Research Article

Numerical Computation of Brinkman Flow with Stable Mixed Element Method

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Herein, we discuss a mixed finite element method applied to the Brinkman equation of fluid motion in porous medium, which covers a field of situation from the Darcy equation to the Stokes problem associated with various perturbation parameters. The finite element based on staggered meshes is shown to be stable and effective with each case as the corresponding error estimates for both velocity and pressure are established. Finally, we present numerical examples confirming the theoretical analysis and the stability of the finite spaces approximation.

1. Introduction

In this paper, we consider an efficient mixed finite method to simulate the Brinkman model that describes the viscous fluid creeping flow in highly heterogeneous porous medium [1]. Firstly, we comment on its mathematical and physical backgrounds. The mathematical framework is that the parameter-dependent model viewed as a generalized Stokes problem can be reduced to be a form of Darcy's law. The derivation of the equation could be explained in some aspect [2, 3]. It is widely used in calculating the hydrodynamic interaction and hindered transport of solid spherical particles to order or disorder fibrous medium [4] and applied to fuel cell dynamics or the boundary layer close to a solid wall [5].

The Brinkman model has been studied by some researchers. The crucial key is to find uniform stable simulate as it covers the fields of both Stokes and Darcy problems. As seen in [2], it does not work as the parameter approaches zero by numerical tests for the common Stokes elements, e.g., $P_2 - P_0$, Crouzeix-Raviart and Mini element, while it fails as the parameter is far away from zero for the classical Raviart-Thomas element approximating. Reference [2] proposes a new nonconforming robust finite element spaces for the unknown variables and detailed argument for the case of perturbation solution. Using this fact, Xu-Zhang [6] give

a divergence-free element based on a special triangulation which could satisfy compatible condition. With different norm, [7, 8] provide a priori and a posteriori error estimates to ensure the Mini element to be uniformly stable. Reference [9] introduces unified stabilized element utilizing orthogonal subscale stabilization and algebraic subgrid scale method with two different weak formulations, both containing a term of the pressure jumps internal boundaries. Reference [10] gives a stable least-square formulation by introducing an auxiliary variable to substitute for velocity flux. Other more novel methods, such as weak Galerkin finite element method [11, 12], multigrid method [13], and virtual method [14], are applied to Brinkman model without considering parameter dependence of solution.

In this paper, we will consider a stable mixed finite method for Brinkman equation associated with various perturbation parameters, which is first constructed to deal with Stokes equation [15]; furthermore, we modified it for Darcy equation and made it more efficient [16]. The remainder of this paper is organized as follows. In Section 2, we introduce the model and corresponding weak formulation. In Section 3, we discuss error estimates and prove convergence of the discrete scheme. Finally, in Section 4, some numerical experiments confirming theoretical result close the paper.

Throughout this paper, the symbol C, C_i denoting generic positive constants, independent of any mesh parameters, may take different values at different occurrences. ε_i denotes generic small positive constant.

2. The Model Problem and Corresponding Weak Formulation

In the domain $\Omega \subset \mathbb{R}^2$, we denote the velocity by $\mathbf{u}$ and pressure by p . The Brinkman model is described by the parametric equations

$$-t^2 \Delta \mathbf{u} + K^{-1} \mathbf{u} + \nabla p = \mathbf{f}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = q, \quad (2)$$

where K is the media permeability and $\mathbf{f}$ and q are given data. For simplicity, we consider the homogenous boundary conditions: for $t > 0$, we let $\mathbf{u} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{n} = 0$ in the limit case of $t = 0$; here $\mathbf{n}$ is defined as the unit outward normal vectors of the boundary.

We begin by recalling some useful preliminaries and notations. We use the classical definitions for the Sobolev spaces $H^k(\Omega)$ with the usual norm and seminorm $\|\cdot\|_k$ and $|\cdot|_k$. Let $L^2(\Omega)$ be the space of square integrable function in Ω with inner product $(\cdot, \cdot)$. We also use the Hilbert space

$$H(\text{div}, \Omega) = \left\{ u \in [L^2(\Omega)]^2; \text{div } u \in L^2(\Omega) \right\}, \quad (3)$$

then $H_0^1(\Omega)$ is defined as the subspace of $H^1(\Omega)$ with zero trace on $\partial\Omega$ and $H_0(\text{div}, \Omega)$ is the subspace of $H(\text{div}, \Omega)$ with zero normal trace on $\partial\Omega$. $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of functions with zero mean value on Ω .

