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Research Article

Pricing of Fixed-Strike Lookback Options on Assets with Default Risk

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In over-the-counter markets, many options on defaultable instruments are influenced by default risks emanating from the possibility that the option writer may not fulfill its contractual obligations. In this paper, we investigate the valuation of fixed-strike lookback options based on the issuer's credit risk. Using double Mellin transforms and the method of images, we have a closed-form solution to fixed-strike lookback options with a default risk. Furthermore, we analyze the values of the vulnerable fixed-strike lookback options with respect to the model parameters and also show that the Monte Carlo simulations and the Implicit Finite Difference Method converge to the closed-form solutions and this verifies the correctness of our formulas.

1. Introduction

Since the global financial and Eurozone crises in 2007–2008, there have been quickly increasing concerns regarding financial derivatives, which are more likely to be exposed to credit default risk in over-the-counter (OTC) markets. In general, because of the unstructured stock exchange in OTC markets, the option holder is always apt to have counterparty credit risk because the option writer of the counterparty may not fulfill the promised contracts at the expiry date. We call options that are prone to a default risk "vulnerable options." The value of a vulnerable option, which is subject to counterparty credit risk, is certainly less than that of an identical nonvulnerable option because of the probability of a default risk. The pricing of vulnerable European options was first proposed by Johnson and Stulz [1] in 1987. Klein [2] provided applications of such vulnerable options to other types of financial problems such that the counterparty may have other liabilities within the capital structure, and derived an analytic pricing formula based on the correlation between the underlying asset of the option and the option writer's assets. Dai and Chiu [3] studied a closed formula of vulnerable European options by taking advantage of a first passage model, and Hung and Liu [4] studied the pricing of vulnerable options when the market is incomplete. In particular, Yang et al. [5] used a stochastic volatility model to obtain the pricing formulae of vulnerable options.

In the paper, we discuss the pricing of fixed-strike lookback options with a default risk of an option's writer. In finance terminology, a fixed-strike lookback option is an option whose payoff is determined based on the maximum (or minimum) price of the underlying asset arising over the life of the option. The pricing of fixed-strike lookback options is tricky and provides a mathematical challenge because the option value at any time depends on the path taken by the underlying asset, as well as simply on the underlying asset at that point. There have been a number of researches on the analytic valuation of lookback options based on the Black-Scholes settings. The analytic solutions to a floating-strike and fixed-strike lookback options were proposed by Goldman et al. [6] and Conze and Viswanathan [7]. Dai et al. [8] dealt with an explicit closed formula for quanto lookback options.

In addition, Wong and Kwok [9] used a new technique which involves a subreplicating portfolio and the corresponding replenishing strategy to carry out the calculations of multistate lookback option prices more easily. In particular, they presented the relation between fixed and floating lookback options. Eberlein and Papapantoleon [10]

used a change of numeraire and a representation for the characteristic triplet of the dual of a Lévy process to find a symmetry relationship between floating-strike and fixed-strike lookback options, and Leung [11] adopted a homotopy analysis technique to derive an analytic pricing formulae for floating-strike and fixed-strike lookback options under Heston's stochastic volatility model. Moreover, Wong and Chan [12] investigated the valuation of lookback options under a multiscale stochastic volatility model driven by a fast-mean reverting process and a slow-varying volatility process. Furthermore, they noted that the lookback option features appear in many insurance products. Particularly, the lookback option is closely related to a dynamic fund protection scheme in terms of a guarantee of the minimum fund value.

To derive the pricing formula for vulnerable fixed-strike lookback options (VFSLOs), we exploit the Mellin transform approach, and the method of images presented in Eltayeb and Kilicman [13] and Buchen [14], respectively. The two methods are actually very useful for finding an explicit solution to the price of VFSLOs, which includes the standard normal cumulative distribution because this helps us obtain a closed-form solution more easily and effectively.

The Mellin transform is an integral transform and can be regarded as a multiplicative version of a two-sided Laplace transform. The analytic formulas for an evaluation of the options have been mostly derived by researchers using probabilistic techniques. Nevertheless, the analytic approach from a Mellin transform allows us to resolve the complexity of the calculation of the option pricing using probabilistic methods. In fact, Panini and Srivastav [15] and Panini and Srivastav [16] obtained the pricing formulas of European and American vanilla options, basket options, and American perpetual options using the Mellin transform. In addition, Yoon and Kim [17] found European vulnerable options under stochastic (Hull White) interest rates, as well as under a constant interest rate, using the double Mellin transform method, and Yoon [18] utilized a Mellin transform to derive a closed-form formula for European options in a Black-Scholes framework with stochastic interest rates. Jeon et al. [19] described a general solution for the inhomogeneous Black-Scholes partial differential equation with mixed boundary conditions using Mellin transform.

