Research Article

Efficient Implementation and Numerical Analysis of Finite Element Method for Fractional Allen-Cahn Equation

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We embed the fractional Allen-Cahn equation into a Galerkin variational framework and thus develop its corresponding finite element procedure and then prove rigorously its mathematical and physical properties for the finite element solution. Combining the merits of the conjugate gradient (CG) algorithm and the Toeplitz structure of the coefficient matrix, we design a fast CG for the linearized finite element scheme to reduce the computation cost and the storage to $O(M \log M)$ and $O(M)$, respectively.

Numerical experiments confirm that the proposed fast CG algorithm recognizes accurately the mass and energy dissipation, the phase separation through a very clear coarsening process, and the influences of different indices of fractional Laplacian and different coefficients $K, \eta$ on the width of the interfaces.

1. Introduction

As a typical phase-field model, the classical Allen-Cahn equation [1]

$$\partial_t u - K \Delta u + \psi'(u) = 0,$$

(1)

was originally derived through the minimization of the Ginzburg-Landau free energy functional

$$\int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \psi(u) \right) dx$$

(2)

to describe the motion of antiphase boundaries in crystalline solids with the double-well potential $\psi(u) = (1/4\eta^2)(u^2 - 1)^2$. Since then, the Allen-Cahn equation has been widely applied to many complicated moving interface problems, for example, vesicle membranes, nucleation of solids, and mixture of two incompressible fluids, etc. (cf. [2–8]), and many and many research results on its theories, applications, and numerics of the Allen-Cahn equation have been achieved; see the reviews [9–15] and the references cited therein.

To recognize the influences of the long-range interactions between particles in those complicated moving interface problems, it could reasonably make physical significance if the Laplacian operator in (1) is replaced by its fractional version of Riesz-type potential to form the fractional Allen-Cahn equation. In this line, [16] proposed a kind of fractional Allen-Cahn model and discussed the solvability in some fractional Sobolev spaces, and [17] developed a fractional extension of the Allen-Cahn phase-field model with its fractional Laplacian defined by Riemann-Liouville fractional derivative that describes the mixture of two incompressible fluids. In [17], the authors also proposed a Petrov-Galerkin spectral method for spacial discretization combined with a stabilized ADI scheme for temporal discretization in the absence of rigorous numerical analysis for the solvability, stability, convergence, and conservation properties of the numerical scheme. Some other numerical methods such as finite difference [18], finite volume [19], and collocation method [20] were established. As far as we know, few research works have been done on the efficient finite element method and its rigorous numerical analysis for the fractional Laplace operator defined as Riesz-type potentials on the whole space $\mathbb{R}$.

In this article, we consider the following fractional Allen-Cahn equation [16]:

$$\partial_t u + K \left([(-\Delta)^{\gamma} u + \psi'(u)]\right) = 0, \quad (x, t) \in \Omega \times (0, T],$$

where $\gamma$ is a positive real number.
where \( \Omega = (0,1), r \in (0,1), K > 0 \) is the diffusion coefficient and the fractional Laplace operator \((-\Delta)^{r}\) is defined as Riesz-type potentials on the whole space \( R \) in Section 2. The unknown \( \eta(x) \) can be viewed as an indicator of the concentration or volume fraction of one fluid at the location \( x \) in the immiscible mixture with the second fluid.

The main objectives of this article are to (1) embed the fractional Allen-Cahn equation (1) into a Galerkin variational framework and thus develop its corresponding finite element procedure; (2) prove rigorously the solvability, the optimal convergence rates, and the new energy equality; (3) combine the Toepplitz structure of the coefficient matrix and the merits of the classic CG algorithm [21] to design a fast CG (FCG) for the linearized finite element scheme, which reduces the computation cost and the storage to \( O(M \log M) \) and \( O(M) \), respectively; and (4) conduct numerical experiments to verify the efficiency of the FCG, which show that the FCG possesses the ideal convergence rates as Newtons algorithm does in the efficiency of the FCG, which show that the FCG possesses the ideal convergence rates as Newtons algorithm does in space and time, preserves the mass, energy dissipation, and energy equality law, and recognizes accurately the phase separation by a very clear coarse graining process. The numerical experiments also test the tunable sharpness, that is, the influences of different fractional indices \( r \) and different coefficients \( \eta \).

The rest of this article is outlined as follows. Section 2 is preliminaries. Section 3 is for the solvability and stability of discrete system. We demonstrate that the discrete solution preserves mass and energy dissipation and satisfies new energy equality in Section 4. Sections 5 and 6 are devoted to convergence procedure and efficient FCG algorithm, respectively. In the last section, numerical experiments are conducted to test the efficiency of the proposed efficient finite element algorithm.

2. Preliminaries

We first briefly revisit the definitions and some properties of fractional Laplace operator.

**Definition 1** (see [22, 23]). For \( r \in (0,1) \) and \( u \in \mathcal{S}(\mathbb{R}) \), where \( \mathcal{S}(\mathbb{R}) \) is the Schwartz class of rapidly decaying functions at infinity, the fractional Laplace operator \((-\Delta)^{r}\) is reformulated by

\[
(-\Delta)^{r} u (x) = C_{r} \lim_{\epsilon \to 0} \int_{R \setminus (R, x, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{1+2r}} \, dy,
\]

where \( C_{r} \) is a constant given by

\[
C_{r} = \frac{4^{r} \Gamma(1/2 + r)}{\pi^{1/2} \Gamma(1 - r)}.
\]

**Definition 2** (see [22, 24, 25]). For \( 0 < r < 1 \), the fractional Sobolev spaces \( H_{0}^{r}(R) \) are defined by

\[
H_{0}^{r}(R) = \begin{cases}
\{ v \in L^{2}(R) : v = 0 \text{ in } R \setminus \Omega \} & r = 0, \\
\{ v \in L^{2}(R) : |v(x) - v(y)|^{2} \in L^{1}(R \times R) \} & 0 < r < 1.
\end{cases}
\]

and equipped with the norm

\[
\| v \|_{H_{0}^{r}(R)}^{2} = C_{r} \int_{R} \int_{R} \frac{|v(x) - v(y)|^{2}}{|x - y|^{1+2r}} \, dx \, dy,
\]

and with equivalent seminorm

\[
\| v \|_{H_{0}^{r}(R)} = \frac{1}{C_{r}} \int_{R} \int_{R} \frac{|v(x) - v(y)|^{2}}{|x - y|^{1+2r}} \, dx \, dy.
\]

The energy space and the energy are defined, respectively, by

\[
H_{r}(R) = \{ v \in H_{0}^{r}(R) : E_{r}(v) < +\infty \},
\]

\[
E_{r}(v) = \frac{K}{2} \| v \|_{H_{0}^{r}(R)}^{2} + \frac{K}{4r^{2}} \int_{R} v^{4} \, dx - \frac{K}{2r^{2}} \int_{R} v^{2} \, dx.
\]

**Definition 3** (see [22]). For \( 0 < r < 1 \), the operator \( T_{r} : H_{0}^{r}(R) \rightarrow H_{0}^{r}(R) \) is defined by

\[
\langle T_{r} u, v \rangle_{H_{0}^{r}} = \frac{C_{r}}{2} \int_{R} \int_{R} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+2r}} \, dx \, dy,
\]

\[
\forall u, v \in H_{0}^{r}.
\]

Here \( H_{0}^{r}(R) \) stands for its dual of \( H_{0}^{r}(R) \).

