alpha \sigma\| \\ &\quad \cdot \|\partial_x^\alpha \partial_i \omega_i\| \leq C\delta \left\{ \|\partial_x^\alpha \sigma\|^2 + \|\nabla\omega\|_2^2 \right\}. \end{aligned} \quad (45)$$

By the Hölder inequality, the Young inequality, Lemma 3, and (12), the third term on the RHS of (43) can be estimated as follows:

$$\begin{aligned} &|\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha (\partial_i \bar{\rho} \omega_i)(t) \rangle| \\ &\leq \sum_{|\beta| \leq |\alpha|} |\langle \partial_x^\alpha \sigma, \partial_x^\beta \partial_i \bar{\rho} \partial_x^{\alpha-\beta} \omega_i \rangle| \\ &\leq C \sum_{|\beta| \leq |\alpha|} \|\partial_x^\alpha \sigma\| \|\partial_x^\beta \partial_i \bar{\rho}\|_{L^3} \|\partial_x^{\alpha-\beta} \omega_i\|_{L^6} \\ &\leq \delta \|\partial_x^\alpha \sigma\|^2 + \frac{C}{\delta} \sum_{|\beta| \leq |\alpha|} \|\partial_x^\beta \partial_i \bar{\rho}\|_1^2 \|\partial_x^{\alpha-\beta} \nabla \omega_i\|^2 \\ &\leq C\delta \left\{ \|\partial_x^\alpha \sigma\|^2 + \|\nabla\omega\|_2^2 \right\}. \end{aligned} \quad (46)$$

By a similar argument, the fourth term on the RHS of (43) has following estimations:

$$\begin{aligned} &|\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha (\bar{\rho} \partial_i \omega_i)(t) \rangle| \\ &= \left\{ \sum_{|\beta|=0} + \sum_{1 \leq |\beta| \leq |\alpha|} \right\} |\langle \partial_x^\alpha \sigma, \partial_x^\beta \bar{\rho} \partial_x^{\alpha-\beta} \partial_i \omega_i \rangle| \\ &\leq C \|\partial_x^\alpha \sigma\| \|\bar{\rho}\|_{L^\infty} \|\partial_x^\alpha \partial_i \omega_i\| \\ &\quad + C \sum_{1 \leq |\beta| \leq |\alpha|} \|\partial_x^\alpha \sigma\| \|\partial_x^\beta \bar{\rho}\|_{L^3} \|\partial_x^{\alpha-\beta} \partial_i \omega_i\|_{L^6} \\ &\leq C\delta \left\{ \|\partial_x^\alpha \sigma\|^2 + \|\nabla\omega\|_2^2 \right\}. \end{aligned} \quad (47)$$

Therefore, putting (44)-(47) into (43) yields

$$I_1^k(t) \leq C\delta \left\{ \|\partial_x^\alpha \sigma\|^2 + \|\nabla\omega\|_2^2 \right\}. \quad (48)$$

For $I_2^k(t)$, from (38), it is obvious that

$$\begin{aligned}
 I_2^k(t) &\leq \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\omega_i \partial_i \omega_j) \rangle \right| + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\sigma \partial_i \partial_i \omega_j) \rangle \right| \\
 &\quad + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\sigma \partial_j \partial_i \omega_i) \rangle \right| + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\sigma \partial_j \sigma) \rangle \right| \\
 &\quad + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (z \partial_j \sigma) \rangle \right| + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\sigma \partial_j z) \rangle \right| \\
 &\quad + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (z \partial_j z) \rangle \right| + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\bar{\rho} \partial_i \partial_i \omega_j) \rangle \right| \quad (49) \\
 &\quad + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\bar{\rho} \partial_j \partial_i \omega_i) \rangle \right| + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\bar{\rho} \partial_j \sigma) \rangle \right| \\
 &\quad + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\sigma \partial_j \bar{\rho}) \rangle \right| + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (z \partial_j \bar{\rho}) \rangle \right| \\
 &\quad + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (\bar{\rho} \partial_j z) \rangle \right| \\
 &\quad + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha ((\sigma + \bar{\rho} + 1)(m_j \partial_i m_j + m_i \partial_i m_j)) \rangle \right|.
 \end{aligned}$$

For the second term on the RHS of (49), based on integration by parts, the Hölder inequality, Lemma 3, and (33), we have

