Research Article

Iterative Methods for Computing the Resolvent of the Sum of a Maximal Monotone Operator and Composite Operator with Applications

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Total variation image denoising models have received considerable attention in the last two decades. To solve constrained total variation image denoising problems, we utilize the computation of a resolvent operator, which consists of a maximal monotone operator and a composite operator. More precisely, the composite operator consists of a maximal monotone operator and a bounded linear operator. Based on recent work, in this paper we propose a fixed-point approach for computing this resolvent operator. Under mild conditions on the iterative parameters, we prove strong convergence of the iterative sequence, which is based on the classical Krasnoselskii–Mann algorithm in general Hilbert spaces. As a direct application, we obtain an effective iterative algorithm for solving the proximity operator of the sum of two convex functions, one of which is the composition of a convex function with a linear transformation. Numerical experiments on image denoising are presented to illustrate the efficiency and effectiveness of the proposed iterative algorithm. In particular, we report the numerical results for the proposed algorithm with different step sizes and relaxation parameters.

1. Introduction

In the last two decades, the total variation (TV) image denoising model proposed by Rudin, Osher, and Fatemi [1] has received considerable attention. Often referred to as the ROF model, this takes the form

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - u\|^2 + \mu \|x\|_{TV}, \quad (1)$$

where $u \in \mathbb{R}^n$ is the observed noisy image, $\|x\|_{TV}$ is the total variation, and $\mu > 0$ is the regularization parameter, which balances the data fidelity term and total variation regularization. In the following, we denote by $x \in \mathbb{R}^n$ the vectorized image $X \in \mathbb{R}^{M \times N}$, where $n = MN$. Because TV regularization has the advantage of maintaining image edges when removing noise, it has been extended to many other important image processing problems, including image

deblurring [2–4], image inpainting [5], image superresolution [6], image segmentation [7], and image reconstruction [8, 9].

Many efficient iterative algorithms have been proposed to solve the ROF model (1). These include the Chambolle gradient projection algorithm and its variants [10–12], the primal-dual hybrid gradient algorithm [13–15], and the split Bregman algorithm [16, 17]. Isotropic total variation (ITV) and anisotropic total variation (ATV) are the most widely employed methods in the literature. It is worth mentioning that TV includes ITV and ATV, which can both be viewed as compositions of a convex function φ with a linear transformation B . That is, $\|x\|_{TV} = \varphi(Bx)$. For specific examples, see [18, 19]. Based on this, Micchelli et al. [20] extended the ROF model (1) to the general convex optimization problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - u\|^2 + \varphi(Bx), \quad (2)$$

where $u \in R^n$, $\phi : R^m \rightarrow (-\infty, +\infty]$ is a proper lower semicontinuous convex function and $B : R^n \rightarrow R^m$ is a linear transformation. The convex optimization problem (2) is equivalent to computing the proximity operator of the function $\phi \circ B$. Recall that the proximity operator of a proper lower semicontinuous convex function $f : R^n \rightarrow (-\infty, +\infty]$ is defined as

$$\text{prox}_f(u) = \arg \min_{x \in R^n} \left\{ \frac{1}{2} \|x - u\|^2 + f(x) \right\}. \quad (3)$$

Thus, Micchelli et al. [20] proved that the above minimization problem (2) can be solved using a fixed-point equation. The advantage of the fixed-point framework is that it provides a convenient analysis of the convergence of the proposed algorithm and enables the development of efficient numerical algorithms via various fixed-point iterations.

Because the pixel values of grayscale images are generally distributed in $[0, 255]$ or $[0, 1]$, it is natural to incorporate this prior information into the ROF model (1). Thus, we obtain the following constrained ROF model:

$$\min_{x \in C} \frac{1}{2} \|x - u\|^2 + \mu \|x\|_{TV}, \quad (4)$$

where C is a nonempty closed and convex set, which could be chosen as the pixel range of the images. To solve the constrained ROF model (4), it is sufficient to consider the following constrained convex optimization problem:

$$\min_{x \in C} \frac{1}{2} \|x - u\|^2 + \phi(Bx). \quad (5)$$

In general, the constrained convex optimization problem (5) can always be reformulated as an unconstrained convex optimization problem. More precisely,

$$\min_{x \in R^n} \frac{1}{2} \|x - u\|^2 + \phi(Bx) + \delta_C(x), \quad (6)$$

where δ_C denotes the indicator function of C , which is defined by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise. As the fixed-point algorithm proposed by Micchelli et al. [20] cannot be applied to solve the problem in (6), this motivated us to develop an efficient iterative algorithm for this purpose. Hence, we apply this algorithm to the constrained total variation denoising model (4).

As the indicator function δ_C belongs to the class of proper lower semicontinuous convex functions, we are motivated to solve the following general convex optimization problem in general Hilbert spaces:

$$\min_{x \in H} \frac{1}{2} \|x - u\|^2 + \phi(Bx) + g(x). \quad (7)$$

where g is a proper lower semicontinuous convex function. To achieve this goal, let us recall the definition of the resolvent operator. Let H be a real Hilbert space and let $T : H \rightarrow 2^H$ be a maximal monotone operator. The resolvent of T is the single-valued operator $J_T = (I + T)^{-1}$. Next, let $f : H \rightarrow (-\infty, +\infty]$ be a proper lower semicontinuous convex

function and let ∂f denote the subdifferential of f , and let $T = \partial f$. Thus, the resolvent of T coincides with the proximity operator prox_f as follows:

$$J_{\partial f}(u) = \text{prox}_f(u). \quad (8)$$

Under certain qualification conditions, the problem considered in (7) can be solved via the resolvent operator $J_{B^* \circ \partial \phi \circ B + \partial g}$. Overall, this leads to the problem of computing the resolvent operator of $S + B^*TB$. More precisely, let $u \in H$ and

$$x := J_{S+B^*TB}(u), \quad (9)$$

where $S : H \rightarrow 2^H$ and $T : G \rightarrow 2^G$ are two maximal monotone operators and $B : H \rightarrow G$ is a bounded linear operator from the Hilbert space H to the Hilbert space G . Following this, let $S = \partial g$ and $T = \partial \phi$. Hence, the problem (9) reduces to the convex optimization problem (7), because $J_{\partial g + B^* \circ \partial \phi \circ B}(u) = \text{prox}_{g + \phi \circ B}(u)$.

1.1. Existing Work. Next, let us briefly review some existing work concerning the computation of resolvent operators. Bauschke and Combettes [21] extended the Dykstra algorithm [22] for computing projections onto the intersection of two closed convex sets to compute the resolvent of the sum of two maximal monotone operators. Hence, they obtained an algorithm for finding the proximity operator of the sum of two proper lower semicontinuous convex functions. Combettes [23] proposed two inexact parallel splitting algorithms for computing the resolvent of a weighted sum of maximal monotone operators. The key idea was to reformulate the weighted sum of maximal monotone operators as a sum of two maximal monotone operators in a product space. The two iterative algorithms were based on extensions of the Douglas-Rachford splitting and Dykstra-like methods, respectively. Furthermore, Combettes [23] applied these algorithms when computing the proximity operator of a weighted sum of proper lower semicontinuous convex functions. In more recent work, Artacho and Campoy [24] generalized the averaged alternating modified reflection algorithm [25] to compute the resolvent of the sum of two maximal monotone operators.