In addition, to deal with the problem of VFSLOs, the method of images becomes a very important tool. We should therefore adopt not only the Mellin transform method but also the method of images mentioned in Buchen [14] for pricing purposes. The method of images is closely related to the reflection principle of the expectation solution. Applying the method of images to partial differential equations (PDEs) enables us to derive the pricing formula of path-dependent options more easily in comparison with the existing method. In other words, using the method of images enables us to transform the PDE of a lookback option under two conditions (boundary and final conditions) into a PDE with the final conditions of the extended range of underlying assets, and we can then find the pricing formula of the options using a Mellin transform. Buchen [14] found the pricing formula of barrier options more easily than with the existing method by using the method of images, and Buchen and Konstandatos [20] studied the pricing of double-barrier options using the identical method. Furthermore, Jeon et al. [21] derived the vulnerable path-dependent option prices using a double Mellin transform and the method of images. Finally, Jeon et al. [22] investigated the closed solution of the vulnerable dynamic fund protection which is known as an insurance instrument by using the image solution of the given PDE.

The remainder of this paper is organized as follows. In Section 2, we formulate the model problem for the VFSLO and induce a related PDE with regard to the option price. Section 3 deals with the closed pricing formula for the VFSLO through the utilization of a Mellin transform and the method of images on the derived PDE. In Section 4, the influence of the default risk of an option writer is analyzed with respect to the model parameters on the option price, and the accuracy and efficiency of our derived explicit integral solution are demonstrated through the uses of a Monte-Carlo simulation and a Finite Difference Method (FDM). Section 5 discusses the concluding remarks.

2. Model Formulation: Vulnerable Fixed-Strike Lookback Options (VFSLOs)

In this section, we deal with the pricing of VFSLO under the Black–Scholes settings. Before we do so, let us consider the probability space denoted by $(\Omega, \mathscr{F}, \mathbb{P})$, where $\{\mathscr{F}_t \mid t \in [0,T]\}$ is the natural filtration generated by $(B_t)_{t\geq 0}$ and T is the expiry date. Let S_t be the underlying asset price, μ_s be the constant drift rate of S_t, and σ_s be the constant positive volatility of the underlying asset. In addition, let V_t be the market value of the option writer's asset with a constant drift rate μ_v and constant positive volatility σ_v. Then, the model dynamics of the underlying asset price, S_t, and the market value of the option writer's asset, V_t, follow geometric Brownian motions:

$$\begin{aligned} dS_t &= \mu_s S_t dt + \sigma_s S_t dB_t^s, \\ dV_t &= \mu_v V_t dt + \sigma_v V_t dB_t^v, \end{aligned} \tag{1}$$

where B_t^s and B_t^v are standard Brownian motions under the physical probability measure $\mathbb{P}$, with $d\langle B^s, B^v\rangle_t = \rho dt$.

Then, using the Girsanov theorem (cf. Oksendal [23]), the model in (1) is converted into the following stochastic differential equations (SDEs)

$$\begin{aligned} dS_t &= rS_t dt + \sigma_s S_t dB_t^{s*}, \\ dV_t &= rV_t dt + \sigma_v V_t dB_t^{v*}, \end{aligned} \tag{2}$$

under the equivalent martingale measure, where r is a positive interest rate with a constant value and B_t^{s*} and B_t^{v*} are the transformed Brownian motions of B_t^s and B_t^v, respectively, satisfying $d\langle B^{s*}, B^{v*}\rangle_t = \rho dt$.

Now, let us proceed with a procedure for the pricing of a vulnerable fixed-strike lookback call option with an option writer's default risk. As shown in Johnson and Stulz [1], let D^* be a critical value such that a default loss occurs if the value of the firm's assets, V_T, is less than D^*, and D is the value of

total liabilities given by D^* and an additional liability owing to the possibility of a counterparty maintaining operation even while V_T is less than D^*. If V_T is greater than or equal to the default boundary D^*, the entire claim is paid out. Otherwise, a default occurs and only the proportion $(1-\alpha)V_T/D$ of the nominal claim is paid out, where the ratio V_T/D represents the value of the firm's assets available to pay the claim. We then call the fixed-strike lookback call option with default risk VFSLCO. The terminal payoff of VFSLCO at time $t = T$ is given by

$$\left(M_T^s - K\right)^+ \cdot \left(\mathbf{1}_{\{V_T \geq D^*\}} + \mathbf{1}_{\{V_T < D^*\}} \frac{(1-\alpha) V_T}{D}\right), \tag{3}$$

where $M_T^s := \max_{0 \leq \gamma \leq T} S_\gamma$.

Then, the no-arbitrage pricing of VFSLCO with the payoff (3) at time t is described by

$$\begin{aligned} VC_{fix}(t,s,v,m) = e^{-r(T-t)} \mathbb{E}^{\mathrm{Q}} \Big[& \left(M_T^s - K\right)^+ \\ & \cdot \left(\mathbf{1}_{\{V_T \geq D^*\}} + \mathbf{1}_{\{V_T < D^*\}} \frac{(1-\alpha) V_T}{D}\right) \mid S_t = s,\ V_t \\ & = v,\ M_t^s = m \Big], \end{aligned} \tag{4}$$

where $(\cdot)^+ \triangleq \max\{\cdot, 0\}$.