$$\begin{aligned}
 &\left| \langle \partial_x^\alpha \omega(t), \partial_x^\alpha (\sigma \partial_i \partial_i \omega_j) \rangle \right| \\
 &= \left\{ \sum_{|\beta|=0} + \sum_{1 \leq |\beta| \leq |\alpha|} \right\} \left| \langle \partial_x^\alpha \omega, \partial_x^\beta \sigma \partial_x^{\alpha-\beta} \partial_i \partial_i \omega_j \rangle \right| \\
 &\leq \left| \langle (\partial_x^\alpha \partial_i \omega)^2, \sigma \rangle \right| + \left| \langle \partial_x^\alpha \omega, \partial_i \sigma \partial_x^\alpha \partial_i \omega_j \rangle \right| \\
 &\quad + \sum_{1 \leq |\beta| \leq |\alpha|-1} \left| \langle \partial_x^\alpha \omega, \partial_x^\beta \sigma \partial_x^{\alpha-\beta} \partial_i \partial_i \omega_j \rangle \right| \\
 &\quad + \left| \langle \partial_x^\alpha \omega, \partial_x^\alpha \sigma \partial_i \partial_i \omega_j \rangle \right| \quad (50) \\
 &\leq C \|\sigma\|_{L^\infty} \|\partial_x^\alpha \partial_i \omega\|^2 + \|\partial_x^\alpha \omega\|_{L^6} \|\partial_i \sigma\|_{L^3} \|\partial_x^\alpha \partial_i \omega_j\| \\
 &\quad + \sum_{1 \leq |\beta| \leq |\alpha|-1} \|\partial_x^\alpha \omega\|_{L^6} \|\partial_x^\beta \sigma\|_{L^3} \|\partial_x^{\alpha-\beta} \partial_i \partial_i \omega_j\| \\
 &\quad + \|\partial_x^\alpha \omega\|_{L^6} \|\partial_x^\alpha \sigma\| \|\partial_i \partial_i \omega_j\|_{L^3} \\
 &\leq C\delta \left\{ \|\partial_x^\alpha \nabla \omega\|^2 + \|\nabla^2 \omega\|_1^2 \right\}.
 \end{aligned}$$

For the 11th term on the RHS of (49), by using the Hölder inequality, the Hardy inequality, Lemma 3, and (12), we obtain

$$\begin{aligned}
 &\left| \langle \partial_x^\alpha \omega(t), \partial_x^\alpha (\sigma \partial_j \bar{\rho}) \rangle \right| \\
 &= \left\{ \sum_{|\beta|=0} + \sum_{1 \leq |\beta| \leq |\alpha|} \right\} \left| \langle \partial_x^\alpha \omega(t), \partial_x^\beta \sigma \partial_x^{\alpha-\beta} \partial_j \bar{\rho} \rangle \right| \\
 &\leq C \|\partial_x^\alpha \omega\|_{L^6} \|(1+|x|) \partial_x^\alpha \partial_j \bar{\rho}\|_{L^3} \left\| \frac{\sigma}{1+|x|} \right\|
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{1 \leq |\beta| \leq |\alpha|} \|\partial_x^\alpha \omega\|_{L^6} \|\partial_x^\beta \sigma\| \|\partial_x^{\alpha-\beta} \partial_j \bar{\rho}\|_{L^3} \leq C\delta \|\partial_x^\alpha \nabla \omega\| \\
 &\cdot \|\nabla \sigma\| + C\delta \sum_{1 \leq |\beta| \leq |\alpha|} \|\partial_x^\alpha \nabla \omega\| \|\partial_x^\beta \sigma\| \leq C\delta \left\{ \|\partial_x^\alpha \nabla \omega\|^2 \right. \\
 &\quad \left. + \|\nabla \sigma\|_1^2 \right\}, \\
 &\left| \langle \partial_x^\alpha \omega(t), \partial_x^\alpha (\sigma m_j \partial_i m_j) \rangle \right| \\
 &= \left\{ \sum_{|\beta|=0} + \sum_{1 \leq |\beta| \leq |\alpha|} \right\} \left| \langle \partial_x^\alpha \omega(t), \right. \\
 &\quad \left. \partial_x^\beta \sigma \partial_x^{\alpha-\beta} (\sigma m_j \partial_i m_j) \rangle \right| \leq C \|\sigma\|_{L^\infty} \|\partial_x^\alpha (m_j \partial_i m_j)\| \\
 &\quad \cdot \|\partial_x^\alpha \omega(t)\| + \sum_{1 \leq |\beta| \leq |\alpha|} \|\partial_x^\alpha \omega\|_{L^6} \|\partial_x^\beta \sigma\| \\
 &\quad \cdot \|\partial_x^{\alpha-\beta} (m_j \partial_i m_j)\|_{L^3} \leq C \|\sigma\|_{L^\infty} \|m\|_{L^\infty} \|\partial_x^\alpha \omega\| \|\partial_x^\alpha \\
 &\quad \nabla m\| + \sum_{1 \leq |\beta| \leq |\alpha|} \|\partial_x^\alpha \nabla \omega\| \|m\|_{L^6} \|\partial_x^\beta \sigma\| \|\partial_x^{\alpha-\beta} \nabla m\|_{L^6} \\
 &\leq C\delta \left\{ \|\partial_x^\alpha \nabla \omega\|^2 + \|\partial_x^\alpha \nabla m\|^2 \right\}, \quad (51)
 \end{aligned}$$

where we used (33). Similarly,

$$\begin{aligned}
 &\left| \langle \partial_x^\alpha \omega(t), \partial_x^\alpha (\sigma m_i \partial_i m_j) \rangle \right| \\
 &\leq C\delta \left\{ \|\partial_x^\alpha \nabla \omega\|^2 + \|\partial_x^\alpha \nabla m\|^2 \right\}, \quad (52) \\
 &\left| \langle \partial_x^\alpha \omega, \partial_x^\alpha (m_j \partial_i m_j + m_i \partial_i m_j) \rangle \right| \\
 &\leq C\delta \left\{ \|\partial_x^\alpha \nabla \omega\|^2 + \|\partial_x^\alpha \nabla m\|^2 \right\}.
 \end{aligned}$$