In contrast, Moudafi [26] proposed a fixed-point algorithm to compute the resolvent of operator B^*TB , where $B : H \rightarrow G$ is a bounded linear operator with the adjoint B^* , $T : G \rightarrow 2^G$ is a maximal monotone operator, and H and G are two Hilbert spaces. When $T = \partial \phi$, this algorithm utilizes the fixed-point approach to computing the proximity operator $\text{prox}_{\phi \circ B}$ proposed by Micchelli et al. [20]. It is clear that the resolvent of the operator B^*TB is a special case of (9) by allowing $S = 0$. In addition, Combettes et al. [27] proposed a dual forward-backward splitting algorithm to solve the convex optimization problem in (7) (see also [28]). The main idea in this case was to first derive the dual problem of (7) and then apply the forward-backward splitting algorithm to solve this problem. Finally, the convergence of the primal iterative sequence was proved using the connection between the primal optimal solution and the dual optimal solution. However, the authors did not apply the obtained iterative

algorithm to solve the constrained ROF model (4). Beck and Teboulle [3] solved the constrained ROF model (4) as a subproblem of image deblurring. Chan et al. [29] applied the alternating direction method of multipliers to solve the constrained total variation image deblurring problem. Their numerical results confirmed that the quality of the restored image could be improved by incorporating prior information of the images as a constraint. However, neither study considered the general resolvent operator of (9).

In this study, we focus on computing the resolvent of the operator $S + B^*TB$ (9), where $S : H \rightarrow 2^H$ and $T : G \rightarrow 2^G$ are two maximal monotone operators and $B : H \rightarrow G$ is a bounded linear operator. We assume that the operator $S + B^*TB$ is maximally monotone. This is true, for example, if $0 \in \text{ri}(B(\text{dom } S) - \text{dom } T)$ [30], where $\text{ri}(C)$ represents a relative interior of the set C . If $S = 0$, then the problem (9) becomes that of computing the resolvent operator of B^*TB . Inspired by the work of Moudafi [26] and Combettes et al. [27], we propose a fixed-point approach to computing resolvent operators (9). First, we show that the resolvent operator (9) can be characterized using a fixed-point equation. Subsequently, we propose a fixed-point algorithm and prove the strong convergence of its iterative sequence. Next, we employ the proposed iterative algorithm to solve the convex optimization problem (7) arising in the field of image denoising. Finally, we conduct numerical experiments on image denoising to validate the effectiveness of the proposed algorithm. In particular, we show how the performance of the algorithm is influenced by the selection of the step size and relaxation parameters.

The remainder of this paper is organized as follows. In Section 2, we introduce some notation and present useful definitions and lemmas. In Section 3, we present the main fixed-point algorithm and prove its strong convergence. In Section 4, we employ the obtained iterative algorithm to solve a particular convex optimization problem, which is related to the calculation of the resolvent operator (9). In Section 5, we present some numerical experiments on image denoising to illustrate the performance of our proposed algorithm. Finally, we provide some conclusions and ideas for future work in Section 6.

2. Preliminaries

In this section, we review some basic definitions and lemmas in monotone operator theory and convex analysis, which will be used throughout this paper. First, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Let I denote the identity operator on H . We use $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x and $x^k \rightarrow x$ to indicate that the sequence $\{x^k\}$ converges strongly to x . Let $L(H, G)$ denote all bounded linear operators from the Hilbert space H to the Hilbert space G . Let $B \in L(H, G)$. Then, the adjoint of B is the unique operator $B^* \in L(G, H)$ such that $\langle Bx, y \rangle = \langle x, B^*y \rangle$.

Let $A : H \rightarrow 2^H$ be a set-valued operator. The sets $\text{dom } A = \{x \in H \mid Ax \neq \emptyset\}$, $\text{ran } A = \{u \in H \mid \exists x \in H, u \in Ax\}$, $\text{gra } A = \{(x, u) \in H \times H \mid u \in Ax\}$, and $\text{zer } A = \{x \in$

$H \mid 0 \in Ax\}$ represent the domain, range, graph, and zero set of A , respectively. The inverse A^{-1} of A is the set-valued operator with the graph $\{(u, x) \in H \times H \mid u \in Ax\}$.

We now introduce some definitions and lemmas, most of which can be found in [28, 31].

Definition 1 (see [28], (maximal monotone operator)). Let $A : H \rightarrow 2^H$ be a set-valued operator. A is said to be monotone if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall (x, u) \in \text{gra } A, (y, v) \in \text{gra } A. \quad (10)$$

Moreover, A is maximally monotone if its graph is not strictly contained in the graph of any other monotone operator.

Definition 2 (see [28], (resolvent and Yosida approximation)). Let $A : H \rightarrow 2^H$ and $\gamma > 0$. The resolvent of γA is $J_{\gamma A} = (I + \gamma A)^{-1}$, and the Yosida approximation of γA is ${}^\gamma A = (1/\gamma)(I - J_{\gamma A})$.

For any $x \in H$, it is easy to derive that ${}^\gamma Ax \in A(J_{\gamma A}x)$.

Lemma 3 (see [28]). Let $A : H \rightarrow 2^H$ be maximally monotone and $\gamma > 0$. Then, the following hold:

- (i) $J_{\gamma A} : H \rightarrow H$, and $I - \gamma_{\gamma A} : H \rightarrow H$ are firmly nonexpansive and maximally monotone.
- (ii) The Yosida approximation ${}^\gamma A$ is γ -cocoercive and maximally monotone.

Definition 4 (see [28]). Let C be a nonempty subset of H and $T : C \rightarrow C$. Then

- (i) T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C$.
- (ii) T is firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in C$.
- (iii) T is α -averaged if there exists a nonexpansive S and $\alpha \in (0, 1)$ such that $T = (1 - \alpha)I + \alpha S$, where I denotes the identity operator.
- (iv) T is β -cocoercive (or β -inversely strong monotone) for $\beta > 0$ if $\langle x - y, Tx - Ty \rangle \geq \beta \|Tx - Ty\|^2 \quad \forall x, y \in C$.

Remark 5. (i) An equivalent definition of firm nonexpansiveness is that $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle \quad \forall x, y \in C$. It is easy to check that firm nonexpansiveness implies nonexpansiveness. If T is firmly nonexpansive, then T is 1-cocoercive.

(ii) An equivalent definition of α -averaged is $\|Tx - Ty\|^2 \leq \|x - y\|^2 - ((1 - \alpha)/\alpha) \|(I - T)x - (I - T)y\|^2$ for all $x, y \in C$. Thus, if T is α -averaged then T is also nonexpansive.

(iii) Let T be β -cocoercive. Then, for any $\gamma > 0$ γT is β/γ -cocoercive. Furthermore, owing to the Cauchy-Schwartz inequality T is $1/\beta$ -Lipschitz continuous; i.e., $\|Tx - Ty\| \leq (1/\beta)\|x - y\|$.

The following lemma provides some useful characterizations between an operator T and $I - T$.

Lemma 6 (see [28]). Let C be a nonempty subset of H and let $T : C \rightarrow C$. Then

- (i) T is nonexpansive if and only if $I - T$ is $1/2$ -cocoercive.
- (ii) T is firmly nonexpansive if and only if $I - T$ is firmly nonexpansive.
- (iii) T is $1/2\nu$ -averaged if and only if $I - T$ is ν -cocoercive, where $\nu > 1/2$.