3. Derivation of the Analytic Valuation of VFSLCO

In this section, we derive the closed formula of a vulnerable fixed-strike lookback call option (VFSLCO) using the Mellin transform approaches and the method of images.

In many cases, the analytic formulas for the option price have been mostly obtained by using probabilistic techniques. However, as you can see in (4), the vulnerable fixed-strike lookback option depends on the underlying asset S_t, the option writer's asset V_t, and the maximum process M_t^s of S_t. Therefore, the closed form solution cannot be obtained simply and directly by using the two-dimensional log-normal density function of the underlying asset S_t and the option writer's asset V_t, and the density function of the maximum process M_t^s. In other words, the following three-dimensional joint probability density functions of S_t, V_t, and M_t^s should be considered as follows:

$$\mathbb{Q}\left(S_T \in ds,\ V_T \in dv,\ M_T^s \leq m\right). \tag{5}$$

Nevertheless, it will be very difficult to derive the desired probability density function before we calculate the expectation of $VC_{fix}(t,s,v,m)$ stated in (4).

Hence, to overcome these difficulties, we use a Mellin transform technique. The analytic approach from a Mellin transform allows us to resolve the complexity of the calculation of the option pricing using probabilistic methods. Of course, in a partial differential equation (PDE) perspective, our problem is formulated as (1+3)-dimensional parabolic PDE with Neumann boundary condition. So, it is not easy to derive a general analytic solution of the PDE. However, to solve this problem, we first reduce the PDE to a (1+2)-dimensional PDE with Neumann boundary condition through an unusual simple transform, and then, by using the double Mellin transform techniques, we are able to derive an explicit form of the solution satisfying the given PDE. Here, the procedure of obtaining the closed solution on fixed-strike lookback options is similar to that of the solution on vulnerable dynamic fund protection stated in Jeon et al. [22].

In Section 2, for $0 \leq t \leq T$, the solution for VFSLCO at time t given by (4) can be represented by

$$\begin{aligned} VC_{fix}(t,s,v,m) = {} & e^{-r(T-t)} \mathbb{E}^{\mathrm{Q}} \Big[\max\left(M_t^s, K\right) \\ & \cdot \left(\mathbf{1}_{\{V_T \geq D^*\}} + \mathbf{1}_{\{V_T < D^*\}} \frac{(1-\alpha) V_T}{D}\right) \mid S_t = s,\ V_t \\ & = v,\ M_t^s = m \Big] \\ & - K e^{-r(T-t)} \mathbb{E}^{\mathrm{Q}} \Big[\left(\mathbf{1}_{\{V_T \geq D^*\}} + \mathbf{1}_{\{V_T < D^*\}} \frac{(1-\alpha) V_T}{D}\right) \mid S_t \\ & = s,\ V_t = v,\ M_t^s = m \Big] = J_2(t,s,v,m) - K J_1(t,v), \end{aligned} \tag{6}$$

where $J_2(t,s,v,m) = e^{-r(T-t)}\mathbb{E}^{\mathrm{Q}}[\max(M_t^s, K) \cdot (\mathbf{1}_{\{V_T \geq D^*\}} + \mathbf{1}_{\{V_T < D^*\}}((1-\alpha)V_T/D)) \mid S_t = s,\ V_t = v,\ M_t^s = m]$, and $J_1(t,v) = e^{-r(T-t)}\mathbb{E}^{\mathrm{Q}}[(\mathbf{1}_{\{V_T \geq D^*\}} + \mathbf{1}_{\{V_T < D^*\}}((1-\alpha)V_T/D)) \mid V_t = v]$.

To compute $J_2(t,s,v,m)$ and $J_1(t,v)$, denoted by (6), we introduce the PDE for each function. The Feynman–Kac formula can enable us to obtain the following two PDEs with respect to $J_2(t,h,v,m)$ and $J_1(t,v)$:

$$\begin{aligned} & \mathscr{L}_2 J_2(t,s,v,m) = 0, \\ & \frac{\partial J_2}{\partial m}(t,m,v,m) = 0, \\ & J_2(T,s,v,m) = \max(m,K) \\ & \qquad \cdot \left(\mathbf{1}_{\{v \geq D^*\}} + \mathbf{1}_{\{v < D^*\}} \frac{(1-\alpha) v}{D}\right), \\ & \mathscr{L}_2 \triangleq \frac{\partial}{\partial t} + \frac{\sigma_s^2}{2} s^2 \frac{\partial^2}{\partial s^2} + \frac{\sigma_v^2}{2} v^2 \frac{\partial^2}{\partial v^2} \\ & \qquad + \rho \sigma_s \sigma_v s v \frac{\partial^2}{\partial s \partial v} + r s \frac{\partial}{\partial s} + r v \frac{\partial}{\partial v} \\ & \qquad - r \cdot \end{aligned} \tag{7}$$

in the region $\{(t,s,v,m) \mid 0 \leq t < T,\ 0 < s \leq m,\ 0 < v, m < \infty\}$, and

$$\begin{aligned} & \mathscr{L}_1 J_1(t,v) = 0, \\ & J_1(T,v) = \mathbf{1}_{\{v \geq D^*\}} + \mathbf{1}_{\{v < D^*\}} \frac{(1-\alpha) v}{D}, \\ & \mathscr{L}_1 \triangleq \frac{\partial}{\partial t} + \frac{\sigma_v^2}{2} v^2 \frac{\partial^2}{\partial v^2} + r v \frac{\partial}{\partial v} - r \cdot \end{aligned} \tag{8}$$

in the region $\{(t,v) \mid 0 \leq t < T,\ 0 < v < \infty\}$, respectively.