The other terms on the RHS of (49) can be estimated in some similar way.

$$\begin{aligned}
 I_2^k(t) &\leq C\delta \left\{ \|\nabla \sigma\|_1^2 + \|\partial_x^\alpha \omega\|^2 + \|\partial_x^\alpha \nabla \omega\|^2 + \|\partial_x^\alpha \nabla m\|^2 \right. \\
 &\quad \left. + \|\nabla \omega\|_2^2 + \|\nabla z\|_1^2 \right\}. \quad (53)
 \end{aligned}$$

Now, let us estimate $I_3^k(t)$, $I_4^k(t)$. Similar to the estimate before, $I_3^k(t)$ and $I_4^k(t)$ hold that

$$I_3^k(t) \leq C\delta \left\{ \|\partial_x^\alpha m\|^2 + \|\partial_x^\alpha \nabla m\|^2 \right\}, \quad (54)$$

$$I_4^k(t) \leq C\delta \left\{ \|\partial_x^\alpha z\|^2 + \|\partial_x^\alpha \nabla z\|^2 + \|\nabla \omega\|_2^2 + \|\nabla z\|_2^2 \right\}. \quad (55)$$

Thus (37) together with (48), (53), (54), and (55) yields

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha (\sigma, \omega, m, z)(t)\|^2 + \frac{\mu_1}{2} \|\nabla \partial_x^\alpha \omega(t)\|^2 \\
 &\quad + \frac{\mu_2}{2} \|\nabla \cdot \partial_x^\alpha \omega(t)\|^2 + \frac{\nu}{2} \|\nabla \partial_x^\alpha m(t)\|^2 \\
 &\quad + \frac{\kappa_1}{2} \|\nabla \partial_x^\alpha z(t)\|^2
 \end{aligned}$$

$$\begin{aligned} &\leq C\delta \left\{ \|\partial_x^\alpha(\sigma, \omega, m, z)\|^2 + \|\partial_x^\alpha \nabla(\omega, m, z)\|^2 \right. \\ &\quad \left. + \|\nabla \sigma\|_1^2 + \|\nabla(\omega, m, z)\|_2^2 \right\}, \end{aligned} \quad (56)$$

where we have taken $\delta > 0$ sufficiently small. Therefore, (42) and (56) give (36). $\square$

Now, the following lemma will give the dissipative estimates for $\nabla \sigma$.

Lemma 9. For $0 \leq k \leq 1$, it holds that

$$\begin{aligned} &\frac{d}{dt} \langle \nabla^k \omega(t), \nabla^k \nabla \sigma(t) \rangle + \frac{\gamma_1}{2} \|\nabla^k \nabla \sigma(t)\|^2 \\ &\leq C \left\{ \|\nabla^k \nabla \omega(t)\|_1^2 + \|\nabla^k \nabla z(t)\|_2^2 \right\} \\ &\quad + C\delta \left\{ \|\nabla \sigma(t)\|_1^2 + \|\nabla(\omega, m, z)(t)\|_2^2 \right\}, \end{aligned} \quad (57)$$

for any $0 \leq t \leq T$.

Proof. From (28)₂, we get

$$\gamma_1 \nabla \sigma = -\omega_t + \mu_1 \Delta \omega + \mu_2 \nabla \nabla \cdot \omega - \gamma_2 \nabla z + S_2. \quad (58)$$

By applying $\partial_x^\alpha (|\alpha| = k)$ to (58), multiplying it by $\nabla \partial_x^\alpha \sigma$, and then integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} \gamma_1 \|\nabla \partial_x^\alpha \sigma\|^2 &= -\langle \partial_x^\alpha \omega_t, \nabla \partial_x^\alpha \sigma \rangle + \mu_1 \langle \partial_x^\alpha \Delta \omega, \nabla \partial_x^\alpha \sigma \rangle \\ &\quad + \mu_2 \langle \partial_x^\alpha \nabla \nabla \cdot \omega, \nabla \partial_x^\alpha \sigma \rangle \\ &\quad - \gamma_2 \langle \partial_x^\alpha \nabla z, \nabla \partial_x^\alpha \sigma \rangle + \langle \partial_x^\alpha S_2, \nabla \partial_x^\alpha \sigma \rangle. \end{aligned} \quad (59)$$

Using integration by parts and (28)₁, we obtain

$$\begin{aligned} -\langle \partial_x^\alpha \omega_t, \nabla \partial_x^\alpha \sigma \rangle &= -\frac{d}{dt} \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha \sigma \rangle + \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha \sigma_t \rangle \\ &= -\frac{d}{dt} \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha \sigma \rangle \\ &\quad - \gamma_1 \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha \nabla \cdot \omega \rangle \\ &\quad + \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha S_1 \rangle. \end{aligned} \quad (60)$$