The following lemma shows that a composition of two averaged operators is also an average. This result first appeared in the work of Ogura and Yamada [32]. Combettes and Yamada [33] subsequently confirmed it with a different proof.

Lemma 7 (see [32]). Let C be a nonempty subset of H . Furthermore, let $T_1 : C \rightarrow C$ be α_1 -averaged and let $T_2 : C \rightarrow C$ be α_2 -averaged. Then $T_1 T_2$ is $(\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2)/(1 - \alpha_1\alpha_2)$ -averaged.

We also make full use of the following lemma.

Lemma 8 (see [28]). Let $x \in H$ and $y \in H$. Let $\alpha \in \mathbb{R}$ and R denote the set of real numbers. Then

$$\begin{aligned} \|\alpha x + (1 - \alpha)y\|^2 + \alpha(1 - \alpha)\|x - y\|^2 \\ = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2. \end{aligned} \quad (11)$$

We end this section by introducing the Krasnoselskii–Mann algorithm. Theorem 9 provides a fundamental tool for studying the convergence of many operator splitting methods.

Theorem 9 (see [28], (Krasnoselskii–Mann algorithm)). Let C be a nonempty closed convex subset of H , and let $T : C \rightarrow C$ be a nonexpansive operator such that $\text{Fix}(T) \neq \emptyset$, where $\text{Fix}(T)$ denotes the fixed-point set of T . Let $\lambda_k \in (0, 1)$ be such that $\sum_{k=0}^{\infty} \lambda_k(1 - \lambda_k) = +\infty$. For any $x_0 \in C$, define

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k T x_k. \quad (12)$$

Then, the following hold:

- (i) $\{x_k\}$ is Fejér monotone with respect to $\text{Fix}(T)$; i.e., $\|x_{k+1} - p\| \leq \|x_k - p\|$ for any $p \in \text{Fix}(T)$.
- (ii) $\|x_k - T x_k\|$ converges strongly to 0.
- (iii) $\{x_k\}$ converges weakly to a fixed point in $\text{Fix}(T)$.

3. Computing the Resolvent Operator (9)

Before presenting our main results, we first introduce some notation. For a fixed $u \in H$, let $\gamma > 0$ define operators $F : G \rightarrow G$ and $Q : G \rightarrow G$ as follows:

$$F(y) = y + B J_S(u - \gamma B^* y), \quad \forall y \in G, \quad (13)$$

and

$$Q(y) = (I - J_{(1/\gamma)T})F(y), \quad \forall y \in G. \quad (14)$$

In addition, the following lemma provides a fixed-point characterization of the resolvent operator (9).

Lemma 10. Let $S : H \rightarrow 2^H$ and $T : G \rightarrow 2^G$ be two maximal monotone operators. Furthermore, let $B \in L(H, G)$ and $u \in H$. Then, for a given $\gamma > 0$,

$$J_{S+B^*TB}(u) = J_S(u - \gamma B^* y) \quad (15)$$

if and only if y is a fixed point of Q .

Proof. It follows from the definition of the resolvent operator that $J_{S+B^*TB}(u) = (I + S + B^*TB)^{-1}(u)$. Thus, we can write

$$J_{S+B^*TB}(u) \in u - (S + B^*TB)J_{S+B^*TB}(u). \quad (16)$$

Let y satisfy (15). Then we have $u - \gamma B^* y \in (I + S)J_{S+B^*TB}(u)$; that is $J_{S+B^*TB}(u) \in u - (\gamma B^* y + S J_{S+B^*TB}(u))$. By (16), we have $y \in (1/\gamma)T B J_{S+B^*TB}(u)$. Thus, $y \in (1/\gamma)T B J_S(u - \gamma B^* y)$. Using the fact that $y \in (1/\gamma)T x \iff x = J_{(1/\gamma)T}(x + y)$, we deduce that

$$B J_S(u - \gamma B^* y) = J_{(1/\gamma)T}(B J_S(u - \gamma B^* y) + y), \quad (17)$$

which implies that y is a fixed point of Q .

Conversely, let y be a fixed point of Q . Then, we have that

$$\begin{aligned} y &= (I - J_{(1/\gamma)T})(y + B J_S(u - \gamma B^* y)) \iff \\ B J_S(u - \gamma B^* y) &= J_{(1/\gamma)T}(y + B J_S(u - \gamma B^* y)) \iff \\ y + B J_S(u - \gamma B^* y) &\in \left(I + \frac{1}{\gamma}T\right)(B J_S(u - \gamma B^* y)) \iff \\ \gamma y &\in T B J_S(u - \gamma B^* y) \implies \\ u &\in u - \gamma B^* y + B^* T B J_S(u - \gamma B^* y) \iff \\ u &\in (I + S)J_S(u - \gamma B^* y) + B^* T B J_S(u - \gamma B^* y) \iff \\ J_{S+B^*TB}(u) &= J_S(u - \gamma B^* y). \end{aligned} \quad (18)$$

This completes the proof. $\square$

Next, we prove the following lemma, which characterizes an important property of the operator F .

Lemma 11. Let $S : H \rightarrow 2^H$ be a maximal monotone operator, and let $B \in L(H, G)$. For a given $u \in H$ and $\gamma > 0$, define an operator $R : G \rightarrow G : y \mapsto -B J_S(u - \gamma B^* y)$. Then

- (i) R is $1/\gamma\|B\|^2$ -cocoercive.
- (ii) For any $\gamma \in (0, 2/\|B\|^2)$, $F = I - R$ is $\gamma\|B\|^2/2$ -averaged.

Proof. (i) Let $y_1, y_2 \in G$. Then, we have that

$$\begin{aligned} \langle y_1 - y_2, R(y_1) - R(y_2) \rangle &= \langle y_1 - y_2, \\ &\quad -B J_S(u - \gamma B^* y_1) + B J_S(u - \gamma B^* y_2) \rangle \\ &= \frac{1}{\gamma} \langle (u - \gamma B^* y_1) - (u - \gamma B^* y_2), J_S(u - \gamma B^* y_1) \\ &\quad - J_S(u - \gamma B^* y_2) \rangle \geq \frac{1}{\gamma} \|J_S(u - \gamma B^* y_1) \end{aligned}$$

$$\begin{aligned}
- J_S(u - \gamma B^* y_2) \|^2 &\geq \frac{1}{\gamma \|B\|^2} \|(-BJ_S(u - \gamma B^* y_1)) \\
- (BJ_S(u - \gamma B^* y_2)) \|^2 &= \frac{1}{\gamma \|B\|^2} \|R(y_1) \\
- R(y_2) \|^2.
\end{aligned} \tag{19}$$

The first inequality follows from the fact that the resolvent operator is firmly nonexpansive, and the second is trivial. Thus, R is $1/\gamma\|B\|^2$ -cocoercive.

(ii) For any $\gamma \in (0, 2/\|B\|^2)$, we know from Lemma 6(iii) that $F = I - R$ is $\gamma\|B\|^2/2$ -averaged. $\square$

Lemma 11 shows that, for any $\gamma \in (0, 2/\|B\|^2)$, F is $\gamma\|B\|^2/2$ -averaged. On the other hand, $I - J_{(1/\gamma)T}$ is firmly nonexpansive and is also $1/2$ -averaged. According to Lemma 7, $Q = (I - J_{(1/\gamma)T}) \circ F$ is $2/(4 - \gamma\|B\|^2)$ -averaged.

Now, we are ready to present our main results.