If $t > 0$, we have $\mathbf{V} = (H_0^1(\Omega))^2$ and $S = L_0^2(\Omega)$; in the singular case that the perturbation parameter $t \rightarrow 0$, the vector space is replaced by $\mathbf{V} = H_0(\text{div}, \Omega)$.

Define two bilinear forms by

$$a(\mathbf{u}, \mathbf{v}) = t^2 (\nabla \mathbf{u}, \nabla \mathbf{v}) + (K^{-1} \mathbf{u}, \mathbf{v}), \quad (4)$$

$$b(\mathbf{v}, w) = (w, \nabla \cdot \mathbf{v}).$$

The variational formulation of the model is represented by the following: find $(\mathbf{u}, p) \in \mathbf{V} \times S$, such that

$$a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \quad (5)$$

$$b(\mathbf{u}, w) = (q, w) \quad \forall q \in S.$$

The associated norm on $\mathbf{V}$ is given by

$$\|\mathbf{u}\|_{\mathbf{V}}^2 = t^2 \|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2 + \|\nabla \cdot \mathbf{u}\|^2. \quad (6)$$

For $0 < t < C$, we find that $\|\mathbf{u}\|_{\mathbf{V}} \leq C_1 \|\mathbf{u}\|_1$. The L-B-B condition holds, as

$$\sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{V}}} \geq C_1 \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1} \geq C_1 \beta \|q\| \quad \forall q \in S. \quad (7)$$

Theorem 1. *There exists a unique solution $(\mathbf{u}, p) \in \mathbf{V} \times S$ to problem (5).*

Proof. The coercivity and continuity of $a(\mathbf{u}, \mathbf{v})$ can be obtained by simple calculation, which means

$$a(\mathbf{u}, \mathbf{u}) \geq C_2 \|\mathbf{u}\|_{\mathbf{V}}^2, \quad \forall \mathbf{u} \in \mathbf{V}_0, \quad (8)$$

$$a(\mathbf{u}, \mathbf{v}) \leq C_3 \|\mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V},$$

where $\mathbf{V}_0$ is a closed subspace of $\mathbf{V}$ via $\mathbf{V}_0 = \{\mathbf{v} \in \mathbf{V} \mid b(\mathbf{v}, q) = 0, \forall q \in S\}$. Combining with the inf-sup condition (7), the Brezzi condition is satisfied. It is clear that problem (5) is well-posed and the solution is unique [7, 17]. $\square$

Next, we discuss the finite element discretization of the variational formulation (5). The finite element spaces $\mathbf{V}_h$ and S_h are initially proposed by Han and Wu to treat with Stokes problem [15]. We find it is stable uniformly in the parameter $t \in (0, K]$. In brief, we firstly recall that τ_h is quasiuniformly rectangular partitioning of the domain Ω ; τ_h^1 and τ_h^2 are staggered rectangular meshes based on τ_h . Then, the discrete spaces are defined as

$$\begin{aligned} V_h^1 &= \{v_h \in C^{(0)}(\overline{\Omega}) : v_h|_{T^1} \in Q_{1,1}(T^1), \forall T^1 \in \tau_h^1\}, \\ V_h^2 &= \{v_h \in C^{(0)}(\overline{\Omega}) : v_h|_{T^2} \in Q_{1,1}(T^2), \forall T^2 \in \tau_h^2\}, \\ S_h & \end{aligned} \quad (9)$$

$$:= \left\{ w_h : w_h|_T = \text{constant}, \forall T \in \tau_h; \int_{\Omega} w_h dx = 0 \right\}.$$

Let

$$\mathbf{V}_h = \mathbf{V} \cap (V_h^1 \times V_h^2). \quad (10)$$

Using the subspaces $\mathbf{V}_h$ and S_h instead of $\mathbf{V}$ and S in the variational problem (5), we obtain the discrete problem: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times S_h$, such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, w_h) &= (q, w_h) \quad \forall w_h \in S_h. \end{aligned} \quad (11)$$