Adding (59) and (60) gives

$$\begin{aligned} &\gamma_1 \|\nabla \partial_x^\alpha \sigma\|^2 + \frac{d}{dt} \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha \sigma \rangle \\ &= \mu_1 \langle \partial_x^\alpha \Delta \omega, \nabla \partial_x^\alpha \sigma \rangle + \mu_2 \langle \partial_x^\alpha \nabla \nabla \cdot \omega, \nabla \partial_x^\alpha \sigma \rangle \\ &\quad - \gamma_2 \langle \partial_x^\alpha \nabla z, \nabla \partial_x^\alpha \sigma \rangle + \langle \partial_x^\alpha S_2, \nabla \partial_x^\alpha \sigma \rangle \\ &\quad - \gamma_1 \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha \nabla \cdot \omega \rangle + \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha S_1 \rangle. \end{aligned} \quad (61)$$

Then by using the Young inequality and integration by parts, it follows from the above equality that

$$\begin{aligned} &\frac{3\gamma_1}{4} \|\nabla \partial_x^\alpha \sigma\|^2 + \frac{d}{dt} \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha \sigma \rangle \\ &\leq C \|\partial_x^\alpha \nabla^2 \omega\|^2 + \|\partial_x^\alpha \nabla \cdot \omega\|^2 + \|\partial_x^\alpha \nabla z\|^2 \\ &\quad + C |\langle \nabla \partial_x^\alpha \omega, \partial_x^\alpha S_1 \rangle| + C |\langle \nabla \partial_x^\alpha \sigma, \partial_x^\alpha S_2 \rangle|. \end{aligned} \quad (62)$$

Similar to the estimates of $I_1^k(t)$ and $I_2^k(t)$ in the proof of Lemma 8, we have

$$|\langle \nabla \partial_x^\alpha \omega, \partial_x^\alpha S_1 \rangle| \leq C \|\nabla \partial_x^\alpha \omega\|^2 + C\delta (\|\nabla \sigma\|^2 + \|\nabla \omega\|_2^2), \quad (63)$$

$$\begin{aligned} |\langle \nabla \partial_x^\alpha \sigma, \partial_x^\alpha S_2 \rangle| &\leq C\delta (\|\nabla \partial_x^\alpha \sigma\|^2 + \|\nabla \sigma\|_1^2 + \|\nabla \omega\|_2^2 \\ &\quad + \|\nabla m\|_2^2 + \|\nabla z\|_1^2). \end{aligned} \quad (64)$$

Putting (63) and (64) into (62) and taking $\delta > 0$ sufficiently small yield

$$\begin{aligned} &\frac{\gamma_1}{2} \|\nabla \partial_x^\alpha \sigma\|^2 + \frac{d}{dt} \langle \partial_x^\alpha \omega, \nabla \partial_x^\alpha \sigma \rangle \\ &\leq C (\|\partial_x^\alpha \nabla \omega\|_1^2 + \|\partial_x^\alpha \nabla z\|^2) \\ &\quad + C\delta (\|\nabla \sigma\|_1^2 + \|\nabla \omega\|_2^2 + \|\nabla m\|_2^2 + \|\nabla z\|_1^2). \end{aligned} \quad (65)$$

Thus, the proof of Lemma 9 is finished. $\square$

Let $0 \leq l \leq 1$. By summing up (36) from $k = l$ to 2, since δ is small, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} &\frac{d}{dt} \sum_{l \leq k \leq 2} \|\nabla^k(\sigma, \omega, m, z)(t)\|^2 \\ &\quad + \sum_{l \leq k \leq 2} \|\nabla^{k+1}(\omega, m, z)(t)\|^2 \leq C_1 \delta \|\nabla \sigma(t)\|_1^2. \end{aligned} \quad (66)$$

Summing up (57) for from $k = l$ to 1, since δ is small, we have

$$\begin{aligned} &\frac{d}{dt} \sum_{l \leq k \leq 1} \langle \nabla^k \omega(t), \nabla^k \nabla \sigma(t) \rangle + C_2 \sum_{l \leq k \leq 1} \|\nabla^k \nabla \sigma(t)\|^2 \\ &\leq C \left\{ \sum_{l \leq k \leq 2} \|\nabla^{k+1} \omega(t)\|^2 + \sum_{l \leq k \leq 1} \|\nabla^{k+1} z(t)\|^2 \right. \\ &\quad \left. + \delta \|\nabla(\sigma, \omega, m, z)(t)\|^2 \right\}, \end{aligned} \quad (67)$$

where $C_2 > 0$ is a constant. Multiplying (67) by $2C_1\delta/C_2 < 1/2$, adding it with (66), taking $l = 0$, and using the smallness of δ , we have

$$\begin{aligned} &\frac{d}{dt} \left\{ \sum_{0 \leq k \leq 2} \|\nabla^k(\sigma, \omega, m, z)(t)\|^2 \right. \\ &\quad \left. + 2\frac{C_1\delta}{C_2} \sum_{0 \leq k \leq 1} \langle \nabla^k \omega(t), \nabla^k \nabla \sigma(t) \rangle \right\} \\ &\quad + C \left\{ \sum_{0 \leq k \leq 1} \|\nabla^{k+1} \sigma(t)\|^2 \right. \\ &\quad \left. + \sum_{0 \leq k \leq 2} \|\nabla^{k+1}(\omega, m, z)(t)\|^2 \right\} \leq 0. \end{aligned} \quad (68)$$

Integrating (68) with respect to t and using the smallness of δ , we get (34). The proof of Proposition 6 is complete.