Theorem 12. Let $S : H \rightarrow 2^H$ and $T : G \rightarrow 2^G$ be two maximal monotone operators. Furthermore, let $B \in L(H, G)$ and $u \in H$. For any $y_0 \in G$, we define the iterative sequences $\{x_k\}$ and $\{y_k\}$ as follows:

$$\begin{aligned}
x_k &= J_S(u - \gamma B^* y_k), \\
y_{k+1} &= (1 - \lambda_k) y_k + \lambda_k (I - J_{(1/\gamma)T})(y_k + Bx_k),
\end{aligned} \tag{20}$$

where $\gamma \in (0, 2/\|B\|^2)$ and $\lambda_k \in (0, (4 - \gamma\|B\|^2)/2)$ such that $\sum_{k=0}^{\infty} \lambda_k (1 - (2/(4 - \gamma\|B\|^2))\lambda_k) = +\infty$. Then, the following hold:

- (i) $\{y_k\}$ converges weakly to a fixed point of Q .
- (ii) Suppose that $\inf \lambda_k > 0$. Then, $\{x_k\}$ converges strongly to the resolvent operator (9).

Proof. (i) Because the resolvent operator $J_{S+B^*TB}(u)$ exists and is unique, the first part of the Lemma 10 implies that Q has a fixed point. Let $Q(y_k) = (I - J_{(1/\gamma)T})(y_k + BJ_S(u - \gamma B^* y_k))$. Then, the iterative sequence $\{y_{k+1}\}$ can then be rewritten as follows:

$$y_{k+1} = (1 - \lambda_k) y_k + \lambda_k Q y_k. \tag{21}$$

Because Q is $2/(4 - \gamma\|B\|^2)$ -averaged, there exists a nonexpansive mapping U such that $Q = (1 - 2/(4 - \gamma\|B\|^2))I + (2/(4 - \gamma\|B\|^2))U$ and $\text{Fix}(U) = \text{Fix}(Q)$. Therefore, the iterative sequence (21) is equivalent to

$$\begin{aligned}
y_{k+1} &= (1 - \lambda_k) y_k \\
&+ \lambda_k \left(\left(1 - \frac{2}{4 - \gamma\|B\|^2} \right) y_k + \frac{2}{4 - \gamma\|B\|^2} U y_k \right) \\
&= \left(1 - \frac{2}{4 - \gamma\|B\|^2} \lambda_k \right) y_k + \frac{2}{4 - \gamma\|B\|^2} \lambda_k U y_k.
\end{aligned} \tag{22}$$

Notice that $(2/(4 - \gamma\|B\|^2))\lambda_k \in (0, 1)$ and $\sum_{k=0}^{\infty} \lambda_k (1 - (2/(4 - \gamma\|B\|^2))\lambda_k) = +\infty$. By Theorem 9, we can conclude the following:

- (1) $\{y_k\}$ is Fejér monotone with respect to $\text{Fix}(U)$.
- (2) $\|y_k - U y_k\| \rightarrow 0$ as $k \rightarrow \infty$.
- (3) $\{y_k\}$ converges weakly to a fixed point of U .

Taking into account the fact that $\text{Fix}(U) = \text{Fix}(Q)$, $\{y_k\}$ also converges weakly to a fixed point of Q , and $\{y_k\}$ is Fejér monotone with respect to $\text{Fix}(Q)$. Therefore, $\lim_{k \rightarrow \infty} \|y_k - y\|$ exists for any $y \in \text{Fix}(Q)$. Further, we have that $\|y_k - Q y_k\| = \|(2/(4 - \gamma\|B\|^2))(y_k - U y_k)\| \rightarrow 0$, as $k \rightarrow \infty$.

(ii) Let $y \in \text{Fix}(Q)$. By Lemma 11 (ii) and Remark 5 (ii), we then have that

$$\begin{aligned}
\|Q y_k - y\|^2 &= \|(I - J_{(1/\gamma)T})(y_k + BJ_S(u - \gamma B^* y_k)) \\
&- (I - J_{(1/\gamma)T})(y + BJ_S(u - \gamma B^* y))\|^2 \\
&\leq \|(y_k + BJ_S(u - \gamma B^* y_k)) \\
&- (y + BJ_S(u - \gamma B^* y))\|^2 \leq \|y_k - y\|^2 \\
&- \frac{1 - \alpha}{\alpha} \|-BJ_S(u - \gamma B^* y_k) \\
&- (-BJ_S(u - \gamma B^* y))\|^2,
\end{aligned} \tag{23}$$

where $\alpha = \gamma\|B\|^2/2$. From Lemma 8, we have that

$$\begin{aligned}
\|y_{k+1} - y\|^2 &= \|(1 - \lambda_k)(y_k - y) + \lambda_k(Q y_k - y)\|^2 \\
&= (1 - \lambda_k) \|y_k - y\|^2 + \lambda_k \|Q y_k - y\|^2 \\
&- \lambda_k (1 - \lambda_k) \|y_k - Q y_k\|^2.
\end{aligned} \tag{24}$$

Substituting (23) into (24), we derive

$$\begin{aligned}
\|y_{k+1} - y\|^2 &\leq \|y_k - y\|^2 \\
&- \frac{1 - \alpha}{\alpha} \lambda_k \|BJ_S(u - \gamma B^* y_k) - BJ_S(u - \gamma B^* y)\|^2 \\
&- \lambda_k (1 - \lambda_k) \|y_k - Q y_k\|^2,
\end{aligned} \tag{25}$$

which implies that

$$\begin{aligned}
\frac{1 - \alpha}{\alpha} \lambda_k \|BJ_S(u - \gamma B^* y_k) - BJ_S(u - \gamma B^* y)\|^2 \\
\leq \|y_k - y\|^2 - \|y_{k+1} - y\|^2 \\
+ \lambda_k (\lambda_k - 1) \|y_k - Q y_k\|^2.
\end{aligned} \tag{26}$$

Because $\inf \lambda_k > 0$, $\lim_{k \rightarrow \infty} \|y_k - y\|$ exists, and $\lim_{k \rightarrow \infty} \|y_k - Q y_k\| = 0$. Taking $k \rightarrow \infty$ in the above inequality (26), we obtain

$$\lim_{k \rightarrow \infty} \|BJ_S(u - \gamma B^* y_k) - BJ_S(u - \gamma B^* y)\| = 0. \tag{27}$$

By Lemma 10, $J_{S+B^*TB}(u) = J_S(u - \gamma B^* y)$, where $y \in \text{Fix}(Q)$. Then, we have that

$$\begin{aligned} \|x_k - J_S(u - \gamma B^* y)\|^2 &= \|J_S(u - \gamma B^* y_k) \\ &\quad - J_S(u - \gamma B^* y)\|^2 \leq \langle (u - \gamma B^* y_k) \\ &\quad - (u - \gamma B^* y), J_S(u - \gamma B^* y_k) - J_S(u - \gamma B^* y) \rangle \\ &= \gamma \langle B^* y - B^* y_k, J_S(u - \gamma B^* y_k) \\ &\quad - J_S(u - \gamma B^* y) \rangle = \gamma \langle y - y_k, BJ_S(u - \gamma B^* y_k) \\ &\quad - BJ_S(u - \gamma B^* y) \rangle \leq \gamma \|y - y_k\| \|BJ_S(u - \gamma B^* y_k) \\ &\quad - BJ_S(u - \gamma B^* y)\|. \end{aligned} \quad (28)$$