3. Convergence Analysis and Error Estimate

In this section, we give the convergence analysis and error estimate. For $t > 0$, the definition of the interpolating $\Pi : \mathbf{V} \rightarrow \mathbf{V}_h$ by $\Pi \mathbf{u} \in \mathbf{V}_h$ is expressed by the formulation [15]

$$\int_l \Pi_h^1 u_1 ds = \int_l u_1 ds \quad \forall l \in L_V, \quad (12)$$

$$\int_l \Pi_h^2 u_2 ds = \int_l u_2 ds \quad \forall l \in L_H,$$

where L_V is the set that contains all vertical edges of the partition τ_h , while L_H contains all horizontal edges. For $t = 0$, the interpolation operator can be redefined as [18]

$$\int_l \Pi_h^1 u_1 ds = \int_l u_1 ds \quad \forall l \in L_{V'}, \quad (13)$$

$$\int_l \Pi_h^2 u_2 ds = \int_l u_2 ds \quad \forall l \in L_{H'},$$

where $L_{V'}$ is a set of edges of elements gotten by bisecting the most bottom element edges and most top element edges of L_V and $L_{H'}$ are gotten by bisecting the most left element edges and most right element edges of L_H . Furthermore, we denote $P_s : S \rightarrow S_h$ as the L^2 projection operator.

Referring to [18], we recall some properties of the interpolating Π and projection P_s .

Lemma 2. (i) There exist two constants C_3 and C_4 independent of h , such that

$$|\mathbf{u} - \Pi\mathbf{u}|_{1,\Omega} \leq C_3 h^{j-1} |\mathbf{u}|_{j,\Omega}, \quad 1 \leq j \leq 2. \quad \forall \mathbf{u} \in \mathbf{V}. \quad (14)$$

$$\|p - P_s p\| \leq C_4 h |p|_{1,\Omega}, \quad \forall p \in S. \quad (15)$$

(ii) For $\mathbf{u} \in (H^1(\Omega))^2$, there exists a constant C_5 independent of h , such that

$$\|\Pi\mathbf{u}\|_{1,\Omega} \leq C_5 \|\mathbf{u}\|_{1,\Omega} \quad \forall \mathbf{u} \in \mathbf{V}. \quad (16)$$

(iii) For any $\mathbf{u} \in \mathbf{V}$, we have

$$\int_{\Omega} q_h \operatorname{div}(\mathbf{u} - \Pi\mathbf{u}) dx = 0, \quad \forall q_h \in S_h. \quad (17)$$

Theorem 3. The discrete inf-sup condition is valid; namely, there is a constant $\beta \geq 0$, such that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \geq \beta \|q_h\|, \quad \forall q_h \in S_h. \quad (18)$$

Proof. Based on the fact that $\|\mathbf{v}_h\|_{\mathbf{V}} \leq C \|\mathbf{v}_h\|_1$, we find

$$\begin{aligned} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}} &\geq \frac{1}{C} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq \frac{1}{C} \frac{b(\Pi\mathbf{v}, q_h)}{\|\Pi\mathbf{v}\|_1} \\ &\geq \frac{1}{CC_5} \frac{b(\mathbf{v}, q_h)}{\|\mathbf{v}\|_1} \end{aligned} \quad (19)$$

which means

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \geq c \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q_h)}{\|\mathbf{v}\|_1} \geq c\beta \|q_h\|, \quad (20)$$

$$\forall q_h \in S_h. \quad \square$$

Theorem 4. If $(\mathbf{u}, p)$ satisfy (5) and $(\mathbf{u}_h, p_h)$ is the solution of (11), then the following error estimate holds:

$$\|\mathbf{u} - \mathbf{u}_h\|_a + \|p - p_h\| \leq Ch(t \|\mathbf{u}\|_2 + h \|\mathbf{u}\|_2 + \|p\|_1). \quad (21)$$

Proof. For the error, we subtract (11) from (5) by $\mathbf{v} = \mathbf{v}_h \in \mathbf{V}_h$ and obtain that

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - (p - p_h, \nabla \cdot \mathbf{v}_h) = 0 \quad (22)$$

$$(\nabla \cdot (\mathbf{u} - \mathbf{u}_h), w_h) = 0 \quad (23)$$

Taking $\mathbf{v}_h = \mathbf{u}_h - \Pi\mathbf{u}$, we have

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_h - \Pi\mathbf{u}) &= (p - p_h, \nabla \cdot (\mathbf{u}_h - \Pi\mathbf{u})) \\ &= (p - q_h, \nabla \cdot (\mathbf{u}_h - \Pi\mathbf{u})), \end{aligned} \quad (24)$$

$$\forall q_h \in S_h.$$

which is derived by combining with the equality (17) and (23). For convenience, we introduce a norm $\|\cdot\|_a$ for the norm on $\mathbf{V}_h$, which is commonly used in the following:

$$\|\mathbf{v}\|_a^2 = \|\mathbf{v}\|^2 + t^2 \|\nabla \mathbf{v}\|^2 = a(\mathbf{v}, \mathbf{v}). \quad (25)$$