It is easily seen that \( T_{r} \) is a symmetric positive definite operator.

**Definition 4** (see [22]). Let \( T > 0 \). If \( u \in C([0,T];H_{0}^{r}) \cap W_{1}^{2,2}([0,T];H_{0}^{r}) \) satisfies

\[
\partial_{t} u + K \left[ (-\Delta)^{r} u + \frac{1}{\eta^{2}} (u^{3} - u) \right] = 0, \quad a.e. \text{ in } H_{0}^{r},
\]

then the \( u \) is called a weak solution to problem (3).

3. Finite Element Procedure

In this section, we construct finite element scheme for (3) based on the weak formulation (12) and prove the solvability and stability of solution of discrete system.
Multiplying (12) by any \( v \in H_0^r \), integrating over \( R \) and combining the homogeneous boundary condition, we obtain the variational formulation of (3) as to find \( u \in H_0^r \) such that

\[
(a) \left( \partial_t u, v \right) + K \left( T_u, v \right) + \frac{K}{\eta^2} \left( u^3, v \right) - \frac{K}{\eta^2} (u, v) = 0, \\
\forall v \in H_0^r, \tag{13}
\]

(b) \( u(x, 0) = u^0(x) \).

Taking \( M \) and \( N \) as integers, we divide \( \Omega = [0, 1] \) uniformly by intervals \( I_i = [x_{i-1}, x_i] \) for \( i = 1, 2, \ldots, M \) with \( x_0 = 0, x_M = 1, h = h_i = x_i - x_{i-1} = 1/M, \) and partition the time interval \([0, T]\) by the nodes \( t_n = nt \) for \( n = 0, 1, \ldots, N \) with the time step \( \tau = T/N \).

Upon the space partition, we define the finite element space as

\[
V_h = \left\{ v_h \in C(\Omega) : v_h|_{I_i} \in P_k(I_i), \quad i = 1, 2, \ldots, M; \quad v_h(0) = v_h(1) = 0; \quad k \geq 1 \right\}, \tag{14}
\]

where \( P_k(I_i) \) being the set of polynomials of degree not bigger than \( k \) over the interval \( I_i \).

Applying the backward Euler scheme to discrete the time derivative \( \partial_t u \), we define the fully discrete finite element procedure of (13) as to find \( u^n_h \in V_h \) such that, for \( v_h \in V_h \),

\[
(a) \left( \frac{u^n_h - u^{n-1}_h}{\tau}, v_h \right) + K \left( T_u^n, v_h \right) + \frac{K}{\eta^2} \left( u^n_h^3, v_h \right) - \frac{K}{\eta^2} (u^n_h, v_h) = 0, \tag{15}
\]

(b) \( u^0_h = R_h u^0 \),

where \( R_h u^0 \) is the elliptic projection of the initial value \( u^0(x) \) to the finite element space.

The dimension of \( V_h \) is \((kM - 1) \times (kM - 1)\); see the review [24]. Assume \( \varphi_i(x), i = 1, 2, \ldots, kM - 1 \) are the basis functions; then, we can express the numerical solution \( u^n_h \) by

\[
u^n_h(x) = \sum_{i=1}^{kM-1} u^n_i \varphi_i(x), \tag{16}
\]

and, thus, the fully discrete finite element scheme (15) is transformed equivalently to the following algebraic equation:

\[
\left( \left( 1 - \frac{\tau K}{\eta^2} \right) A + \tau KB \right) U^n + \frac{\tau K}{\eta^2} H(U^n) U^n = A U^{n-1}, \tag{17}
\]

where \( U^n = (u^n_0, u^n_1, \ldots, u^n_{kM-1})^T \) is the unknown vector, and the matrices \( A, B, H(U^n) \) are

\[
A = \left[ \left( \varphi_i, \varphi_j \right) \right]_{(kM-1) \times (kM-1)}, \\
B = \left[ \left( T \varphi_i, \varphi_j \right) \right]_{(kM-1) \times (kM-1)}, \tag{18}
\]

\[
H(U^n) = \left[ \left( u^n_h^3 \varphi_i, \varphi_j \right) \right]_{(kM-1) \times (kM-1)}. \]

It is easily verified that \( A, B, \) and \( H(U^n) \) are symmetric and positive matrices.

By using the contraction mapping principle, we prove the existence and uniqueness of the fully discrete finite element scheme (15).

**Theorem 5.** There exists a unique solution \( U^n, n = 1, 2, \ldots, N \) to (17) for sufficiently small \( \tau > 0 \).

**Proof.** Selecting the time step \( \tau \) sufficiently small such that \( 0 < \tau CK/\eta^2 < 1 \) and noticing that \( \tau K > 0 \) and the matrices \( A \) and \( B \) are positive definite, we know that the matrix

\[
E := \left( 1 - \frac{\tau K}{\eta^2} \right) A + \tau KB \tag{19}
\]

is positive definite and thus invertible for sufficiently small \( \tau > 0 \). Therefore, we solve \( U^n \) from (17)

\[
U^n = -E^{-1} \frac{\tau K}{\eta^2} H(U^n) U^n + E^{-1} A U^{n-1}. \tag{20}
\]

Define the mapping \( \mathcal{T} : X \in R^{kM-1} \to Y \in R^{kM-1} \) by

\[
Y = -E^{-1} \frac{\tau K}{\eta^2} H(X)X + E^{-1} A U^{n-1}. \tag{21}
\]

The mapping \( \mathcal{T} \) is well defined for given \( U^{n-1} \) due to the positiveness of the matrix \( E \).

We shall use a corollary of the well-known contraction mapping principle [26] to prove the mapping \( \mathcal{T} \) having a unique fixed point in a bounded domain \( \mathcal{U} = \{X \in R^{kM-1} : \|X\| \leq L \} \) of \( R^{kM-1} \).