The first inequality follows from the fact that the resolvent operator is firmly nonexpansive, and the final inequality is derived from the Cauchy-Schwarz inequality. Taking $k \rightarrow \infty$ on the right of (28), we have that $\|x_k - J_S(u - \gamma B^* y)\| \rightarrow 0$. Thus, we can conclude that the iterative sequence $\{x_k\}$ converges strongly to the resolvent operator of $J_{S+B^*TB}(u)$. This completes the proof. $\square$

Remark 13. We observe that $(4 - \gamma\|B\|^2)/2 > 1$ for any $\gamma \in (0, 2/\|B\|^2)$. Taking $\lambda_k = 1$ in (20), we obtain a simple iterative algorithm to compute the resolvent operator problem (9). More precisely, for any $y_0 \in G$ the iterative sequences $\{x_k\}$ and $\{y_k\}$ are generated as follows:

$$\begin{aligned} x_k &= J_S(u - \gamma B^* y_k), \\ y_{k+1} &= (I - J_{(1/\gamma)T})(y_k + Bx_k), \end{aligned} \quad (29)$$

where $\gamma \in (0, 2/\|B\|^2)$. The iterative algorithm (29) is equivalent to the Picard iteration scheme, which can be easily applied in practice. In fact, the iterative scheme (29) can be rewritten as

$$y_{k+1} = (I - J_{(1/\gamma)T})(y_k + BJ_S(u - \gamma B^* y_k)). \quad (30)$$

With the help of the relation $J_{\gamma T} + \gamma J_{\gamma^{-1}T^{-1}} \circ \gamma^{-1}I = I$, we obtain from (30) that

$$y_{k+1} = \frac{1}{\gamma} J_{\gamma T^{-1}}(\gamma(y_k + BJ_S(u - \gamma B^* y_k))). \quad (31)$$

Let $\bar{y}_k = \gamma y_k$. Then, the iterative scheme (31) is equivalent to

$$\bar{y}_{k+1} = J_{\gamma T^{-1}}(\bar{y}_k + \gamma BJ_S(u - B^* \bar{y}_k)). \quad (32)$$

Remark 14. We observe that the resolvent operator of $S + B^*TB$ in (9) is equivalent to the following monotone inclusion problem:

$$\begin{aligned} \text{find } x \in H, \\ \text{such that } 0 \in x - u + Sx + B^*TBx, \end{aligned} \quad (33)$$

where u , S , T , and B are the same as in (9). This monotone inclusion (33) can be solved using existing methods for more

general monotone inclusion problems (such as [34–37]). However, these iterative algorithms are not identical to our proposed algorithm. To illustrate this point, we will explain the proposed algorithm from the perspective of duality. In fact, the dual monotone inclusion of (33) is

$$\begin{aligned} \text{find } y \in G, \\ \text{such that } 0 \in T^{-1}y - BJ_S(u - B^* y). \end{aligned} \quad (34)$$

By Lemma 11, we know that $-BJ_S(u - B^* y)$ is $1/\|B\|^2$ -cocoercive. Thus, y is a solution of the dual monotone inclusion (34), and $x = J_S(u - B^* y)$ is a solution of the primal monotone inclusion (33). Next, we apply the relaxed forward-backward splitting algorithm (see, e.g., Theorem 26.14 of [28]) to solve the dual monotone inclusion (34). More precisely, let $y_0 \in G$ and set

$$y_{k+1} = (1 - \lambda_k) y_k + \lambda_k J_{\gamma T^{-1}}(y_k + \gamma BJ_S(u - B^* y_k)), \quad (35)$$

which is equivalent to the iterative algorithm introduced in Theorem 12. For convenience, we summarize the differences between the proposed algorithm and existing algorithms in Table 1.

Let $S = 0$ in Theorem 12. Then, we obtain the following corollary.

Corollary 15. Let $T : G \rightarrow 2^G$ be a maximal monotone operator and $B \in L(H, G)$, and let $u \in H$. For any $y_0 \in G$, we define the iterative sequences $\{x_k\}$ and $\{y_k\}$ as follows:

$$\begin{aligned} x_k &= u - \gamma B^* y_k, \\ y_{k+1} &= (1 - \lambda_k) y_k + \lambda_k (I - J_{(1/\gamma)T})(y_k + Bx_k), \end{aligned} \quad (36)$$

where γ and λ_k are the same as in Theorem 12. Then, the following hold:

- (i) $\{y_k\}$ converges weakly to a fixed point of Q , where $Q(\cdot) = (I - J_{(1/\gamma)T})(Bu + (I - \gamma BB^*)(\cdot))$.
- (ii) If $\inf \lambda_k > 0$, then $\{x_k\}$ converges strongly to the resolvent of $J_{B^*TB}(u)$ and $J_{B^*TB}(u) = u - \gamma B^* y$, $y \in \text{Fix}(Q)$.

Remark 16. Moudafi [26] proposed the following iterative algorithm to solve the resolvent operator $J_{B^*TB}(u)$. Let $y_0 \in G$. Then, the iterative sequence $\{y_k\}$ is defined by

$$\begin{aligned} y_{k+1} &= (1 - \lambda_k) y_k \\ &\quad + \lambda_k (I - J_{(1/\gamma)T})(Bu + (I - \gamma BB^* y_k)), \end{aligned} \quad (37)$$

where $\gamma \in (0, 2/\|B\|^2)$ and $\lambda_k \in (0, 1)$ such that $\sum_{k=0}^{+\infty} \lambda_k(1 - \lambda_k) = +\infty$. Moudafi proved that $J_{B^*TB}(u) = u - \gamma B^* y^\infty$, where y^∞ is the weak limit point of $\{y_k\}$. Furthermore, he presented several applications of the obtained iterative algorithm.

Corollary 15 extends some of the results from [26] in two aspects. (i) The range of $\{\lambda_k\}$ is expanded and (ii) we obtain strong convergence of the iterative sequence $\{x_k\}$ (which is not studied in [26]).

Let $B = I$ in Theorem 12. Then, we obtain the following corollary.

Corollary 17. Let $S : H \rightarrow 2^H$ and $T : G \rightarrow 2^G$ be two maximal monotone operators. In addition, let $u \in H$. For any $y_0 \in G$, we define the iterative sequences $\{x_k\}$ and $\{y_k\}$ as follows:

$$\begin{aligned} x_k &= J_S(u - \gamma y_k), \\ y_{k+1} &= (1 - \lambda_k) y_k + \lambda_k (I - J_{(1/\gamma)T})(y_k + x_k), \end{aligned} \quad (38)$$

where $\gamma \in (0, 2)$ and $\lambda_k \in (0, (4 - \gamma)/2)$ such that $\sum_{k=0}^{\infty} \lambda_k (1 - (2/(4 - \gamma))\lambda_k) = +\infty$. Then, the following hold:

- (i) $\{y_k\}$ converges weakly to a fixed point of Q , where $Q(\cdot) = (I - J_{(1/\gamma)T})(\cdot + J_S(u - \gamma \cdot))$.
- (ii) Suppose that $\inf \lambda_k > 0$. Then, $\{x_k\}$ converges strongly to the resolvent operator $J_{S+T}(u)$.