Then, from (24) and ε -inequality, we further have

$$\begin{aligned} &\|\mathbf{u}_h - \Pi\mathbf{u}\|_a^2 \\ &= a(\mathbf{u} - \Pi\mathbf{u}, \mathbf{u}_h - \Pi\mathbf{u}) - (p - q_h, \nabla \cdot (\mathbf{u}_h - \Pi\mathbf{u})) \\ &= a(\mathbf{u} - \Pi\mathbf{u}, \mathbf{u}_h - \Pi\mathbf{u}) \\ &\quad - (p - P_s p, \nabla \cdot (\mathbf{u}_h - \Pi\mathbf{u}) - \nabla \cdot \Pi'(\mathbf{u}_h - \Pi\mathbf{u})) \\ &\leq \varepsilon_1 \|\mathbf{u}_h - \Pi\mathbf{u}\|_a^2 + \frac{1}{\varepsilon_1} \|\mathbf{u} - \Pi\mathbf{u}\|_a^2 \\ &\quad + Ch \|p - P_s p\|_0 \|\nabla \cdot (\mathbf{u}_h - \Pi\mathbf{u})\|_1 \\ &\leq \varepsilon_1 \|\mathbf{u}_h - \Pi\mathbf{u}\|_a^2 + \frac{1}{\varepsilon_1} \|\mathbf{u} - \Pi\mathbf{u}\|_a^2 \\ &\quad + Ch \|p - P_s p\|_0 \|\nabla \cdot (\mathbf{u}_h - \Pi\mathbf{u})\|_0 \\ &\leq \varepsilon_1 \|\mathbf{u}_h - \Pi\mathbf{u}\|_a^2 + \frac{1}{\varepsilon_1} \|\mathbf{u} - \Pi\mathbf{u}\|_a^2 + \varepsilon_2 \|p - P_s p\|^2 \\ &\quad + \frac{1}{\varepsilon_2} \|\mathbf{u}_h - \Pi\mathbf{u}\|^2 \end{aligned} \quad (26)$$

where $\varepsilon_1, \varepsilon_2$ are arbitrary positive constants and combine with the quality of the interpolation Π' , which is based on the fact that $|\nabla \cdot (\mathbf{u}_h - \Pi\mathbf{u})|_1 \leq C$ as $\partial_{x_i}(\nabla \cdot (\mathbf{u}_h - \Pi\mathbf{u})) = c$ on each $T \in \tau_h^i$. Now, we can obtain that

$$\begin{aligned} \|\mathbf{u}_h - \Pi\mathbf{u}\|_a &\leq C(\|\mathbf{u} - \Pi\mathbf{u}\|_a) + \|p - P_s p\| \\ &\leq Ch(t \|\mathbf{u}\|_2 + h \|\mathbf{u}\|_2 + \|p\|_1). \end{aligned} \quad (27)$$

For $P_s p - p_h$, there exist a $\mathbf{v} \in (H_0^1(\Omega))^2$ and a constant $c_2 > 0$ independent of $P_s p - p_h$ [17], satisfying

$$\begin{aligned} \nabla \cdot \mathbf{v} &= P_s p - p_h, \\ \|\mathbf{v}\|_{1,\Omega} &\leq c_2 \|P_s p - p_h\|. \end{aligned} \quad (28)$$

Then, we have

$$\begin{aligned} \|P_s p - p_h\|^2 &= (P_s p - p_h, \nabla \cdot \mathbf{v}) = (P_s p - p_h, \nabla \cdot \Pi\mathbf{v}) \\ &= (P_s p - p, \nabla \cdot \Pi\mathbf{v}) + a(\mathbf{u} - \mathbf{u}_h, \Pi\mathbf{v}) \\ &\leq Ch \|p\|_1 \|\Pi\mathbf{v}\|_1 + C \|\mathbf{u} - \mathbf{u}_h\|_a \|\Pi\mathbf{v}\|_1 \end{aligned} \quad (29)$$

which means

$$\|P_s p - p_h\| \leq Ch(t \|\mathbf{u}\|_2 + h \|\mathbf{u}\|_2 + \|p\|_1). \quad (30)$$