First, to solve PDE (7), we can use two approaches, which are the splitting direction technique proposed by Wong and Kwok [24] and the Mellin transform method using the method of image. The splitting direction technique is used to derive the joint density functions of multivariate terminal asset prices associated with the presence of barriers based upon the lognormal assumption. If we substitute the function $\partial J_2/\partial m$ by J_2^* after we differentiate the function J_2 with respect to m on the PDEs (7), J_2^* becomes PDEs of a barrier option with multiassets, and then J_2^* is a solution of the PDEs. On the PDEs, we can take into consideration the splitting direction technique or the Mellin transform method using the method of images. However, we think that dealing with the splitting direction method to solve this on PDEs (7) is a little more difficult and complicated than that of the Mellin transform method using the method of images; we are trying to use the Mellin transform method and the method of images.

So, we consider Lemmas 3.1 and 3.2 described in Jeon et al. [22]. As seen in Lemma 3.1, if we put $J_2(t,s,v,m) = \overline{J_2}(t,s,v,\max(m,K))$, the given PDE (7) is transformed into

$$\begin{aligned}
&\mathscr{L}_2\overline{J_2}(t,s,v,m) = 0,\\
&\overline{J_2}(T,s,v,m) = \overline{h}(v,m)\\
&\qquad \triangleq m\cdot\left(\mathbf{1}_{\{v\geq D^*\}} + \mathbf{1}_{\{v<D^*\}}\frac{(1-\alpha)v}{D}\right),\\
&\frac{\partial\overline{J_2}}{\partial m}(t,m,v,m) = 0
\end{aligned} \tag{9}$$

on domain $\{(t,s,v,m) \mid 0 \leq t < T,\ 0 < s \leq m,\ 0 < v,m < \infty\}$. The process of solving the above PDE (9) is not hard because, using Lemma 3.1 in Jeon et al. [22], if we differentiate both sides of PDE (7) with respect to m and put $I(t,s,v,m) = (\partial\overline{J_2}/\partial m)(t,s,v,m)$, then we can obtain the PDE of barrier option with two underlying assets in terms of I. To solve this problem, we use the method of images mentioned in Lemma 3.2 of Jeon et al. [22]. Finally, from the procedure of the proof of Theorem 3.1 in Jeon et al. [22], we can have the following explicit solution of PDE $\overline{J_2}(t,s,v,m)$.

Corollary 1. *The solution $\overline{J_2}(t,s,v,m)$ of PDE (9) can be described using*

$$\begin{aligned}
\overline{J_2}(t,s,v,m) &= m\mathcal{N}_2\left(-d_1\left(\tau,\frac{s}{m}\right),d_2\left(\tau,\frac{v}{D^*}\right),-\rho\right)\\
&+\frac{(1-\alpha)vm}{D}\mathcal{N}_2\left(-d_1\left(\tau,\frac{s}{m}\right)-\rho\sigma_v\sqrt{\tau},-d_2\left(\tau,\right.\right.\\
&\left.\left.\frac{v}{D^*}\right)-\sigma_v\sqrt{\tau},\rho\right)+s\mathcal{N}_2\left(d_1\left(\tau,\frac{s}{m}\right)+\sigma_s\sqrt{\tau},d_2\left(\tau,\right.\right.\\
&\left.\left.\frac{v}{D^*}\right)+\rho\sigma_s\sqrt{\tau},\rho\right)+\frac{(1-\alpha)sv}{D}\\
&\cdot e^{(r+\rho\sigma_s\sigma_v)\tau}\mathcal{N}_2\left(d_1\left(\tau,\frac{s}{m}\right)+(\sigma_s+\rho\sigma_v)\sqrt{\tau},\right.\\
&\left.-d_2\left(\tau,\frac{v}{D^*}\right)-(\sigma_v+\rho\sigma_s)\sqrt{\tau},-\rho\right)\\
&+s\int_0^{s/m}u^{-(1+k_s)}\left[e^{-r\tau}\mathcal{N}_2\left(-d_1\left(\tau,\frac{1}{u}\right),\right.\right.\\
&d_2\left(\tau,\frac{u^{-2\rho\sigma_v/\sigma_s}v}{D^*}\right),-\rho\Bigg)+\frac{(1-\alpha)v}{D}\\
&\cdot\mathcal{N}_2\left(-d_1\left(\tau,\frac{1}{u}\right)-\rho\sigma_v\sqrt{\tau},-d_2\left(\tau,\frac{u^{-2\rho\sigma_v/\sigma_s}v}{D^*}\right)\right.\\
&\left.\left.-\sigma_v\sqrt{\tau},\rho\right)\right]du,
\end{aligned} \tag{10}$$

where $d_1(\cdot,\cdot)$ and $d_2(\cdot,\cdot)$ are given by $d_1(t,x) = (\log x + (r-\sigma_s^2/2)t)/\sigma_s\sqrt{t}$ and $d_2(t,y) = (\log y + (r-\sigma_v^2/2)t)/\sigma_v\sqrt{t}$, respectively, and $\mathcal{N}(\cdot)$ is a standard normal cumulative distribution function.