Remark 18. The obtained iterative algorithm (38) for computing the resolvent operator $J_{S+T}(u)$ is different from those proposed by Bauschke and Combettes [21, 23]. In fact, similar to (32), the iterative scheme (38) can be rewritten as

$$\begin{aligned} x_k &= J_S(u - \bar{y}_k), \\ \bar{y}_{k+1} &= (1 - \lambda_k) \bar{y}_k + \lambda_k J_{\gamma T^{-1}}(\bar{y}_k + \gamma x_k), \end{aligned} \quad (39)$$

where $\bar{y}_k = \gamma y_k$.

Bauschke and Combettes [21] proposed a Dykstra-like algorithm to compute the resolvent operator $J_{S+T}(u)$. Let $x_0 = u$, $p_0 = 0$, and $q_0 = 0$. Then, the Dykstra-like algorithm is defined by

$$\begin{aligned} y_k &= J_T(x_k + p_k), \\ p_{k+1} &= x_k + p_k - y_k, \\ x_{k+1} &= J_S(y_k + q_k), \\ q_{k+1} &= y_k + q_k - x_{k+1}. \end{aligned} \quad (40)$$

In addition, Bauschke and Combettes proved that $\{x_k\}$ and $\{y_k\}$ generated by (40) converge strongly to $J_{S+T}(u)$. Following some simple calculation, the Dykstra-like algorithm (40) can be rewritten as follows:

$$\begin{aligned} x_{k+1} &= J_S(J_T(x_k + p_k) + q_k), \\ p_{k+1} &= J_{T^{-1}}(x_k + p_k), \\ q_{k+1} &= J_{S^{-1}}(J_T(x_k + p_k) + q_k). \end{aligned} \quad (41)$$

On the other hand, Combettes [23] proposed an inexact Douglas–Rachford splitting algorithm and an inexact Dykstra-like algorithm for computing the resolvent of the sum of a finite family of maximal monotone operators. For the resolvent of the sum of two maximal monotone operators, the inexact Dykstra-like algorithm without errors coincided with the iterative algorithm (40). For simplicity, we have presented the inexact Douglas–Rachford splitting algorithm

without errors for computing the resolvent of the sum of two maximal monotone operators. Let $z_0 \in H$, and define the iterative sequences as follows:

$$\begin{aligned} y_k &= J_{(2\gamma/(2+\gamma))S} \left(\frac{2}{2+\gamma} z_k + \frac{\gamma}{2+\gamma} u \right), \\ p_k &= J_{(2\gamma/(2+\gamma))T} \left(\frac{2}{2+\gamma} (2y_k - z_k) + \frac{\gamma}{2+\gamma} u \right), \\ z_{k+1} &= z_k + \lambda_k (p_k - y_k), \end{aligned} \quad (42)$$

where $\gamma > 0$, and $\{\lambda_k\} \subseteq (0, 2)$ such that $\sum_{k=0}^{\infty} \lambda_k (2 - \lambda_k) = +\infty$ and $\inf \lambda_k > 0$. Next, the sequences $\{y_k\}$ and $\{p_k\}$ converge strongly to $J_{S+T}(u)$. In fact, this iterative algorithm (42) is equivalent to applying the Douglas–Rachford splitting algorithm to the monotone inclusion of

$$0 \in \left(\frac{1}{2}x - \frac{1}{2}u + Sx \right) + \left(\frac{1}{2}x - \frac{1}{2}u + Tx \right). \quad (43)$$

Comparing (39), (40), and (42), we find that the obtained iterative sequences generated by all algorithms converge strongly to the resolvent operator $J_{S+T}(u)$. The iterative algorithms (39) and (42) do not have any requirements for the initial value, whereas (40) requires the selection of a fixed initial value. Unlike (39) and (42), the Dykstra-like algorithm (40) does not require tuning of the parameters.

4. Application to Convex Optimization Problem

In this section, we apply the obtained results to solve a particular convex optimization problem that has been studied in the literature.

For convenience, we introduce some notation. A function $f : H \rightarrow (-\infty, +\infty]$ is called proper if $\text{dom } f = \{x \in H \mid f(x) < +\infty\} \neq \emptyset$. $\Gamma_0(H)$ denotes the class of proper lower semicontinuous convex functions from H to $(-\infty, +\infty]$. The Fenchel-conjugate of f is defined by $f^* : u \mapsto \sup_{x \in H} \langle x, u \rangle - f(x)$.

Problem 19. Let $r \in G$ and $u \in H$, and let $f : H \rightarrow (-\infty, +\infty]$ and $g : G \rightarrow (-\infty, +\infty]$ be proper lower semicontinuous convex functions. Furthermore, let $L : H \rightarrow G$ be a nonzero bounded linear operator such that $r \in \text{core}(L(\text{dom } f) - \text{dom } g)$, where the core of a subset $C \subseteq H$ is defined by $\text{core } C = \{x \in C \mid \text{cone}(C - x) = H\}$ (see, e.g., Definition 6.9 of [28]). Consider the following convex optimization problem:

$$\min_{x \in H} \frac{1}{2} \|x - u\|^2 + f(x) + g(Lx - r). \quad (44)$$

The minimization problem (44) is equivalent to the proximity operator of $f(x) + g(Lx - r)$. As a direct application of the resolvent operator (9), we obtain the following convergence theorem from Theorem 12.

FIGURE 1: Test images. (a) “Barbara”: pixel range $[12, 246]$, (b) “Lena”: pixel range $[24, 245]$, (c) “Boat”: pixel range $[0, 255]$, and (d) “Goldhill”: pixel range $[10, 235]$.

Theorem 20. Under the conditions of Problem 19, let $y_0 \in G$ and set $\{x_k\}$ and $\{y_k\}$ as

$$\begin{aligned} x_k &= \text{prox}_f(u - \gamma L^* y_k), \\ y_{k+1} &= (1 - \lambda_k) y_k \\ &\quad + \lambda_k (I - \text{prox}_{(1/\gamma)g})(y_k + Lx_k - r), \end{aligned} \quad (45)$$

where $\gamma \in (0, 2/\|L\|^2)$, and $\lambda_k \in (0, (4 - \gamma\|L\|^2)/2)$ such that $\sum_{k=0}^{\infty} \lambda_k (1 - (2/(4 - \gamma\|L\|^2))\lambda_k) = +\infty$ and $\inf \lambda_k > 0$. Then, the iterative sequence $\{x_k\}$ converges strongly to a solution of (44).