Based on (16), we get that

$$\|p - p_h\| \leq Ch(t \|\mathbf{u}\|_2 + h \|\mathbf{u}\|_2 + \|p\|_1). \quad (31)$$

□

TABLE 1: Numerical error of $\|\mathbf{u}\|_{L^2}$ for Example 1.

t	2^{-2}		2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	error	rate	error	rate	error	rate	error	rate	error	rate
2^{-2}	3.29e-1	—	7.89e-2	2.0587	1.95e-2	2.0133	4.88e-3	2.0031	1.22e-3	2.0008
2^{-4}	2.40e-1	—	4.88e-2	2.3006	1.16e-2	2.0752	2.86e-3	2.0184	7.12e-4	2.0050
2^{-6}	2.33e-1	—	4.56e-2	2.3569	1.05e-2	2.1107	2.59e-3	2.0284	6.43e-4	2.0065
2^{-8}	2.33e-1	—	4.56e-2	2.3572	1.05e-2	2.1144	2.56e-3	2.0377	6.34e-4	2.0134
2^{-10}	2.33e-1	—	4.56e-2	2.3572	1.05e-2	2.1143	2.56e-3	2.0378	6.34e-4	2.0145
2^{-12}	2.33e-1	—	4.56e-2	2.3571	1.05e-2	2.1143	2.56e-3	2.0377	6.34e-4	2.0144
0	2.37e-1	—	4.58e-2	2.3757	1.05e-2	2.1206	2.56e-3	2.0395	6.34e-4	2.0145

In general, the solution $(\mathbf{u}, p)$ depends on the parameter t ; the conclusion above will deteriorate as t approaches zero. As the introduction in [2], we recall some notation in the following:

$$H_+^1 = \left\{ g \in H^1 \cap L_0^2 \mid \int_{\Omega} \frac{|g(x)|^2}{|x - x_j|^2} dx < \infty, \quad j = 1, 2, \dots, N \right\} \quad (32)$$

with

$$\|g\|_{1,+}^2 = \|g\|_1^2 + \sum_{j=1} \int_{\Omega} \frac{|g(x)|^2}{|x - x_j|^2} dx, \quad (33)$$

where $x_1, \dots, x_n$ denote the vertices of the domain Ω .

$$\text{rot} \mathbf{v} = \frac{\partial \mathbf{v}_2}{\partial x_1} - \frac{\partial \mathbf{v}_1}{\partial x_2}. \quad (34)$$

Theorem 5. *When the solution $(\mathbf{u}, p)$ of system (1)-(2) depends on t , then there is a constant C independent of t and h , such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_a + \|p - p_h\| \leq Ch^{1/2} (\|\mathbf{f}\|_{\text{rot}} + \|q\|_{1,+}), \quad (35)$$

Proof. Now, we give some new regularity about the solution. In the case $t = 0$, system (5) reduces to elliptical about $(\mathbf{u}^0, p^0)$. Referring to [2], we derive that

$$\|\mathbf{u}^0\|_1 + \|p^0\|_1 \leq c (\|\mathbf{f}\|_{\text{rot}} + \|q\|_0), \quad (36)$$

while the solution $(\mathbf{u}, p)$ in case of $t > 0$ has the new regularity

$$\begin{aligned} t^2 \|\mathbf{u}\|_2 + t \|\mathbf{u}\|_1 + \|\mathbf{u} - \mathbf{u}^0\|_1 + \|p - p^0\|_1 \\ \leq Ct^{1/2} (\|\mathbf{f}\|_{\text{rot}} + \|q\|_{1,+}). \end{aligned} \quad (37)$$

Applying with (37), we conclude that

$$\|\mathbf{u} - \Pi \mathbf{u}\| + t \|\mathbf{u} - \Pi \mathbf{u}\|_1 \leq Ch^{1/2} (\|\mathbf{f}\|_{\text{rot}} + \|q\|_{1,+}). \quad (38)$$