Proof. The proof of the derivation of $\overline{J_2}(t,s,v,m)$ is similar to that of Theorem 3.1 given by Jeon et al. [22]. Refer to this article. □

Theorem 2 (derivation of the pricing formula of VFSLCO). *For the given $S_t = s$, $V_t = v$, and $M_t^s := \max_{0\leq\gamma\leq t}S_\gamma = m$, the pricing of VFSLCO at time t is expressed by*

$$\begin{aligned}
VFSLCO_t &= m^\star e^{-r\tau}\mathcal{N}_2\left(-d_1\left(T-t,\frac{s}{m^\star}\right),d_2\left(T-t,\right.\right.\\
&\left.\left.\frac{v}{D^*}\right),-\rho\right)+\frac{(1-\alpha)vm^\star}{D}\mathcal{N}_2\left(-d_1\left(T-t,\frac{s}{m^\star}\right)\right.\\
&\left.-\rho\sigma_v\sqrt{T-t},-d_2\left(T-t,\frac{v}{D^*}\right)-\sigma_v\sqrt{T-t},\rho\right)\\
&+s\mathcal{N}_2\left(d_1\left(T-t,\frac{s}{m^\star}\right)+\sigma_s\sqrt{T-t},d_2\left(T-t,\right.\right.\\
&\left.\left.\frac{v}{D^*}\right)+\rho\sigma_s\sqrt{T-t},\rho\right)+\frac{(1-\alpha)sv}{D}\\
&\cdot e^{(r+\rho\sigma_s\sigma_v)(T-t)}\mathcal{N}_2\left(d_1\left(T-t,\frac{s}{m^\star}\right)+(\sigma_s+\rho\sigma_v)\right.\\
&\cdot\sqrt{T-t},-d_2\left(T-t,\frac{v}{D^*}\right)-(\sigma_v+\rho\sigma_s)\sqrt{T-t},\\
&\left.-\rho\right)+s\int_0^{s/m^\star}\theta^{-(1+k_s)}\left[e^{-r\tau}\mathcal{N}_2\left(-d_1\left(T-t,\frac{1}{\theta}\right),\right.\right.\\
&d_2\left(T-t,\frac{\theta^{-2\rho\sigma_v/\sigma_s}v}{D^*}\right),-\rho\Bigg)+\frac{(1-\alpha)v}{D}\\
&\cdot\mathcal{N}_2\left(-d_1\left(T-t,\frac{1}{\theta}\right)-\rho\sigma_v\sqrt{T-t},\right.\\
&\left.\left.-d_2\left(T-t,\frac{\theta^{-2\rho\sigma_v/\sigma_s}v}{D^*}\right)-\sigma_v\sqrt{T-t},\rho\right)\right]d\theta\\
&-Kv\frac{(1-\alpha)v}{D}
\end{aligned}$$

$$\cdot \mathcal{N}\left(-\frac{\log\left(v/D^*\right)+\left(r+\sigma_v^2/2\right)(T-t)}{\sigma_v\sqrt{(T-t)}}\right)$$
$$-e^{-r(T-t)}K\mathcal{N}\left(\frac{\log\left(v/D^*\right)+\left(r-\sigma_v^2/2\right)(T-t)}{\sigma_v\sqrt{(T-t)}}\right), \tag{11}$$

where $m^\star = \max(m, K)$*, and* $d_1(\cdot,\cdot)$ *and* $d_2(\cdot,\cdot)$*, are presented by Corollary 1.*

Proof. Using Appendix A.1 and Appendix B.1 described in Jeon et al. [22], the solution $J_1(t, v)$ of PDE (8) is represented as

$$\begin{aligned} J_1(t,v) &= \int_0^\infty \left(\mathbf{1}_{\{y>D^*\}}+\mathbf{1}_{\{y<D^*\}}\frac{(1-\alpha)\,y}{D}\right) \\ &\quad\cdot \mathcal{B}^1\left(T-t,\frac{v}{y}\right)\frac{1}{y}dy = v\frac{(1-\alpha)\,v}{D} \\ &\quad\cdot \mathcal{N}\left(-\frac{\log\left(v/D^*\right)+\left(r+\sigma_v^2/2\right)\tau}{\sigma_v\sqrt{\tau}}\right) \\ &\quad+e^{-r\tau}\mathcal{N}\left(\frac{\log\left(v/D^*\right)+\left(r-\sigma_v^2/2\right)\tau}{\sigma_v\sqrt{\tau}}\right), \end{aligned} \tag{12}$$

where $\tau = T - t$ and $\mathcal{N}(\cdot)$ is a standard normal cumulative distribution function.