Proof. Let $S = \partial f$, $T = \partial g(\cdot - r)$, and $B = L$ in (9). Because $f \in \Gamma_0(H)$ and $g \in \Gamma_0(G)$, S and T are maximal monotone operators. Furthermore, note that $J_S = (I + \partial f)^{-1} = \text{prox}_f$ and $J_{(1/\gamma)T} = (I + (1/\gamma)\partial g(\cdot - r))^{-1} = \text{prox}_{(1/\gamma)g}(\cdot - r) + r$. Hence, the iterative sequences (45) follow directly from (20). By Theorem 12, we conclude that $\{x_k\}$ converges strongly to a solution of (44). This completes the proof. $\square$

Remark 21. The Moreau identity states that, for any $\lambda > 0$, $\text{prox}_{\lambda f}(u) + \lambda \text{prox}_{(1/\lambda)f^*}((1/\lambda)u) = u$, where f^* denotes the

Fenchel conjugate of f . It follows from this that the iterative sequences (45) can be rewritten as

$$\begin{aligned} x_k &= \text{prox}_f(u - \gamma L^* y_k), \\ y_{k+1} &= (1 - \lambda_k) y_k + \frac{\lambda_k}{\gamma} \text{prox}_{\gamma g^*}(\gamma y_k + \gamma(Lx_k - r)). \end{aligned} \quad (46)$$

Let $\bar{y}_k = \gamma y_k$. Then, (46) is reduced to

$$\begin{aligned} x_k &= \text{prox}_f(u - L^* \bar{y}_k), \\ \bar{y}_{k+1} &= (1 - \lambda_k) \bar{y}_k + \lambda_k \text{prox}_{\gamma g^*}(\bar{y}_k + \gamma(Lx_k - r)). \end{aligned} \quad (47)$$

Combettes et al. [27] proposed the following iterative algorithm to solve the optimization problem (44). For any $y_0 \in G$, set $\{x^k\}$ and $\{y^k\}$ as

$$\begin{aligned} x_k &= \text{prox}_f(u - L^* y_k) + b_k, \\ y_{k+1} &= (1 - \lambda_k) y_k \\ &\quad + \lambda_k \text{prox}_{\gamma_k g^*}(y_k + \gamma_k(Lx_k - r) + a_k), \end{aligned} \quad (48)$$

where $\lambda_k \in [\epsilon, 1]$, $\gamma_k \in (\epsilon, 2/\|L\|^2 - \epsilon)$, and $\epsilon \in (0, \min(1, 1/\|L\|^2))$. Under the conditions that $\sum_{k=0}^{\infty} \|a_k\| < \infty$

FIGURE 2: SNR, PSNR, and objective function values against the iteration number for the ROF model. (a)–(c) $\gamma = 1/4$, (d)–(f) $\gamma = 1/8$, and (g)–(i) $\gamma = 1/16$.

FIGURE 3: SNR, PSNR, and objective function values against the iteration number for the nonnegative ROF model (N-ROF). (a)–(c) $\gamma = 1/4$, (d)–(f) $\gamma = 1/8$, and (g)–(i) $\gamma = 1/16$.

FIGURE 4: SNR, PSNR, and objective function against the iteration number for the bounded ROF model (B-ROF). (a)–(c) $\gamma = 1/4$, (d)–(f) $\gamma = 1/8$, and (g)–(i) $\gamma = 1/16$.

FIGURE 5: (a) The noisy image of “Barbara.” (b) Denoised image from the ROF model, (c) denoised image from the N-ROF model, and (d) denoised image from the B-ROF model.

TABLE 1: Comparison of the proposed algorithm with existing algorithms.

Method	Iterative algorithm	Operator splitting algorithms
Primal-dual	[34, 35]	Forward-backward-forward splitting
	[36]	Variable metric forward-backward splitting
	[37]	Forward-backward-half forward splitting
Dual	Proposed (20)	Relaxed forward-backward splitting

$+\infty$ and $\sum_{k=0}^{\infty} \|b_k\| < +\infty$, the authors proved that $\{x_k\}$ converges strongly to a solution of (44).

Remark 22. (1) Comparing (47) with (48), the range of λ_k in (47) is clearly larger than that of the iterative sequence (48) introduced by Combettes et al. [27] when the variable step size γ_k is constant; i.e., $\gamma_k \equiv \gamma$. In addition, Theorem 20 also recovers the main results of Proposition 28.16 in [28]. However, we require that λ_k satisfies $\sum_{k=0}^{\infty} \lambda_k(1 - (2/(4 - \gamma\|L\|^2))\lambda_k) = +\infty$, while Proposition 28.16 in [28] requires that $0 < \inf \lambda_k \leq \sup \lambda_k < (4 - \gamma\|L\|^2)/2$. It is obvious that

our conditions on λ_k are weaker than those of Proposition 28.16 in [28].

(2) Although our proposed iterative sequences (45) are error-free, it is not difficult to add error sequences in corresponding locations, as in (48). Because the proof is almost identical to that of Theorem 12, we have omitted it here.

5. Numerical Experiments

In this section, we present numerical experiments to verify the effectiveness of the proposed iterative algorithms for

FIGURE 6: (a) The noisy image of “Lena.” (b) Denoised image from the ROF model, (c) denoised image from the N-ROF model, and (d) denoised image from the B-ROF model.

solving the constrained total variation model (4) for image denoising problems. All experiments are conducted on a Lenovo laptop with an Intel 2.3 GHz CPU and 4 GB RAM. We run the program with MATLAB R2014a.

We select “Barbara,” “Lena,” “Boat,” and “Goldhill” as the test images (see Figure 1). Gaussian noise of mean 0 and standard deviation 30 is added to the clear images.

We use the signal-to-noise (SNR) and peak-signal-to-noise (PSNR) to evaluate the quality of the restored images. These are defined by

$$SNR = 20 \log_{10} \frac{\|f\|_2}{\|f - \tilde{x}\|_2}, \quad (49)$$

and

$$PSNR = 20 \log_{10} \frac{255 \sqrt{MN}}{\|f - \tilde{x}\|_2}, \quad (50)$$

where f is the original image, $\tilde{x}$ is the restored image, and M and N are the row and column size of the image f ,

respectively. The iterative algorithm is stopped when the following criterion is satisfied:

$$\frac{\|y_{k+1} - y_k\|_2}{\|y_k\|_2} \leq \epsilon, \quad (51)$$

where ϵ is a given small constant. We tuned the regularization parameter μ and set it to 15 for optimal denoised image quality.

We aim mainly to solve the constrained total variation (TV) image denoising problem (4). In particular, we choose the anisotropic total variation as the regularization term during testing. By using the indicator function, the constrained (TV) denoising problem (4) can be reformulated as the following unconstrained optimization problem:

$$\min_{x \in R^n} \frac{1}{2} \|x - f\|_2^2 + \mu \|x\|_{TV} + \delta_C(x), \quad (52)$$

where δ_C is the indicator function. It is clear that the optimization problem (52) is a special case of (44). In fact,

FIGURE 7: (a) The noisy image of “Boat.” (b) Denoised image from the ROF model, (c) denoised image from the N-ROF model, and (d) denoised image from the B-ROF model.

if $f(x) = \delta_C(x)$, $g(Lx) = \|x\|_{TV}$, and $r = 0$, then the proposed iteration scheme (45) can be employed to solve the constrained TV image denoising problem (4). It is well known that $\|L\|^2 \approx 8$, where L is the usual gradient operator of the total variation (see [10, 20]).

Let $C = \mathbb{R}^n$. Then, (4) (also (52)) is reduced to the well-known ROF model (1). We select C as both the nonnegative and the bounded sets. Moreover, the nonnegative set C is defined as $\{x \in \mathbb{R}^n \mid x_i \geq 0\}$, and the bounded set C as $\{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 255\}$. The corresponding minimization problems (52) are referred to as the nonnegative ROF (N-ROF) model and bounded ROF (B-ROF) model, respectively. It is worth mentioning that the proximity operator of the indicator function is the projection operator, which has closed-form solutions based on our choices. Therefore, in (45) the proximity operator of $f(x) = \delta_C(x)$ is the orthogonal projection onto the closed convex set C . That is, $\text{prox}_{\delta_C}(u) = P_C(u)$.