Combining with (27), (36), and (37), we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_a &\leq C (\|\mathbf{u} - \Pi \mathbf{u}\|_a + \|p - P_s p\|) \\ &\leq C (\|\mathbf{u} - \Pi \mathbf{u}\| + t \|\mathbf{u} - \Pi \mathbf{u}\|_1 \\ &\quad + h (\|p - p^0\|_1 + \|p^0\|_1)) \leq Ch^{1/2} (\|\mathbf{f}\|_{\text{rot}} + \|q\|_{1,+}) \\ &\quad + h (\|\mathbf{f}\|_{\text{rot}} + \|q\|_0 + t^{1/2} (\|\mathbf{f}\|_{\text{rot}} + \|q\|_{1,+})) \\ &\leq Ch^{1/2} (\|\mathbf{f}\|_{\text{rot}} + \|q\|_{1,+}). \end{aligned} \quad (39)$$

Finally, from (29) and (39), we have

$$\begin{aligned} \|p - p_h\| &\leq \|p - P_s p\| + \|P_s p - p_h\| \\ &\leq C (h \|p\|_1 + \|\mathbf{u} - \mathbf{u}_h\|_a) \\ &\leq Ch^{1/2} (\|\mathbf{f}\|_{\text{rot}} + \|q\|_{1,+}). \end{aligned} \quad (40)$$

□

4. Numerical Examples

In this section, we present some numerical results for the model problem (1)-(2).

For simplicity, we assume that the domain is a unit square: $\Omega = (0, 1) \times (0, 1)$. Firstly, we take into account that the analytical solution is independent of t ; then, we observe the convergence of the numerical error as the parameter t decreases 1 to zero. Lastly, we introduce several permeability cases to test the performance of the proposed method.

We need to compute the following useful errors: $\|\mathbf{u}\|_{L^2}$, $\|\mathbf{u}\|_a$, $\|p\|_{L^2}$, $\|P_s p\|_{L^2}$ and corresponding convergence rates.

Example 1. The analytical solution of the problem is as follows:

$$\begin{aligned} \mathbf{u} &= \text{curl} \sin^2(\pi x) \sin^2(\pi y) \\ p &= \cos(\pi x) \cos(\pi y) \\ \mathbf{f} &= \mathbf{u} - t^2 \Delta \mathbf{u} - \nabla p. \end{aligned} \quad (41)$$

Then, the error estimates are listed in Tables 1–4, while the convergence rates are plotted with respect to h in Figure 1.

TABLE 2: Numerical error of $\|p\|_{L^2}$ for Example 1.

t	h									
	2^{-2}		2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	error	rate	error	rate	error	rate	error	rate	error	rate
2^{-2}	1.64e-1	—	8.17e-2	1.0092	4.03e-2	1.0193	2.01e-2	1.0061	1.00e-2	1.0016
2^{-4}	1.58e-1	—	8.01e-2	0.9846	4.01e-2	0.9983	2.00e-3	0.9997	1.00e-4	0.9999
2^{-6}	1.58e-1	—	8.00e-2	0.9841	4.01e-2	0.9976	2.00e-3	0.9995	1.00e-4	0.9999
2^{-8}	1.58e-1	—	8.00e-2	0.9842	4.01e-2	0.9976	2.00e-3	0.9996	1.00e-4	0.9999
2^{-10}	1.58e-1	—	8.00e-2	0.9842	4.01e-2	0.9976	2.00e-3	0.9995	1.00e-4	0.9999
2^{-12}	1.58e-1	—	8.00e-2	0.9842	4.01e-2	0.9976	2.00e-3	0.9995	1.00e-4	0.9999
0	1.59e-1	—	8.00e-2	0.9904	4.01e-2	0.9984	2.00e-3	0.9996	1.00e-4	0.9999

TABLE 3: Numerical error of $\|u\|_a$ for Example 1.

t	h									
	2^{-2}		2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	error	rate	error	rate	error	rate	error	rate	error	rate
2^{-2}	3.51e+0	—	1.76e+0	0.9935	8.84e-1	0.9963	4.42e-1	0.9989	2.21e-1	0.9997
2^{-4}	3.14e+0	—	1.59e+0	0.9843	7.96e-1	0.9953	3.99e-1	0.9987	1.99e-1	0.9996
2^{-6}	3.16e+0	—	1.59e+0	0.9926	7.92e-1	1.0027	3.96e-1	1.0007	1.98e-1	1.0002
2^{-8}	3.16e+0	—	1.59e+0	0.9932	7.92e-1	1.0040	3.96e-1	1.0014	1.98e-1	1.0004
2^{-10}	3.16e+0	—	1.59e+0	0.9933	7.92e-1	1.0041	3.96e-1	1.0014	1.98e-1	1.0004
2^{-12}	3.16e+0	—	1.59e+0	0.9933	7.92e-1	1.0041	3.96e-1	1.0015	1.98e-1	1.0004
0	3.21e+0	—	1.59e+0	1.0092	7.93e-1	1.0076	3.96e-1	1.0020	1.98e-1	1.0005

FIGURE 1: Convergence of Example 1 when $t = 0$ and $t = 0.25$.