4. Decay Rates

In this section, we get decay rates of solutions to problem (25). Let $\mathbb{A}$ be the following 8×8 matrix of the differential operators of the form

$$\mathbb{A} = \begin{pmatrix} 0 & \gamma_1 \operatorname{div} & 0 & 0 \\ \gamma_1 \nabla & -\mu_1 \Delta - \mu_2 \nabla \operatorname{div} & 0 & \gamma_2 \nabla \\ 0 & 0 & -\nu \Delta & 0 \\ 0 & \gamma_2 \operatorname{div} & 0 & -\kappa_1 \Delta \end{pmatrix}. \quad (69)$$

If we set

$$\begin{aligned} \widetilde{U} &= (\widetilde{\sigma}, \widetilde{\omega}, \widetilde{m}, \widetilde{z})^t, \\ U(0) &= (\sigma_0, \omega_0, m_0, z_0)^t, \end{aligned} \quad (70)$$

the linearized problem of (25) is written as

$$\begin{aligned} \widetilde{U}_t + \mathbb{A} \widetilde{U} &= 0, \quad \text{for } t > 0, \\ \widetilde{U}|_{t=0} &= U(0). \end{aligned} \quad (71)$$

By taking the Fourier transform of (71) with respect to the x -variable and solving the ordinary differential equation with respect to t , we have

$$\widetilde{U}(t) = E(t) U(0), \quad t \geq 0, \quad (72)$$

where $E(t) = e^{-t\mathbb{A}} (t \geq 0)$ is the semigroup generated by the linear operator $-\mathbb{A}$. We define $E(t)f = \mathcal{F}^{-1}(e^{-t\widehat{\mathbb{A}}} \widehat{\xi} \widehat{f}(\xi, t))$ for function $f(x, t)$ with

$$\begin{aligned} \widehat{\mathbb{A}}(\xi) &= \begin{pmatrix} 0 & \sqrt{-1}\gamma_1 \xi^t & 0 & 0 \\ \sqrt{-1}\gamma_1 \xi & \mu_1 |\xi|^2 + \mu_2 \xi \xi^t & 0 & \sqrt{-1}\gamma_2 \xi \\ 0 & 0 & \nu |\xi|^2 & 0 \\ 0 & \sqrt{-1}\gamma_2 \xi^t & 0 & \overline{\kappa} |\xi|^2 \end{pmatrix}, \\ &\text{for } \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}. \end{aligned} \quad (73)$$

In what following, we denote the solutions to the nonlinear problem (25) by

$$U(t) = (\sigma(t), \omega(t), m(t), z(t))^t. \quad (74)$$

Then (28) can be rewritten as follows:

$$\begin{aligned} U_t + \mathbb{A}U &= S(U), \quad \text{for } t > 0, \\ U|_{t=0} &= U(0), \end{aligned} \quad (75)$$

where $S(U) \triangleq (S_1, S_2, S_3, S_4)^t$. By using the Duhamel principle, we have

$$U(t) = E(t)U_0 + \int_0^t E(t-\tau)S(U)(\tau)d\tau, \quad t \geq 0. \quad (76)$$

Let $\varphi_0(\xi)$ be a function in $C_0^\infty(\mathbb{R}^3)$ such that

$$\varphi_0(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \frac{R}{2}, \\ 0, & \text{for } |\xi| \geq R, \end{cases} \quad (77)$$

where R is some fixed positive constant. Based on the Fourier transform and (77), we can define the low frequency and the high frequency decomposition $(f_L(x), f_H(x))$ for $f(x)$ as follows:

$$\begin{aligned} f_L &= \mathcal{F}^{-1}(\varphi_0(\xi) \widehat{f}(\xi)), \\ f_H &= f - f_L. \end{aligned} \quad (78)$$

By noticing definitions (77) and (78), using the Plancherel theorem, we can obtain the following estimates:

$$\|\nabla f\| \leq C(\|\nabla f_L\| + \|\nabla f_H\|), \quad (79)$$

$$C\|\nabla f_H\| \leq \|\nabla^k f_H\|,$$

$$C\|\nabla^k f_H\| \leq \|\nabla^k f\|, \quad (80)$$

for $k \geq 1$.

Denote $U_L(t) \triangleq \mathcal{F}^{-1}(\varphi_0(\xi) \widehat{U}(\xi, t))$ and $E_L(t)f \triangleq \mathcal{F}^{-1}(\varphi_0(\xi) e^{-t\widehat{\mathbb{A}}(\xi)} \widehat{f}(\xi, t))$ as the definition (78).