Therefore, from (6), we have

$$\begin{aligned} VFSLCO_t &= \left(J_2(t,s,v,m)-KJ_1(t,v)\right) \\ &= \left(\overline{J_2}(t,s,v,\max(m,K))-J_1(t,v)\right), \end{aligned} \tag{13}$$

and from (10) and (12), we obtain the desired result. □

4. Implications

This section examines the price change of VFSLO with regard to the model parameters. Section 3 dealt with the explicit integral solution for VFSLO by using the method of the image and Mellin transform techniques. The existence of the closed-form solution for VFSLO is important because it enables us to analyze the sensitivities of option prices in terms of the model parameters in a more precise and effective manner. To investigate the influence of the price on VFSLO directly, we utilize the closed-form formula for the VFSLO stated in Theorem 2. For the numerical computing, we choose diverse values of the option parameters to achieve a meaningful analysis of the behavior of the option price.

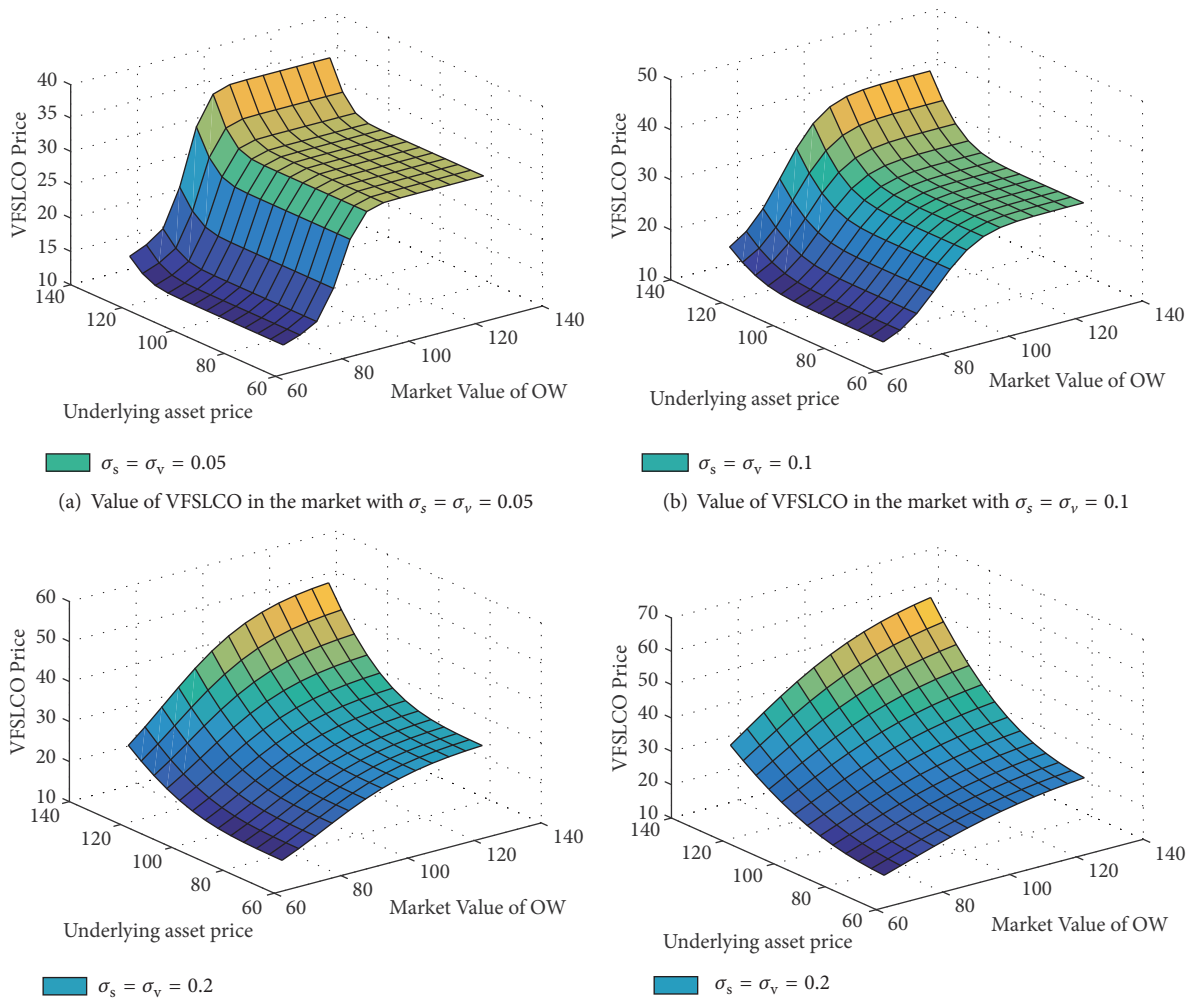
Figure 1 exhibits the price change of VFSLCO in terms of the underlying asset price and the market value of the option's writer (OW). As shown in the figure, when the underlying asset price and the market value of the option's writer (OW) both increase, the value of VFSLCO increases. This is because the greater the underlying asset price increases, the larger the maximum value presented by $M_t^s := \max_{0\le\gamma\le t}S_\gamma$; an increase in the maximum value then leads to an increase in the VFSLCO. In addition, as shown through the payoff of VFSLCO described in (2), the larger the market value of the option writer is, the less likely a default risk will occur, which ultimately increases the value of VFSLCO. Here, for a numerical computation, the following parameters are taken: $T = 1$, $S = 100$, $K = 100$, $M = 100$, $V = 100$, $r = 0.03$, $\rho = 0.3$, $D = 90$, $\alpha = 0.5$, and $D^* = 90$.