FIGURE 2: The true and computed pressure solutions of Example 1.

TABLE 4: Numerical error of $\|P_s p\|_{L^2}$ for Example 1.

t	h									
	2^{-2}		2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	error	rate	error	rate	error	rate	error	rate	error	rate
2^{-2}	4.92e-2	—	1.79e-2	1.4621	4.76e-3	1.9068	1.21e-3	1.9813	3.02e-4	1.9962
2^{-4}	2.16e-2	—	7.14e-3	1.5973	1.92e-3	1.8963	4.89e-4	1.9727	1.23e-4	1.9931
2^{-6}	2.05e-2	—	6.56e-3	1.6422	1.77e-3	1.8935	4.54e-4	1.9587	1.15e-4	1.9856
2^{-8}	2.04e-2	—	6.49e-3	1.6510	1.74e-3	1.9033	4.45e-4	1.9633	1.13e-4	1.9829
2^{-10}	2.04e-2	—	6.49e-3	1.6516	1.73e-3	1.9044	4.44e-4	1.9652	1.12e-4	1.9853
2^{-12}	2.04e-2	—	6.49e-3	1.6517	1.73e-3	1.9045	4.44e-4	1.9653	1.12e-4	1.9985
0	2.57e-2	—	7.04e-3	1.8699	1.80e-3	1.9695	4.51e-4	1.9926	1.13e-4	1.9982

TABLE 5: Numerical error of $\|\mathbf{u}\|_{L^2}$ for Example 2.

t	h									
	2^{-2}		2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	error	rate	error	rate	error	rate	error	rate	error	rate
2^{-2}	1.63e-1	—	5.13e-3	1.6777	1.42e-3	1.8477	2.07e-4	1.9282	5.47e-5	1.9656
2^{-4}	4.22e-2	—	1.81e-2	1.2193	6.77e-3	1.4218	2.09e-3	1.6923	5.82e-4	1.8462
2^{-6}	1.17e-1	—	4.74e-2	1.3052	1.59e-2	1.5723	6.37e-3	1.3227	2.43e-3	1.3928
2^{-8}	1.68e-1	—	1.05e-2	0.6784	5.41e-2	0.9530	2.13e-3	1.3458	6.46e-3	1.7204
2^{-10}	1.71e-1	—	1.17e-1	0.5410	8.00e-2	0.5547	5.05e-2	0.6650	2.62e-2	0.9479
2^{-12}	1.71e-1	—	1.17e-1	0.5378	8.17e-2	0.5288	5.69e-2	0.5212	3.92e-2	0.5373

TABLE 6: Numerical error of $\|p\|_{L^2}$ for Example 2.

t	h									
	2^{-2}		2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	error	rate	error	rate	error	rate	error	rate	error	rate
2^{-2}	2.46e-2	—	1.28e-2	0.9439	6.39e-3	0.9972	3.19e-3	1.0021	1.59e-3	1.0010
2^{-4}	1.18e-2	—	6.79e-3	0.7994	3.41e-3	0.9935	1.65e-3	1.0529	8.06e-4	1.0302
2^{-6}	1.57e-2	—	6.65e-3	1.2433	2.74e-2	1.2787	1.24e-3	1.1456	5.23e-4	1.2441
2^{-8}	1.76e-2	—	8.96e-3	0.9766	4.08e-3	1.1342	1.61e-3	1.3455	6.39e-3	1.3293
2^{-10}	1.78e-2	—	9.29e-3	0.9373	4.60e-3	1.0130	2.22e-3	1.0534	1.00e-3	1.1452
2^{-12}	1.78e-2	—	9.31e-1	0.9352	4.65e-3	1.0022	2.30e-3	1.0130	1.14e-3	1.0196

Figure 2 gives the true and approximate pressure. We can find that all the error orders are optimal and consistent with the theoretical results of Theorem 4.