Then from (76), we have

$$U_L(t) = E_L(t)U_0 + \int_0^t E_L(t-\tau)S(U)(\tau)d\tau. \quad (81)$$

To estimate $U_L(t)$, we need the $L^p - L^q$ type of the time decay estimates on the low-frequency part of the semigroup $E(t)$. Notice that the first, the second, and the fourth equations in (71) have been studied in [29]. The magnetic field $H(t, x)$ satisfies the heat equation, which can be solved in Fourier space, $\widehat{H} = \exp(-|\xi|^2 t) \widehat{H}_0$, whose decay estimate has been proved in many papers, for instance, [26, 27]. Then the semigroup $E(t)$ has the following properties on the decay in time.

Lemma 10 (see [29]). *Let $k \geq 0$ be integers and $1 \leq p \leq 2 \leq q \leq \infty$, then for any $t > 0$, it holds that*

$$\begin{aligned} \|\nabla^k E_L(t)f\| &\leq C(k, p, q)(1+t)^{-(3/2)(1/p-1/q)-k/2} \|f\|_{L^p}. \end{aligned} \quad (82)$$

4.1. Decay Estimates of the Low-Frequency Part. In this subsection, we have the decay estimates of $U_L(t)$ with the $L^p - L^q$ type estimate (82) as follows.

Lemma 11. For $k \geq 0$, it holds that

$$\begin{aligned} & \|\nabla^k U_L(t)\| \\ & \leq C(1+t)^{-(3/2)(1/p-1/2)-k/2} K_0 \\ & \quad + C(\delta + \varepsilon) \int_0^t (1+t-\tau)^{-3/4-k/2} \|\nabla U(\tau)\|_1 d\tau, \end{aligned} \quad (83)$$

where $K_0 = \|U_0\|_{L^p}$ is finite.

Proof. From (81) and Lemma 10, we have

$$\begin{aligned} & \|\nabla^k U_L(t)\| \\ & \leq C(1+t)^{-(3/2)(1/p-1/2)-k/2} \|U_0\|_{L^p} \\ & \quad + C \int_0^t (1+t-\tau)^{3/4-k/2} \|S(U)(\tau)\|_{L^1} d\tau. \end{aligned} \quad (84)$$

For the terms including $\bar{\rho}$ on the RHS of (84), by using the Hölder inequality, (12), and the Hardy inequality, we get

$$\begin{aligned} \|\nabla \bar{\rho} \cdot \omega\|_{L^1} & \leq \|(1+|x|) \nabla \bar{\rho}\| \left\| \frac{\omega}{1+|x|} \right\| \leq C\varepsilon \|\nabla \omega\|, \\ \|\bar{\rho} \nabla \cdot \omega\|_{L^1} & \leq \|\bar{\rho}\| \|\nabla \cdot \omega\| \leq C\varepsilon \|\nabla \omega\|. \end{aligned} \quad (85)$$

Thus, it holds that

$$\|\nabla \cdot (\bar{\rho} \omega)\|_{L^1} \leq C\varepsilon \|\nabla \omega\|. \quad (86)$$

Similarly, the above inequalities give that

$$\begin{aligned} \|\bar{\rho} \left(m \cdot \nabla m - \frac{1}{2} \nabla |m|^2 \right)\|_{L^1} & \leq \varepsilon \|\nabla m\|_1, \\ \|\nabla (\bar{\rho} \sigma)\|_{L^1} & \leq C\varepsilon \|\nabla \sigma\|_1, \\ \|\nabla (\bar{\rho} z)\|_{L^1} & \leq C\varepsilon \|\nabla z\|_1, \\ \|\bar{\rho} \nabla \cdot \omega\|_{L^1} & \leq C\varepsilon \|\nabla^2 \omega\|, \\ \|\bar{\rho} \Delta z\|_{L^1} & \leq C\varepsilon \|\nabla^2 z\|. \end{aligned} \quad (87)$$

The other terms can be estimated easily. Then, from (38) and the a priori assumption (33), we have

$$\begin{aligned} \|S(U)(t)\|_{L^1} & \leq C \left(\|(\sigma, \omega, m)\|_1 \|\nabla(\sigma, \omega, m, z)\|_1 \right. \\ & \quad + \|\nabla \cdot (\bar{\rho} \omega)\|_{L^1} + \|\nabla (\bar{\rho} \sigma)\|_{L^1} \\ & \quad + \left\| \bar{\rho} \left(m \cdot \nabla m - \frac{1}{2} \nabla |m|^2 \right) \right\|_{L^1} + \|\nabla (\bar{\rho} z)\|_{L^1} \\ & \quad \left. + \|\bar{\rho} \nabla^2(\omega, z)\|_{L^1} \right) \leq C(\delta + \varepsilon) \|\nabla(\sigma, \omega, m, z)\|_1. \end{aligned} \quad (88)$$