Figure 2 shows the sensitivities of VFSLO with regard to the critical level for the credit loss of the option's writer, D^*. We can see that the prices of VFSLO with respect to the market value of the option's writer tend to decrease as D^* increases for three cases of the set of the volatilities, (σ_s, σ_v). This is because the growth in critical level D^* enhances the possibility of the firm defaulting, resulting in a decrease in the option values.

Figure 3 plots the behaviors of VFSLCO against the firm's default rate, α. Note that as α increases, the value of VFSLCO in terms of the market value of the option's writer falls for three cases of the set of the volatility, (σ_s, σ_v). This implies that a larger default rate α results in a smaller value of VFSLCO when a default event occurs. In addition, we can observe that the larger the values of the set of the volatilities, (σ_s, σ_v), the bigger the sensitivities of the option price with respect to the market value of the option writer. In particular, it can be observed that, for large market values, as the set of volatility is $(\sigma_s, \sigma_v) = (0.1, 0.1)$, the influence of the option price is very insignificant to the default rate. However, it is interesting to note that, as the set of volatility (σ_s, σ_v) increases, the impact of the option price begins to show significant behaviors with respect to the default rate.

Until now, we investigated the sensitivity of VFSLO with respect to the model parameters by using the closed-form solution of VFSLO described through Theorem 2. Next, we will try and illustrate the importance of a closed-form formula in dealing with derivative pricing. Actually, the existence of a closed-form solution allows us to create a crucial contribution to the efficiency and accuracy of the pricing of VFSLO. To demonstrate the accuracy of the closed-form solution stated in Theorem 2, we compare our closed-form formula with the numerical solution using a Monte-Carlo simulation and a Finite Difference Method (FDM).

Table 1 shows the relative error 1 of the option price between the Monte-Carlo simulation and closed-form solution, and the relative error 2 of the option price between the numerical solution presented by Implicit-FDM (implicit method for Finite Difference Method) and closed-form solution. Here, the relative error 1 is given by |(Closed-form solution − MC simulation)/(Closed-form solution)| and the relative error 2 is expressed by |(Closed-form solution − Numerical Solution by FDM)/(Closed-form solution)|. We utilize an Intel i7-6700 CPU (3.40 GHz, 16 GB RAM) to implement the computations for the Monte-Carlo simulation, the Implicit-FDM, and closed-form solution. The Monte-Carlo simulation is carried out from 10,000 paths to 100,000 paths. As seen from Table 1, as the number of simulations increases, the numerical solution given by the Monte-Carlo simulation, which is a good approximation of a real solution in the market, comes



(a) Value of VFSLCO in the market with $\sigma_s = \sigma_v = 0.05$

(b) Value of VFSLCO in the market with $\sigma_s = \sigma_v = 0.1$

(c) Value of VFSLCO in the market with $\sigma_s = \sigma_v = 0.2$

(d) Value of VFSLCO in the market with $\sigma_s = \sigma_v = 0.3$

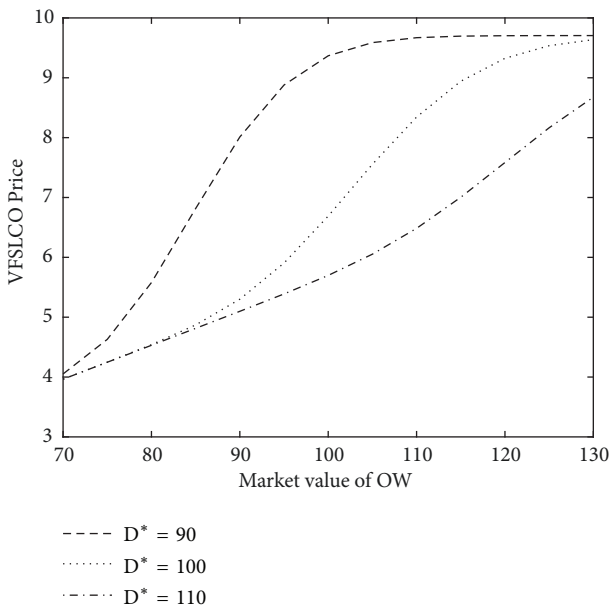
Figure 1: Value of VFSLCO against market value of option writer (OW) and underlying fund price.