Example 2. The solution is dependent on perturbation parameter t [taking a case of [2] as example], and

the analytical solution of the problem is as follows:

$$\begin{aligned} \mathbf{u} &= t \text{curl} e^{-xy/t}, \\ p &= t e^{-x/t}. \end{aligned} \quad (42)$$

Then, the error estimates are listed in Tables 5–8, while the convergence rates are plotted with respect to h in Figure 3. It

TABLE 7: Numerical error of $\|\mathbf{u}\|_a$ for Example 2.

t	h									
	2^{-2}		2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	error	rate	error	rate	error	rate	error	rate	error	rate
2^{-2}	5.97e-2	—	3.24e-2	0.8801	1.69e-2	0.9368	8.68e-3	0.9655	4.39e-3	0.9816
2^{-4}	6.70e-2	—	3.89e-2	0.7846	2.30e-2	0.7567	1.29e-2	0.8299	6.93e-3	0.9015
2^{-6}	1.21e-1	—	5.81e-2	1.0551	3.00e-2	0.9542	1.79e-2	0.7418	1.10e-2	0.7019
2^{-8}	1.68e-1	—	1.05e-1	0.6733	5.61e-2	0.9077	2.72e-2	1.0464	1.43e-2	0.9267
2^{-10}	1.71e-1	—	1.18e-3	0.5405	8.01e-2	0.5528	5.08e-3	0.6591	2.72e-2	0.8991
2^{-12}	1.71e-2	—	1.18e-1	0.5378	8.17e-2	0.5286	5.69e-2	0.5206	3.93e-2	0.5353

TABLE 8: Numerical error of $\|P_s p\|_{L^2}$ for Example 2.

t	h									
	2^{-2}		2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	error	rate	error	rate	error	rate	error	rate	error	rate
2^{-2}	5.78e-3	—	2.38e-3	1.2802	7.44e-4	1.6785	2.07e-4	1.8445	5.47e-5	1.9222
2^{-4}	1.02e-2	—	4.60e-3	1.1446	1.66e-3	1.4703	5.00e-4	1.7296	1.37e-4	1.8671
2^{-6}	1.57e-2	—	6.63e-3	1.2474	2.64e-3	1.3316	1.07e-3	1.3008	3.67e-4	1.5438
2^{-8}	1.76e-2	—	8.96e-3	0.9765	4.08e-3	1.1342	1.61e-2	1.3466	6.32e-4	1.3442
2^{-10}	1.78e-2	—	9.29e-3	0.9373	4.60e-3	1.0130	2.21e-3	1.0534	1.00e-3	1.1452
2^{-12}	1.78e-2	—	9.31e-3	0.9352	4.65e-3	1.0022	2.30e-3	1.0130	1.14e-3	1.0195

FIGURE 3: Convergence of Example 2 when $t = 2^{-2}$ and $t = 2^{-12}$.

FIGURE 4: (a) Profile of permeability; (b) velocity component u_1 ; and (c) velocity component u_2 .

FIGURE 5: Two profiles of permeability: (a) open foam; (b) fibrous.

is shown that the convergence rate for $\mathbf{u}$ is $O(h^{1/2})$ which is coincident with our theoretical analysis of Theorem 5.

In the following, we give two tests for the reliability of the proposed method on several high contrast of permeability. The corresponding boundary conditions can be set as $\mathbf{u} = (1, 0)$ and $\mathbf{f} = 0$.

Example 3. We solve the Brinkman equations on a 100×100 fine mesh with a high-contrast permeability. In Figure 4(a), the red color denotes regions of media with high permeability $K^{-1} = 1$, and the blue region is low permeability $K^{-1} = 10^5$. The computed first and second components of the velocity are shown in Figures 4(b) and 4(c). The results look rather similar to [11].

Example 4. Using Darcy and Brinkman models, we test the effects of numerical modeling on fluid flow in porous

media with different sets of permeability distribution. A description of the open foam and fibrous can be shown in Figures 5(a) and 5(b) on a 100×100 fine mesh. The computed velocity component u_1 of Darcy and Brinkman models is presented in Figure 6. We can find that the diffusion effect of Brinkman model is apparent, while the difference in the upper horizontal boundary is visible as no vertical flow is imposed. These results demonstrate that the numerical method is feasible and the simulation is reasonable [13].

Data Availability

The [numerical results] data used to support the findings of this study are included within the article. These data [numeric results] were obtained by the authors by writing MATLAB codes in order to verify the correctness of the theoretical analysis. There is no data cited.

FIGURE 6: Velocity component u_1 for two sets of permeability distribution.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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