Thus, putting (88) into (84), we get (83). $\square$

4.2. Decay Rates for the Nonlinear Problem. In this subsection, from the a priori estimates and the low frequency and high frequency decomposition (78), we derive a new Lyapunov-type inequality, firstly. From (66) and (67), by taking $l = 1$, similar to the estimate (68), we get

$$\begin{aligned} & \frac{d\mathcal{H}(t)}{dt} + \|\nabla^2(\sigma, \omega, m, z)(t)\|^2 + \|\nabla^2(\omega, m, z)(t)\|_1^2 \\ & \leq C\delta \|\nabla(\sigma, \omega, m, z)(t)\|^2, \end{aligned} \quad (89)$$

where $\mathcal{H}(t) = \sum_{1 \leq k \leq 2} \|\nabla^k(\sigma, \omega, m, z)(t)\|^2 + \langle \nabla \omega(t), \nabla^2 \sigma(t) \rangle$ is an energy functional which is equivalent to $\|\nabla(\sigma, \omega, z)(t)\|_1^2$; that means, there exists a positive constant C_3 such that

$$\begin{aligned} & \frac{1}{C_3} \|\nabla(\sigma, \omega, m, z)(t)\|_1^2 \leq \mathcal{H}(t) \\ & \leq C_3 \|\nabla(\sigma, \omega, m, z)(t)\|^2. \end{aligned} \quad (90)$$

By using (79) and (80), from (89), there exists a constant $C_4 > 0$ such that

$$\begin{aligned} & \frac{d\mathcal{H}(t)}{dt} t + C_4 \|\nabla(\sigma_H, \omega_H, m_H, z_H)(t)\|^2 \\ & \quad + \frac{1}{2} \|\nabla^2(\sigma, \omega, m, z)(t)\|^2 + \frac{1}{2} \|\nabla^2(\omega, m, z)(t)\|_1^2 \\ & \leq C\delta \left\{ \|\nabla(\sigma_L, \omega_L, m_L, z_L)(t)\|^2 \right. \\ & \quad \left. + \|\nabla(\sigma_H, \omega_H, m_H, z_H)(t)\|^2 \right\}. \end{aligned} \quad (91)$$

It follows from (91) and the smallness of δ that

$$\begin{aligned} & \frac{d\mathcal{H}(t)}{dt} + C_5 \|\nabla(\sigma_H, \omega_H, m_H, z_H)(t)\|^2 \\ & \quad + \frac{1}{2} \|\nabla^2(\sigma, \omega, m, z)(t)\|^2 + \frac{1}{2} \|\nabla^2(\omega, m, z)(t)\|_1^2 \\ & \leq C\delta \|\nabla(\sigma_L, \omega_L, m_L, z_L)(t)\|^2, \end{aligned} \quad (92)$$

where $C_5 > 0$ is a constant. Adding $C_5 \|\nabla(\sigma_L, \omega_L, m_L, z_L)(t)\|^2$ to the both sides of (92), with the definition of $\mathcal{H}(t)$, there exists a constant $D > 0$ such that

$$\frac{d\mathcal{H}(t)}{dt} + D\mathcal{H}(t) \leq C \|\nabla U_L(t)\|^2. \quad (93)$$

To prove the decay estimates stated in Theorem 1, we first define

$$M(t) = \sup_{0 \leq \tau \leq t} (1+\tau)^{3(1/p-1/2)+1} \mathcal{H}(\tau). \quad (94)$$

$M(t)$ is obviously nondecreasing and

$$\begin{aligned} \|\nabla U(\tau)\|_1 & \leq C\sqrt{\mathcal{H}(\tau)} \\ & \leq C(1+\tau)^{-(3/2)(1/p-1/2)-1/2} \sqrt{M(t)}, \end{aligned} \quad (95)$$

$$0 \leq \tau \leq t.$$

Then, from (83), by using Lemma 4 and (95), we obtain

$$\begin{aligned} \|\nabla U_L(t)\| &\leq CK_0(1+t)^{-(3/2)(1/p-1/2)-1/2} + C(\delta + \varepsilon) \\ &\cdot \int_0^t (1+t-\tau)^{-5/4} \\ &\cdot (1+\tau)^{-(3/2)(1/p-1/2)-1/2} d\tau \sqrt{M(t)} \\ &\leq C(1+t)^{-(3/2)(1/p-1/2)-1/2} (K_0 + (\delta + \varepsilon) \sqrt{M(t)}). \end{aligned} \quad (96)$$

By Gronwall's inequality and (96), from (93), hence

$$\begin{aligned} \mathcal{H}(t) &\leq \mathcal{H}(0)e^{-Dt} + C_2 \int_0^t e^{-D(t-\tau)} \|\nabla U_L(\tau)\|^2 d\tau \leq \mathcal{H}(0)e^{-Dt} \\ &+ C \int_0^t e^{-D(t-\tau)} (1+\tau)^{-(3/2)(1/p-1/2)-1} d\tau (K_0^2 + (\delta + \varepsilon)^2 M(t)) \quad (97) \\ &\leq (1+t)^{-(3/2)(1/p-1/2)-1} (\mathcal{H}(0) + K_0^2 + (\delta + \varepsilon)^2 M(t)), \end{aligned}$$

where $\mathcal{H}(0)$ is equivalent to $\|\nabla(\sigma_0, \omega_0, m_0, z_0)\|_1^2$.