For this purpose, we let \( X_i \in \mathcal{U} \) and \( Y_i = T(X_i) \) for \( i = 1, 2 \). Then,

\[
\|Y_1 - Y_2\| = \|\mathcal{T}(X_1) - \mathcal{T}(X_2)\| \leq - \frac{\tau K}{\eta^2} \|E^{-1}(H(X_1)X_1 - H(X_2)X_2)\|. \tag{22}
\]

Noticing that the matrix \( H(X) \) is Lipschitz continuously with respect to \( X \), we obtain

\[
\|Y_1 - Y_2\| \leq \frac{\tau K}{\eta^2} \|E^{-1}\| \|H(X_1)X_1 - H(X_2)X_2\| \leq \frac{\tau K}{\eta^2} \|E^{-1}\| \|H(X_1)X_1 - H(X_2)X_2\| \leq \frac{\tau K}{\eta^2} \|E^{-1}\| \sup_{X_1,X_2 \in \mathcal{U}} \|H(X_1)\| + \sup_{X_1,X_2 \in \mathcal{U}} \|X_1 - X_2\| \|X_1 - X_2\| \tag{23}
\]

Thus, if \( \tau K/\eta^2 < 1 \), then \( \mathcal{T} \) is a contraction mapping and

\[
\|Y_1 - Y_2\| \leq \frac{\tau K}{\eta^2} \|E^{-1}\| \|X_1 - X_2\| \|X_1 - X_2\|. \tag{24}
\]

By the contraction mapping principle, there is a unique fixed point \( U^n \) of \( \mathcal{T} \), and

\[
\|U^n - U^{n-1}\| \leq \frac{\tau K}{\eta^2} \|E^{-1}\| \|X_1 - X_2\| \|X_1 - X_2\|. \tag{25}
\]
Select \( \tau \) to be small enough such that
\[
0 < \tau < \left( \frac{K}{\eta^2} \| E^{-1} \| \sup_{X_1, X_2 \in W} (\| H (X_1) \| + C \| X_2 \|) \right)^{-1},
\]
and we have
\[
\| Y_1 - Y_2 \| < \| X_1 - X_2 \|,
\]
which shows that the mapping \( \mathcal{T} \) is a contractive mapping. This, together with an application of the well-known contraction mapping principle, completes the proof. \( \square \)

**Theorem 6.** Assume that \( \tau \) is small enough such that the conditions of Theorem 5 hold. Then, the fully discrete finite element scheme (15) is stable in the following sense, for \( J = 1, 2, \ldots, N, \)
\[
\| u_h^n \|_2^2 + 2 \tau K \sum_{i=1}^J |u_h^i|_{H^1}^2 + \frac{2 \tau K}{\eta^2} \sum_{i=1}^J \| u_h^i \|_{L^4}^4
\leq e^{2TK/\eta^2} \| u_0^0 \|_2^2 .
\]
Proof. Take \( v_h = u_h^n \) in (15) to obtain
\[
(\nu_h^n - \nu_h^{n-1})/\tau + K \langle T, \nu_h^n, \nu_h^n \rangle
+ K \eta^3 \left( (u_h^3, u_h^3) - \frac{K}{\eta^2} (u_h^n, u_h^n) \right) = 0,
\]
and
\[
(\nu_h^n - \nu_h^{n-1}, \nu_h^n) \geq \frac{1}{2} \| \nu_h^n \|_2^2 - \frac{1}{2} \| \nu_h^{n-1} \|_2^2 ,
\]
we have
\[
\| u_h^n \|_2^2 - \| u_h^{n-1} \|_2^2 + 2 \tau K |u_h^i|_{H^1}^2
\leq \frac{2 \tau K}{\eta^2} \left( \| u_h^i \|_{L^4}^4 - \| u_h^i \|_2^2 \right) \leq 0.
\]
Adding all the terms from \( n = 1 \) to \( n = J, \) we get
\[
\| u_h^J \|_2^2 - \| u_h^0 \|_2^2 + 2 \tau K \sum_{i=1}^J |u_h^i|_{H^1}^2
+ \frac{2 \tau K}{\eta^2} \left( \sum_{i=1}^J \| u_h^i \|_{L^4}^4 - \sum_{i=1}^J \| u_h^i \|_2^2 \right) \leq 0.
\]
Namely,
\[
\| u_h^J \|_2^2 + 2 \tau K \sum_{i=1}^J |u_h^i|_{H^1}^2 + \frac{2 \tau K}{\eta^2} \sum_{i=1}^J \| u_h^i \|_{L^4}^4
\leq \frac{2 \tau K}{\eta^2} \sum_{i=1}^J \| u_h^i \|_2^2 + \| u_0^0 \|_2^2 .
\]
Applying the discrete Gronwall inequality, we have
\[
\| u_h^J \|_2^2 + 2 \tau K \sum_{i=1}^J |u_h^i|_{H^1}^2 + \frac{2 \tau K}{\eta^2} \sum_{i=1}^J \| u_h^i \|_{L^4}^4
\leq e^{2TK/\eta^2} \| u_0^0 \|_2^2 \leq e^{2TK/\eta^2} \| u_0^0 \|_2^2 .
\]
This completes the proof. \( \square \)

**4. Properties Preserved by the Finite Element Solution**

In this section, we demonstrate that the finite element solution preserves the energy dissipation law and satisfies a redefined energy equality.