5.1. Numerical Results and Discussion. First, we describe the impacts of the parameters for the iterative step size γ and relaxation variable λ_k on the proposed iterative algorithm (45). According to Theorem 20, we can choose $\gamma \in (0, 1/4)$ and $\lambda_k \in (0, 2 - 4\gamma)$. Figures 2, 3, and 4 demonstrate the performance of the proposed iterative algorithm for solving (52) with different choices of γ and λ_k . The test image was “Barbara,” and the stopping criterion ϵ was set as 10^{-2} .

As shown in Figures 2–4, when the iterative step size γ is fixed, a larger relaxation parameter λ_k leads to a faster convergence of the iterative algorithm. Table 2 presents the corresponding numerical results in terms of the SNR, PSNR, and number of iterations (*Iter*).

Because the prior pixel information of the image is introduced as a constraint, the performance of the constrained ROF model is superior to that of the unconstrained model. Numerical results confirm the advantages of the constrained ROF model. We can see from Table 2 that the iterative step

FIGURE 8: (a) The noisy image of “Goldhill.” (b) Denoised image from the ROF model, (c) denoised image from the N-ROF model, and (d) denoised image from the B-ROF model.

size γ has a significant impact on the performance of the algorithm. Furthermore, the experimental results demonstrate that a larger γ always leads to a faster convergence of the iterative algorithm. Thus, we choose $\gamma = 1/4$ and $\lambda_k = 1$ in the following comparison.

Next, we focus on investigating the performances of the constrained and unconstrained ROF models for the test images from Figure 1. The numerical results are presented in Table 3. We notice that the SNR and PSNR values slowly decrease with an increasing number of iterations. Because more iterations do not improve the quality of the image, we should stop the iterative algorithm in the early stages. Figures 5–8 present denoised images for $\epsilon = 10^{-2}$ in Table 3. From the visual point of view, the images restored by the constrained model are closer to the original images than those restored by

the unconstrained model. This further confirms the benefits of introducing constraints in the image recovery model.

6. Conclusions

The total variation can be viewed as a composition of a convex function with a linear transformation. Thus, Micchelli et al. [20] introduced a fixed-point algorithm based on proximity operators to produce a total variation model for image denoising (1). Inspired by the work of Moudafi [26], we studied the calculation of the resolvent of the sum of a maximal monotone operator and a composite operator (9) to produce a constrained total variation model (4). Subsequently, we proposed a fixed-point algorithm for this resolvent operator. Based on the fixed-point theory of

TABLE 2: Numerical results for different choices of γ and λ_k in terms of the SNR, PSNR, and number of iterations (*Iter*) for the test image “Barbara.”

γ	λ_k	ROF			N-ROF			B-ROF		
		SNR	PSNR	<i>Iter</i>	SNR	PSNR	<i>Iter</i>	SNR	PSNR	<i>Iter</i>
1/16	0.2	18.7102	24.5975	41	18.7073	24.5946	41	18.7063	24.5936	41
	0.4	18.8872	24.7745	34	18.8925	24.7798	35	18.8924	24.7797	35
	0.6	18.9328	24.8200	31	18.9331	24.8204	31	18.9332	24.8205	31
	0.8	18.9520	24.8393	29	18.9528	24.8401	29	18.9530	24.8403	29
	1	18.9633	24.8506	28	18.9644	24.8516	28	18.9646	24.8519	28
	1.2	18.9680	24.8553	26	18.9710	24.8582	27	18.9712	24.8585	27
	1.5	18.9745	24.8618	25	18.9758	24.8631	25	18.9761	24.8634	25
	1.75	18.9775	24.8648	24	18.9789	24.8661	24	18.9791	24.8664	24
1/8	0.2	18.8906	24.7778	34	18.8897	24.7770	34	18.8896	24.7769	34
	0.4	18.9529	24.8401	29	18.9536	24.8409	29	18.9538	24.8411	29
	0.6	18.9683	24.8556	26	18.9695	24.8568	26	18.9697	24.8570	26
	0.8	18.9753	24.8625	24	18.9766	24.8639	24	18.9769	24.8641	24
	1	18.9795	24.8667	23	18.9809	24.8691	23	18.9811	24.8684	23
	1.2	18.9816	24.8688	22	18.9831	24.8703	22	18.9833	24.8706	22
	1.5	18.9829	24.8702	20	18.9844	24.8717	20	18.9847	24.8720	20
	0.2	18.9543	24.8416	28	18.9547	24.8420	28	18.9548	24.8421	28
1/4	0.4	18.9762	24.8635	24	18.9774	24.8647	24	18.9777	24.8649	24
	0.6	18.9813	24.8686	21	18.9834	24.8707	22	18.9837	24.8710	22
	0.8	18.9836	24.8709	20	18.9851	24.8724	20	18.9854	24.8726	20
	1	18.9841	24.8714	19	18.9856	24.8729	19	18.9859	24.8732	19

TABLE 3: Numerical results in terms of the SNR, PSNR, and number of iterations (*Iter*) on different test images for different tolerance values ϵ .

Images	Model	$\epsilon = 10^{-2}$			$\epsilon = 10^{-3}$			$\epsilon = 10^{-4}$		
		SNR	PSNR	<i>Iter</i>	SNR	PSNR	<i>Iter</i>	SNR	PSNR	<i>Iter</i>
"Barbara"	ROF	18.9841	24.8714	19	18.9800	24.8672	71	18.9784	24.8656	216
	N-ROF	18.9856	24.8729	19	18.9816	24.8688	71	18.9800	24.8672	216
	B-ROF	18.9859	24.8732	19	18.9818	24.8691	71	18.9802	24.8675	216
"Lena"	ROF	22.4595	28.1409	21	22.4552	28.1366	78	22.4522	28.1336	242
	N-ROF	22.4605	28.1419	21	22.4563	28.1377	78	22.4534	28.1347	242
	B-ROF	22.4616	28.1430	21	22.4573	28.1387	78	22.4544	28.1358	242
"Boat"	ROF	21.6226	26.9652	20	21.6180	26.9606	74	21.6162	26.9588	228
	N-ROF	21.6326	26.9752	20	21.6278	26.9704	74	21.6260	26.9686	228
	B-ROF	21.6335	26.9761	20	21.6287	26.9713	74	21.6269	26.9695	228
"Goldhill"	ROF	20.9503	27.3164	20	20.9353	27.3015	73	20.9328	27.2989	223
	N-ROF	20.9527	27.3189	20	20.9379	27.3040	73	20.9353	27.3014	223
	B-ROF	20.9577	27.3239	20	20.9429	27.3091	73	20.9403	27.3065	223

nonexpansive mappings, we proved the strong convergence of the obtained iterative sequence. The advantage of the fixed-point approach is that it provides the potential to develop additional fast iterative algorithms. Numerical simulations on image denoising illustrated the performance of the proposed algorithm. In particular, we found that the step size had a significant impact on the convergence speed of the algorithm. In general, when the iterative step size was fixed, larger relaxation parameters resulted in a faster iterative algorithm convergence. Numerical results also confirmed that the constrained ROF model achieved a superior performance compared with the unconstrained ROF model.

Finally, we wish to note that the constrained TV model (4) can also be derived using other iterative algorithms, such as the primal-dual Chambolle–Pock algorithm [15], the alternating direction method of multipliers [29, 38, 39], and the preconditioned primal-dual algorithm [40, 41]. We have not presented the corresponding numerical results here. Thus, we will further examine the convergence rate of our proposed iterative algorithm and include these comparative results in future work.

Data Availability

The image data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

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