In terms of the definition of $M(t)$, it follows from (97) that

$$M(t) \leq C(\mathcal{H}(0) + K_0^2) + C(\delta + \varepsilon)^2 M(t), \quad (98)$$

which implies that if δ and ε are sufficiently small then

$$M(t) \leq C(\mathcal{H}(0) + K_0^2). \quad (99)$$

By applying the definition of $\mathcal{H}(t)$ and $M(t)$, we have

$$\|\nabla(\sigma, \omega, m, z)(t)\|_1 \leq C(1+t)^{-(3/2)(1/p-1/2)-1/2}. \quad (100)$$

Then, we have (9).

Then, we define the temporal energy functional

$$\begin{aligned} \mathcal{H}_1(t) &= \|(\sigma, \omega, m, z)(t)\|_2^2 \\ &+ \frac{2C_1\delta}{C_2} \sum_{0 \leq k \leq 2} \langle \nabla^k \sigma(t), \nabla^k \nabla \omega(t) \rangle, \end{aligned} \quad (101)$$

where $\mathcal{H}_1(t)$ is equivalent to $\|(\sigma, \omega, m, z)(t)\|_2^2$, for the positive constants δ can be sufficiently small. Then, from (68), we have

$$\begin{aligned} \frac{d\mathcal{H}_1(t)}{dt} + \|\nabla(\sigma, \omega, m, z)(t)\|^2 + \|\nabla(\omega, m, z)(t)\|_1^2 \\ \leq 0. \end{aligned} \quad (102)$$

Similar to the estimate (93), from (102), there exists a positive constant D_1 such that

$$\frac{d\mathcal{H}_1(t)}{dt} + D_1 \mathcal{H}_1(t) \leq C \|U_L(t)\|^2. \quad (103)$$

Taking $k = 0$ in (83) and using (100), for $1 \leq p \leq 2$, we obtain

$$\begin{aligned} \|U_L(t)\| &\leq C(1+t)^{-(3/2)(1/p-1/2)} K_0 + C(\delta + \varepsilon) \\ &\cdot \int_0^t (1+t-\tau)^{-3/4} (1+\tau)^{-(3/2)(1/p-1/2)-1/2} d\tau \\ &\leq C(1+t)^{-(3/2)(1/p-1/2)} (K_0 + C), \end{aligned} \quad (104)$$

where let $r_1 = 3/4$, $r_2 = (3/2)(1/p-1/2) + 1/2$ and $\eta = 1/4$ in Lemma 4.

From (103), by using Gronwall's inequality and (104), we get

$$\begin{aligned} \mathcal{H}_1(t) &\leq \mathcal{H}_1(0)e^{-D_1 t} + C \int_0^t e^{-D_1(t-\tau)} \|U_L(\tau)\|^2 d\tau \\ &\leq \mathcal{H}(0)e^{-D_1 t} \\ &+ (C + CK_0^2) \int_0^t e^{-D_1(t-\tau)} (1+\tau)^{-3(1/p-1/2)} d\tau \\ &\leq C(1+t)^{-3(1/p-1/2)} (\mathcal{H}_1(0) + C + K_0^2), \end{aligned} \quad (105)$$

where $\mathcal{H}_1(0)$ is equivalent to $\|(\sigma_0, \omega_0, m_0, z_0)\|_2^2$. Then, it follows from the definition of $\mathcal{H}_1(t)$ that

$$\|(\sigma, \omega, m, z)(t)\| \leq C(1+t)^{-(3/2)(1/p-1/2)}. \quad (106)$$

Finally, by the interpolation, for any $2 \leq q \leq 6$, plugging (106) into (100) yields

$$\begin{aligned} \|U(t)\|_{L^q} &\leq \|U(t)\|_{L^6}^{\eta_0} \|U(t)\|^{1-\eta_0} \\ &\leq C \|\nabla U(t)\|^{\eta_0} \|U(t)\|^{1-\eta_0} \\ &\leq C(1+t)^{-(3/2)(1/p-1/q)}, \end{aligned} \quad (107)$$

where $\eta_0 = 3(q-2)/2q$. Then, we get (10).

On the other hand, by using (25), Lemma 3, (9), and (12), we have

$$\begin{aligned} \|\partial_t(\sigma, \omega, m, z)(t)\|_{L^2} &\leq C \|\nabla \cdot \omega(t)\|_{L^2} + \|S_1(t)\|_{L^2} \\ &+ C \|\nabla \sigma(t)\|_{L^2} + C \|\nabla z(t)\|_{L^2} + C \|\Delta \omega(t)\|_{L^2} \\ &+ C \|\nabla \nabla \cdot \omega(t)\|_{L^2} + \|S_2(t)\|_{L^2} + C \|\Delta m(t)\|_{L^2} \\ &+ \|S_3(t)\|_{L^2} + C \|\Delta z(t)\|_{L^2} + \|S_4(t)\|_{L^2} \\ &\leq C \{ \|\nabla \sigma(t)\|_{L^2} + \|\nabla \omega(t)\|_{H^1} + \|\nabla m(t)\|_{H^1} \\ &+ \|\nabla z(t)\|_{H^1} \} \leq C(1+t)^{-(3/2)(1/p-1/2)-1/2}, \end{aligned} \quad (108)$$

for any $0 \leq t \leq T$.

This completes the proof of Theorem 1.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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