**Theorem 7.** The finite element solution \( u_h^n \) of (15) preserves the energy dissipation law in the following sense, for \( J = 1, 2, \ldots, N, \)
\[
E_r (u_h^n) \leq E_r (u_h^{n-1}) \leq \cdots \leq E_r (u_0) .
\]
Proof. Taking \( v_h = (u_h^n - u_h^{n-1})/\tau \) in (15), we have
\[
\left( \frac{u_h^n - u_h^{n-1}}{\tau} - \frac{u_h^n - u_h^{n-1}}{\tau} \right) + K \left( \frac{T, u_h^n, u_h^n}{\tau} \right)
\]
\[
+ \frac{K}{\eta^3} \left( (u_h^3, u_h^3) - \frac{K}{\eta^2} (u_h^n, u_h^n) \right) = 0.
\]
Due to the fact that
\[
\| u_h^n \|_2^2 = (u_h^n, u_h^n) - (u_h^n)^T A (u_h^n - u_h^{n-1})
\]
and
\[
\| u_h^n \|_2^2 = (u_h^n, u_h^n) - (u_h^n)^T B (u_h^n - u_h^{n-1})
\]
\[
\leq \frac{1}{2} \left( (u_h^n)^T A (u_h^n - u_h^{n-1}) \right)
\leq \frac{1}{2} \left( (u_h^n)^T A (u_h^n - u_h^{n-1}) \right)
\]
and
\[
\langle T, u_h^n, u_h^n - u_h^{n-1} \rangle = (u_h^n)^T B (u_h^n - u_h^{n-1})
\]
\[
\leq \frac{1}{2} \left( (u_h^n)^T B (u_h^n - u_h^{n-1}) \right).
\]
By applying the inequality \( ab \leq (1/2)(a^2 + b^2) \), we get
\[
((u_h^n)^3, u_h^n - u_h^{n-1}) = ((u_h^n)^2, u_h^n - u_h^{n-1})
\]
\[
\geq \frac{1}{2} \left( ((u_h^n)^2, (u_h^n)^2 - (u_h^{n-1})^2) \right)
\]
\[
\geq \frac{1}{4} \left( (u_h^n)^4 - (u_h^{n-1})^4, 1 \right).
\]

Then, combining the above inequalities with (35), we derive
\[
\tau \left\| u_h^n - u_h^{n-1} \right\|_\tau^2 + \frac{K}{2} \left( \left\| u_h^n_{lH} \right\|_2^2 - \left\| u_h^{n-1}_{lH} \right\|_2^2 \right)
\]
\[
+ \frac{K}{4\eta^2} \left( \left\| u_h^n \right\|_L^4 - \left\| u_h^{n-1} \right\|_L^4 \right)
\]
\[
- \frac{K}{2\eta^2} \left( \left\| u_h^n \right\|_L^2 - \left\| u_h^{n-1} \right\|_L^2 \right) \leq 0.
\]

Therefore, we have
\[
\frac{K}{2} \left\| u_h^n_{lH} \right\|_2^2 + \frac{K}{4\eta^2} \left\| u_h^n \right\|_L^4 - \frac{K}{2\eta^2} \left\| u_h^n \right\|_L^2 \leq 0.
\]

By a simple calculation, we have
\[
\langle T, u_h^n - u_h^{n-1} \rangle = \frac{1}{2} \left\| u_h^n_{lH} \right\|_2^2 - \frac{1}{2} \left\| u_h^{n-1}_{lH} \right\|_2^2
\]
\[
+ \frac{1}{2} \left\| u_h^n - u_h^{n-1} \right\|_2^2
\]
\[
\geq \frac{1}{4} \left\| u_h^n - u_h^{n-1} \right\|_2^2.
\]

Substituting the above equalities into (43) and combining (10), we have
\[
\tau E_r(u_h^n) = \tau E_r(u_h^{n-1}) + \left\| u_h^n - u_h^{n-1} \right\|_2^2
\]
\[
+ \frac{\tau K}{4\eta^2} \left\| u_h^n - u_h^{n-1} \right\|_H^2 + \frac{\tau K}{4\eta^2} \left\| (u_h^n - u_h^{n-1})^2 - (u_h^{n-1})^2 \right\|_2^2
\]
\[
+ \frac{\tau K}{4\eta^2} \left\| (u_h^n - u_h^{n-1})^2 - (u_h^{n-1})^2 \right\|_2^2
\]
\[
= 0.
\]

This completes the proof.

### 5. Convergence Analysis

In this section, we shall conduct convergence analysis for the fully discrete finite element scheme (15).

Define the elliptic projection \( R_h u \) of the exact solution \( u \) as
\[
\langle T, (u - R_h u), v_h \rangle = 0, \quad \forall v_h \in V_h.
\]
Theorem 9. Assume that \( u \in H^2(\Omega) \) and \( s > r \) [27],

\[
\|u - R_h u\| \leq C h^{\min(k+1-r,s-r)} \|u\|_H.
\]

Let \( t = t_n \) and \( v = v_h \) in (13), and subtract it from (15) to obtain the error equation

\[
(a) \left( u^n_t - \frac{u^n - u^{n-1}_t}{\tau}, v_h \right) + K \left( T_n (u^n - u^n_h), v_h \right) + \frac{K}{\eta^2} \left( (u^n)^3 - (u^n_h)^3, v_h \right) = 0,
\]

\[
\forall v_h \in V_h,
\]

(b) \( u^n_0 = R_h u^0 \).

Denote

\[
u^n - R_h u^n - u^n_0 = S^n,
\]

\[
u^n_0 - R_h u^n_0 - u^n_0 = \rho^n + \theta^n,
\]

then

\[
u^n - u^n_0 = \nu^n - R_h u^n + R_h u^n - u^n_0 = \rho^n + \theta^n.
\]

The following estimate is valid for \( u \in H^s(\Omega) \) and \( s > r \) [27],

\[
\|u - R_h u\| \leq C h^{\min(k+1-r,s-r)} \|u\|_H.
\]

Let \( t = t_n \) and \( v = v_h \) in (13), and subtract it from (15) to obtain the error equation

\[
(a) \left( u^n_t - \frac{u^n - u^{n-1}_t}{\tau}, v_h \right) + K \left( T_n (u^n - u^n_h), v_h \right) + \frac{K}{\eta^2} \left( (u^n)^3 - (u^n_h)^3, v_h \right) = 0,
\]

\[
\forall v_h \in V_h,
\]

(b) \( u^n_0 = R_h u^0 \).

Denote

\[
u^n - \frac{u^n - u^{n-1}_t}{\tau} = S^n,
\]

\[
u^n - R_h u^n = \rho^n,
\]

\[
R_h u^n - u^n_0 = \theta^n,
\]

then

\[
u^n - u^n_0 = \nu^n - R_h u^n + R_h u^n - u^n_0 = \rho^n + \theta^n.
\]

Theorem 9. Assume that \( u \in H^2(0,T;L^2(R)) \cap H^3(0,T;H_1(R)) \). Then, there exists integers \( m (1 \leq m \leq N) \) and constants \( C_0 \) and \( C_1 \) independent of the step parameters \( h \) and \( \tau \) such that

\[
\|
u^n - u^n_0\|^2 + 2K \tau \sum_{i=1}^m |u^n - u^n_0|^2,
\]

\[
\leq C_0 \tau^2 + C_1 h^{2[\min(k+1,1)-\tau]}
\]

Here

\[
C_0 = C_0 (\|u_t\|_{L^2(0,T;L^2)})^2,
\]

\[
C_1 = C_1 (\|u_t\|_{L^2(0,T;H^2)}), \|u_t\|_{L^2(0,T;H^2)}).
\]

Proof. Take \( v_h = \theta^n \) in (50) to obtain

\[
\left( \frac{\theta^n - \theta^{n-1}}{\tau}, \theta^n \right) + K \left( (R_h u^n)^3 - R_h (u^n)^3, \theta^n \right)
\]

\[
+ \frac{K}{\eta^2} \left( (R_h u^n)^3 - (u^n)^3, \theta^n \right) = (-S^n, \theta^n) + \left( \frac{\rho^n - \rho^{n-1}}{\tau}, \theta^n \right)
\]

\[
+ \frac{K}{\eta^2} \left( (R_h u^n)^3 - (u^n)^3, \theta^n \right) + \frac{K}{\eta^2} \left( \rho^n, \theta^n \right)
\]

\[
+ \frac{K}{\eta^2} \left( \theta^n, \theta^n \right).
\]

We first estimate the left-hand side of (55).

Applying the Hölder inequality and the Young inequality, we have

\[
\left( \frac{\theta^n - \theta^{n-1}}{\tau}, \theta^n \right) = \frac{1}{\tau} \left( ((\theta^n, \theta^n) - (\theta^{n-1}, \theta^n))
\]

\[
\geq \frac{1}{\tau} \left( \|\theta^n\|^2 - \frac{\|\theta^{n-1}\|^2}{2} - \|\theta^n\|^2 \right)
\]

\[
= \frac{1}{2\tau} \left( \|\theta^n\|^2 - \|\theta^{n-1}\|^2 \right).
\]

By using the monotonicity of function \( ((x_1)^3 - (x_2)^3)(x_1 - x_2) \geq 0 \), we have

\[
\left( (R_h u^n)^3 - (u^n)^3, \theta^n \right) \geq 0.
\]

And then, we estimate the right-hand side of (55). Combining the Hölder inequality and the Young inequality, we have

\[
(-S^n, \theta^n) \leq \|S^n\| \|\theta^n\|
\]

\[
= \frac{1}{\tau} \int_{t_{n-1}}^{t_n} (t - t_{n-1}) u_{tt} (t) dt \|\theta^n\|
\]

\[
\leq \frac{1}{\tau} \left( \int_0^1 \int_{t_{n-1}}^{t_n} u_{tt} (t) dt dx \right)^{1/2} \|\theta^n\|
\]

\[
= \tau^{1/2} \left( \int_{t_{n-1}}^{t_n} \|u_{tt} (t)\|^2 dt \right)^{1/2} \|\theta^n\|
\]

\[
\leq \tau \int_{t_{n-1}}^{t_n} \|u_{tt} (t)\|^2 dt + \|\theta^n\|^2.
\]

Similarly, we get

\[
\left( \frac{\rho^n - \rho^{n-1}}{\tau}, \theta^n \right) \leq \left( \frac{\rho^n - \rho^{n-1}}{\tau}, \theta^n \right)
\]

\[
= \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \rho (t) dt \|\theta^n\|
\]

\[
\leq \frac{1}{\tau^{1/2}} \left( \int_{t_{n-1}}^{t_n} \|\rho (t)\|^2 dt \right)^{1/2} \|\theta^n\|
\]

\[
\leq \frac{1}{2\tau} \int_{t_{n-1}}^{t_n} \|\rho (t)\|^2 dt + \frac{1}{2} \|\theta^n\|^2.
\]

The third term on the right-hand side of (55) is estimated by

\[
((R_h u^n)^3 - (u^n)^3, \theta^n) \leq C_1 \|\rho^n\| \|\theta^n\|
\]

\[
\leq C_1 \|\rho^n\|^2 + \frac{C_1}{2} \|\theta^n\|^2
\]

and

\[
(\rho^n, \theta^n) \leq \|\rho^n\| \|\theta^n\| \leq \frac{1}{2} \|\rho^n\|^2 + \frac{1}{2} \|\theta^n\|^2.
\]
Then, taking these inequalities into (55), we obtain
\[
\|\theta^m\|^2 - \|\theta^{m-1}\|^2 + 2K\tau |\theta'|^2_r \\
\leq \tau^2 \int_{t_{m-1}}^{t_m} \|u(t)\|^2 dt + \int_{t_{m-1}}^{t_m} \|\rho(t)\|^2 dt \\
+ \frac{(C_1 + 1)K}{\eta^2} \tau \|\rho''\|^2 \\
+ \left(2 + \frac{C_1 + 3K}{\eta^2}\right) \tau \|\theta''\|^2.
\]
(62)

Noticing \(\theta^0 = u^0_R - R_k u_0 = 0\) and adding all the terms from \(n = 1\) to \(n = N\), we have
\[
\|\theta^m\|^2 + 2K\tau \sum_{i=1}^m |\theta'|^2_r \\
\leq e^{(2\tau^2 + C_1 + 3K)\tau^2} \left( \tau^2 \int_0^T \|u(t)\|^2 dt \\
+ \int_0^T \|\rho(t)\|^2 dt + \frac{C_1 K + K}{\eta^2} \sum_{i=1}^N \|\rho_i\|^2 \right). \\
\]
(63)

If we take \(C = \max \{ (1/3)e^{(2\tau^2 + C_1 + 3K)\tau^2}, e^{(2\tau^2 + C_1 + 3K)\tau^2}, (C_1 K + K)/\eta^2 \} e^{(2\tau^2 + C_1 + 3K)\tau^2} \), then
\[
\|\theta^m\|^2 + 2K\tau \sum_{i=1}^m |\theta'|^2_r \\
\leq C^2 \tau^2 \|u^m\|^2 \|\theta^m\|^2 \\
+ C\tau^{2\min\{k+1,1\}} \|u^m\|^2 \|\theta^m\|^2 \\
+ C\tau^{2\min\{k+1,1\}} \|\theta^m\|^2,
\]
(65)

Then, using the elliptic projection estimate (49) and the triangle inequality, we obtain
\[
\|u^m - u^{m-1}\|^2 + 2K\tau \sum_{i=1}^m |u^m - u^{m-1}|^2_r \\
\leq \|\rho''\|^2 + \|\theta''\|^2 + 2K\tau \sum_{i=1}^m |\theta'|^2_r + 2K\tau \sum_{i=1}^m |\rho'|^2_r \\
\leq C_2 \tau^2 + C_1 h^{2(\min\{k+1,1\}-r)}.
\]
(66)

This concludes the proof.

**Remark 10.** Since our proof for the stability, energy dissipation, new energy-equality and error deduction of the discrete scheme (15) is independent of the dimensional argument, the analysis and conclusions in this article can be extended to multidimensional models, maybe with minor modification.

### 6. Fast-Conjugate-Gradient Algorithm (FCG)

The fully discrete finite element scheme (15) is nonlinear and can be solved by Newton’s iteration algorithm. Here we linearize the nonlinear terms by replacing \((u^k)^2\) by \(1/(u^k)\) and solve this linearized version by the conjugated gradient algorithm (CG). We find that if the CG algorithm is directly used, the computation cost and storage will reach up to \(O(M^2)\) since the coefficient matrix \(B\) is nonsparse due to the nonlocality of the fractional Laplace operator. We also find that if the linear finite element space is employed, the matrix \(B\) is a Toeplitz matrix (see Section 7.1), which makes it possible to reduce the computation cost and storage to \(O(M \log M)\) by a delicate combination of the CG algorithm, the fast Fourier transform (FFT) and its Toeplitz structure of the matrix.

In this section, we depict the general ideas for the combination of CG, FFT, and the Toeplitz matrix and design a fast CG algorithm (FCG). For a good review of a fast algorithm generating from FFT and the Toeplitz matrix, we refer to [28].

Noticing what causes the computation cost up to \(O(M^2)\) is a direct use of the matrix-vector multiplication in the classical CG algorithm; as a remedy, we can obtain the matrix-vector multiplication \(B_{kM-1}x\) for a vector \(x \in \mathbb{R}^{kM-1}\) and \((kM - 1) \times (kM - 1)\) Toeplitz matrix \(B_{kM-1}\) through the following two steps [29, 30]:

1. assembling the \((kM - 1) \times (kM - 1)\) Toeplitz matrix \(B_{kM-1}\) into a \(2(kM - 1) \times 2(kM - 1)\) cyclic matrix \(C_{2(kM-1)}\);

2. extracting the Toeplitz matrix-vector multiplication \(B_{kM-1}x\) for a vector \(x \in \mathbb{R}^{kM-1}\) via

\[
C_{2(kM-1)}x = \begin{pmatrix} B_{kM-1} & D_{kM-1}
\end{pmatrix} \begin{pmatrix} x \\
0
\end{pmatrix} = \begin{pmatrix} B_{kM-1}x \\
D_{kM-1}x
\end{pmatrix}.
\]

In (67), \(D_{kM-1}\) is determined simply by the matrix \(B_{kM-1}\) and the computational cost and the storage of the matrix-vector multiplication \(B_{kM-1}x\) are \(O(M \log M)\) and \(O(M)\) [30]. This will reduce the computation cost and the storage from \(O(M^2)\) to \(O(M \log M)\) and \(O(M)\) compared to the direct use of matrix-vector multiplication.

Upon this analysis, we write down the improved algorithm, called the fast CG (FCG), sentence by sentence in Algorithm 1, and present the corresponding conclusion by Theorem II.

**Theorem 11.** The computational cost and storage of the fast conjugated gradient algorithm (FCG) are reduced from \(O(M^2)\) and \(O(M^2)\) to \(O(M \log M)\) and \(O(M)\) compared with the traditional CG algorithm.
Algorithm 1: The FCG Algorithm.

7. Numerical Experiments

In this section, we linearize the nonlinear term in the fully discrete finite element scheme (15) by taking \((u_{n-1}^h)^3\) to replace \((u_n^h)^3\); we carry out two numerical experiments to verify the result convergence analysis and the physical property of the solution of discrete system.

7.1. The Linear Finite Element System. For simplicity, we take the linear finite element system to compute algebraic equation (17).

For \(k = 1\), the inner nodal basis function is \(\varphi_i(x)\) at \(x_i, i = 1, 2, \ldots, M - 1\), with the following structure:

\[
\varphi_i(x) = \begin{cases} 
\frac{x - x_{i-1}}{h}, & x \in (x_{i-1}, x_i), \\
\frac{x_{i+1} - x}{h}, & x \in (x_i, x_{i+1}), \\
0, & \text{else}. 
\end{cases} \tag{68}
\]

For \(i, j = 1, 2, \ldots, M - 1\), the matrix \(A\) is

\[
A = \left[ (\varphi_i, \varphi_j) \right]_{(M-1) \times (M-1)}
\]

\[
(\left( u_{n-1}^h \right)^2 \varphi_i, \varphi_j) = \begin{cases} 
\frac{30}{30} + \frac{2}{5} + \frac{10}{15}, & |j - i| = 0, \\
\frac{30}{20} + \frac{5}{20} + \frac{10}{15}, & |j - i| = 1, \\
0, & |j - i| = m, m = 2, \ldots, M - 2. 
\end{cases} \tag{71}
\]

For \(i, j = 1, 2, \ldots, M - 1\), the entries of the matrix \(B\) are respectively

\[
\langle T, \varphi_i, \varphi_j \rangle = \frac{C_r}{2} \int \int_{R} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x - y|^{1+2r}} dxdy
\]
\[
\mathcal{C}_r = \frac{2}{r} \left\{ \int_{\Omega} \int_{\Omega} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x-y|^{1+2r}} \, dx \, dy \right. \\
+ \int_{\Omega} \int_{\Omega} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x-y|^{1+2r}} \, dx \, dy \left. + \int_{\Omega} \int_{\Omega} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x-y|^{1+2r}} \, dx \, dy \right\}. \tag{72}
\]

After calculating the above integrals (72), we obtain that the \(ij\)–entries of matrix \(B\) are solved for \(r \neq 1/2\)

\[
\langle T \varphi_i, \varphi_j \rangle = \begin{cases}
\frac{h_{1-2r}^2}{D} \left\{ -4 + 2^{3-2r} \right\}, & |j-i| = 0, \\
\frac{h_{1-2r}^2}{D} \left\{ \frac{7}{2} - 2^{4-2r} + \frac{3^{3-2r}}{2} \right\}, & |j-i| = 1, \\
\frac{h_{1-2r}^2}{D} \left\{ 3m^{3-2r} - 2(m+1)^{3-2r} - 2(m-1)^{3-2r} + \frac{(m-2)^{3-2r}}{2} + \frac{(m+2)^{3-2r}}{2} \right\}, & |j-i| = m, m = 2, \ldots, M-2.
\end{cases}
\tag{73}
\]

where

\[
D = \frac{r(1-2r)(2-2r)(3-2r)}{C_r},
\]

For \(r = 1/2\),

\[
C_r = \frac{4^r r! (1/2 + r)}{\pi^{3/2} r! (1-r)}
\tag{74}
\]

\[
\langle T \varphi_i, \varphi_j \rangle = \begin{cases}
4C_r \ln 2, & |i-j| = 0, \\
C_r \left( \frac{7}{2} \ln 3 - \ln 2 \right), & |i-j| = 1, \\
C_r \left( 14 \ln 2 - 9 \ln 3 \right), & |i-j| = 2, \\
C_r \left\{ 3m^{3-2r} \ln (m) + \frac{(m-2)^2}{2} \ln (m-2) + \frac{(m+2)^2}{2} \ln (m+2) - 2(m-1)^{3-2r} \ln (m-1) - 2(m+1)^{3-2r} \ln (m+1) \right\}, & |i-j| = m, m = 3, \ldots, M-2.
\end{cases}
\tag{75}
\]

### 7.2. Tests on the Efficiency of the Finite Element Procedure and the FCG

**Example 12.** Assuming \(\Omega = [0, 1], T = 1, K = 1\), the analytic solution is

\[
u(x,t) = \chi (1-x) \epsilon \chi \in H^1_0(\Omega),\tag{76}
\]

where \(\gamma \in (0, 1/2)\), is selected as close to 1/2 as possible. The source term \(f\) should be expressed as follows:

\[
f = \chi (1-x) \epsilon \chi + KC_r \left\{ \frac{\chi^{1-2r} \epsilon \chi}{2r (1-2r)} + \frac{(1-x)^{1-2r} \epsilon \chi}{2r (1-2r)} \right\}
\]

We apply Example 12 to verify the error convergence rate of the linear finite element system and to test the efficiency of the FCG for \(r = 1/3, 2/3\).

The numerical solution \(U_h(x)\) is calculated by the FCG for \(T = 1\). The numerical solution and the exact solution are identical at \(t = 1\) in Figure 1, which show that the FCG is accurate.
Table 1: Spatial errors and convergence rates for $\|u - u_h\|_{H^r}$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$h$</th>
<th>Newton rate</th>
<th>Gauss rate</th>
<th>CG rate</th>
<th>FCG rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/3$</td>
<td>$2^{-3}$</td>
<td>1.247E-2</td>
<td>1.336E-2</td>
<td>1.336E-2</td>
<td>1.336E-2</td>
</tr>
<tr>
<td></td>
<td>$2^{-4}$</td>
<td>4.026E-3</td>
<td>4.204E-3</td>
<td>4.204E-3</td>
<td>4.204E-3</td>
</tr>
<tr>
<td></td>
<td>$2^{-5}$</td>
<td>1.276E-3</td>
<td>1.312E-3</td>
<td>1.312E-3</td>
<td>1.312E-3</td>
</tr>
<tr>
<td></td>
<td>$2^{-6}$</td>
<td>4.047E-4</td>
<td>4.121E-4</td>
<td>4.121E-4</td>
<td>4.121E-4</td>
</tr>
<tr>
<td></td>
<td>$2^{-7}$</td>
<td>1.299E-4</td>
<td>1.313E-4</td>
<td>1.313E-4</td>
<td>1.313E-4</td>
</tr>
<tr>
<td></td>
<td>$2^{-4}$</td>
<td>1.808E-2</td>
<td>1.814E-2</td>
<td>1.814E-2</td>
<td>1.814E-2</td>
</tr>
<tr>
<td></td>
<td>$2^{-5}$</td>
<td>7.544E-3</td>
<td>7.556E-3</td>
<td>7.556E-3</td>
<td>7.556E-3</td>
</tr>
<tr>
<td></td>
<td>$2^{-6}$</td>
<td>3.164E-3</td>
<td>3.167E-3</td>
<td>3.167E-3</td>
<td>3.167E-3</td>
</tr>
<tr>
<td></td>
<td>$2^{-7}$</td>
<td>1.361E-3</td>
<td>1.362E-3</td>
<td>1.362E-3</td>
<td>1.362E-3</td>
</tr>
</tbody>
</table>

Table 2: Spatial errors and convergence rates for $\|u - u_h\|$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$h$</th>
<th>Newton rate</th>
<th>Gauss rate</th>
<th>CG rate</th>
<th>FCG rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/3$</td>
<td>$2^{-3}$</td>
<td>6.117E-3</td>
<td>7.269E-3</td>
<td>7.270E-3</td>
<td>7.434E-3</td>
</tr>
<tr>
<td></td>
<td>$2^{-4}$</td>
<td>1.755E-3</td>
<td>2.031E-3</td>
<td>2.030E-3</td>
<td>2.030E-3</td>
</tr>
<tr>
<td></td>
<td>$2^{-5}$</td>
<td>4.800E-4</td>
<td>5.476E-4</td>
<td>5.475E-4</td>
<td>5.540E-4</td>
</tr>
<tr>
<td></td>
<td>$2^{-6}$</td>
<td>1.286E-4</td>
<td>1.451E-4</td>
<td>1.451E-4</td>
<td>1.468E-4</td>
</tr>
<tr>
<td></td>
<td>$2^{-7}$</td>
<td>3.434E-5</td>
<td>3.833E-5</td>
<td>3.833E-5</td>
<td>3.891E-5</td>
</tr>
<tr>
<td>$2/3$</td>
<td>$2^{-3}$</td>
<td>6.517E-3</td>
<td>7.322E-3</td>
<td>7.321E-3</td>
<td>7.497E-3</td>
</tr>
<tr>
<td></td>
<td>$2^{-4}$</td>
<td>2.220E-3</td>
<td>2.411E-3</td>
<td>2.414E-3</td>
<td>2.414E-3</td>
</tr>
<tr>
<td></td>
<td>$2^{-5}$</td>
<td>7.480E-4</td>
<td>7.937E-4</td>
<td>7.937E-4</td>
<td>8.050E-4</td>
</tr>
<tr>
<td></td>
<td>$2^{-6}$</td>
<td>2.567E-4</td>
<td>2.675E-4</td>
<td>2.675E-4</td>
<td>2.711E-4</td>
</tr>
</tbody>
</table>

Table 1 show the spatial convergence rates with respect to $\|u - u_h\|_{H^r}$ for $r = 1/3, 2/3$ respectively. By comparing the convergence rates of FCG to the convergence rate of Newton, Gauss, and CG, respectively, we conclude that the FCG algorithm is efficient. The convergence rates of $\|u - u_h\|_{H^r}$ are at least 1.6 and 1.2 for $r = 1/3, 2/3$, respectively, which are almost equal to the theoretical error rate $\min\{1+\gamma, 2\} - r$.

Table 2 shows the spatial convergence rates with respect to $\|u - u_h\|$ for $r = 1/3, 2/3$, respectively. The convergence rates of $\|u - u_h\|$ are at least 1.9 and 1.5 for $r = 1/3, 2/3$, respectively. We observe that the spatial convergence rates decrease as the spatial fractional order $r$ increases.

Remark 13. Recalling $L_2$ error estimate for second order elliptic equation, one can obtain the estimate of $\|u - u_h\|$ one order higher than that for $|u - u_h|$. However, in the case of the fractional order elliptic equation, we only obtain the convergence rate of $\|u - u_h\|$ dependent on the index $r$. From the numerical Table 2, the convergence order of $\|u - u_h\|$ conforms to the following relation.

Figure 1: Comparison of the initial value, the exact solution, and the numerical solution for $r = 1/3, K = 1, \eta = 1$. 
Table 3: Temporal errors and convergence rates.

<table>
<thead>
<tr>
<th>r</th>
<th>Newton</th>
<th>Gauss</th>
<th>CG</th>
<th>FCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>1.173E-2</td>
<td>2.245E-2</td>
<td>2.245E-2</td>
<td>2.246E-2</td>
</tr>
<tr>
<td>1/8</td>
<td>6.012E-3</td>
<td>1.788E-2</td>
<td>1.020</td>
<td>1.788E-2</td>
</tr>
<tr>
<td>1/10</td>
<td>3.044E-3</td>
<td>1.486E-2</td>
<td>1.017</td>
<td>1.486E-2</td>
</tr>
<tr>
<td>1/12</td>
<td>1.532E-3</td>
<td>1.270E-2</td>
<td>1.014</td>
<td>1.270E-2</td>
</tr>
<tr>
<td>1/14</td>
<td>7.696E-4</td>
<td>1.110E-2</td>
<td>1.012</td>
<td>1.110E-2</td>
</tr>
<tr>
<td>2/3</td>
<td>6.768E-3</td>
<td>1.375E-2</td>
<td>1.376E-2</td>
<td>1.376E-2</td>
</tr>
<tr>
<td>1/8</td>
<td>3.478E-3</td>
<td>1.103E-2</td>
<td>0.988</td>
<td>1.103E-2</td>
</tr>
<tr>
<td>1/10</td>
<td>1.766E-3</td>
<td>9.210E-3</td>
<td>0.991</td>
<td>9.211E-3</td>
</tr>
<tr>
<td>1/12</td>
<td>8.935E-4</td>
<td>7.904E-3</td>
<td>0.992</td>
<td>7.904E-3</td>
</tr>
<tr>
<td>1/14</td>
<td>4.527E-4</td>
<td>6.922E-3</td>
<td>0.993</td>
<td>6.922E-3</td>
</tr>
</tbody>
</table>

Table 4: CPU time of Newton, Gauss, CG, and FCG.

<table>
<thead>
<tr>
<th>h</th>
<th>Newton</th>
<th>Gauss</th>
<th>CG</th>
<th>FCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>r = 1/3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2^{-5}</td>
<td>0.38s</td>
<td>0.25s</td>
<td>0.22s</td>
<td>0.34s</td>
</tr>
<tr>
<td>2^{-6}</td>
<td>1.1m</td>
<td>3.77m</td>
<td>16.63s</td>
<td>4.94s</td>
</tr>
<tr>
<td>2^{-8}</td>
<td>19.3m</td>
<td>1.94h</td>
<td>2.27m</td>
<td>14.71s</td>
</tr>
<tr>
<td>2^{-9}</td>
<td>5.19h</td>
<td>69.9h</td>
<td>23.6m</td>
<td>1.74m</td>
</tr>
<tr>
<td>2^{-10}</td>
<td>15.2h</td>
<td>-</td>
<td>6.79h</td>
<td>25.35m</td>
</tr>
</tbody>
</table>

Figure 2: The relation of spatial convergence rates and fractional order r.

Figure 2 shows that the convergence rate of $\|u - u_h\|$ dependent on the index r, as the index r increases, the rates of spatial convergence decreases and then increases.

Table 3 shows the temporal convergence rates as $r = 1/3, r = 2/3$, respectively. The temporal convergence rates are almost equal to 1 for $r = 1/3, r = 2/3$ at $h = 2^{-10}$, which are consistent with the theoretical expectation for the backward Euler scheme.

Table 4 shows the efficiency of the FCG algorithm. It is easily obtained that the CPU time consumed by the FCG is much less than that of the Newton, Gauss, and CG as h decreases. It can be seen that the CPU time of FCG is 25.35 minutes while the Newton’s 15.2 hours and the CG’s 6.79 hours for $h = 2^{-10}, r = 1/3$. These show that the FCG is an efficiency algorithm.

7.3. Tests on Physical Properties of Finite Element Solution

**Example 14.** Letting $h = 1/100, \tau = 1/500$, the initial value is

$$u(x,0) = 0.9 \sin(2\pi x).$$ (78)

This example is used to test the physical properties such as the energy laws and the new energy equality, the mass dissipation, the phase separation, and the tunable sharpness in Figures 3–8.

We present the energy dissipation in Figure 3 and the new energy equality in Figure 4 for $r = 0.3, \eta = 0.85$, and $K = 1$. In Figure 3, the energy dissipates to zero as time increases. In
Figure 3: Energy dissipation with time increasing.

Figure 4: The new energy equality for $r = 0.3$, $\eta = 0.85$, and $K = 1$.

Figure 5: The mass dissipation for $r = 0.3$, $\eta = 0.85$, and $K = 1$.

Figure 6: The coarse graining process with respect to different $t$.

Figure 7: The tunable sharpness with respect to different $r$.

Figure 4, the energy jumps in scale $10^{-7}$ at the beginning until $t = 1$ and keeps the same at the end; this demonstrates that the finite element solution to (15) almost satisfies new equality equation. We plot the mass changes for $r = 0.3$, $\eta = 0.85$, and $K = 1$ in Figure 5. The mass value dissipates as time increases; this property is coinciding with the classical Allen-Cahn equation, and this also explains the applicability of fractional Laplacian for Allen-Cahn equation in some way. We present the physical phenomenon of a coarse graining process as the time increases for $r = 0.6$, $\eta = 0.1$, and $K =$
0.001 in Figure 6. The two phases are connected in a smooth steep line; this indicates that the phase separation is complete.

Figure 7 shows that the interface gets sharper and sharper as $r$ decreases. Figure 8 shows that the interface gets sharper and sharper as $\eta$ decreases. The above two groups of experimental results show that the tunable sharpness of the interface is related to different parameters $r$ and $\eta$.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

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