Table 1: Change in the relative errors for the Monte-Carlo simulation and the numerical solution by Finite Difference Method (FDM) against closed-form solution of VFSLCO with respect to the number of simulations. Here, the relative error 1 is given by |(Closed-form solution − MC simulation)/(Closed-form solution)| and the relative error 2 is expressed by |(Closed-form solution − Numerical Solution by FDM)/(Closed-form solution)|. The parameter values are given by $T = 1$, $S = 100$, $K = 100$, $M = 100$, $V = 100$, $r = 0.03$, $\rho = 0.3$, $D = 90$, $\alpha = 0.5$, $D^* = 90$, and $\sigma_s = \sigma_v = 0.2$.

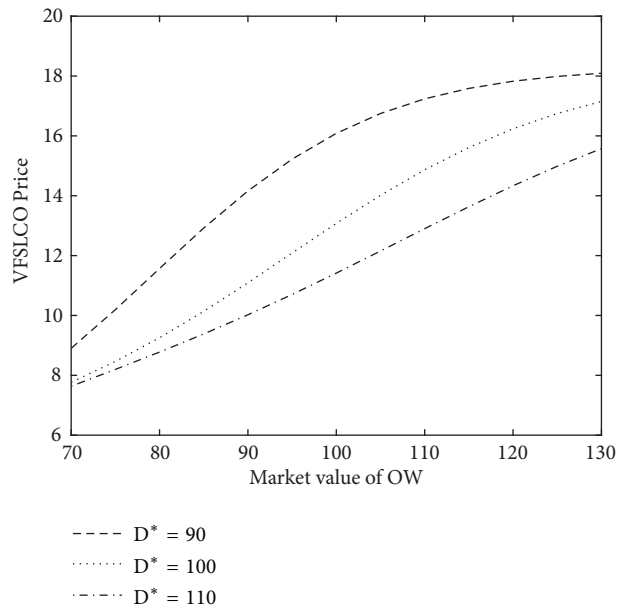
No. of Sim.	Monte-Carlo Simulation	Closed-form Solution	Numerical Solution by FDM	Relative error 1	Relative error 2
10000	15.4915	16.08468	16.1701	0.0368	0.0053
20000	15.7183	16.08468	16.1701	0.0227	0.0053
50000	15.8312	16.08468	16.1701	0.0157	0.0053
100000	15.9428	16.08468	16.1701	0.0088	0.0053

close to the closed-form solution. Comparing it with the result of the Implicit-FDM with the relative errors, this implies that the closed-form formula presented in Theorem 2 becomes an accurately computed solution for VFSLO.

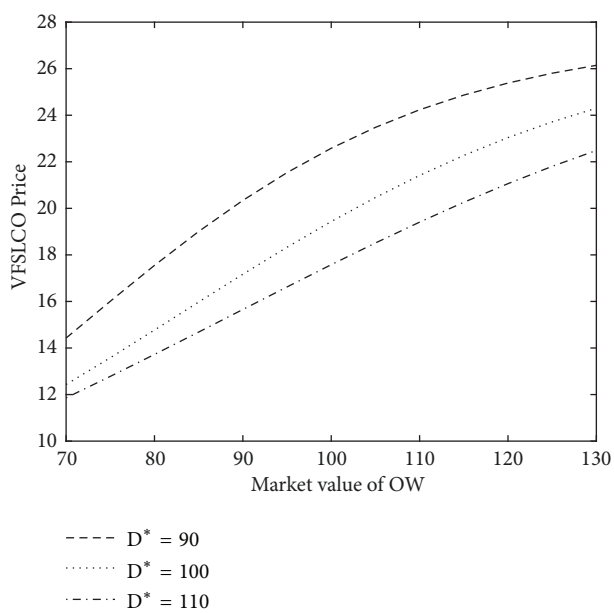
Table 2 shows the change of the relative errors for the Monte-Carlo simulation and the Implicit-FDM versus our closed-form solution with regard to the underlying asset price, the market value of the option's writer, and the maximum value, respectively. Here, the Monte-Carlo simulation is carried out using 100,000 paths. As shown from the relative errors in Tables 2–4, the accuracy of the closed-form solution for VFSLO is well illustrated using the Monte-Carlo simulation and the Implicit-FDM for all cases of the underlying asset price, the market value of the option writer, and the maximum value. Therefore, it implies that the closed-form solution obtained in Theorem 2 verifies the



(a) Value of VFSLCO in the market with $D^* = 90, 100, 110$: $\sigma_s = \sigma_v = 0.1$



(b) Value of VFSLCO in the market with $D^* = 90, 100, 110$: $\sigma_s = \sigma_v = 0.2$



(c) Value of VFSLCO in the market with $D^* = 90, 100, 110$: $\sigma_s = \sigma_v = 0.3$

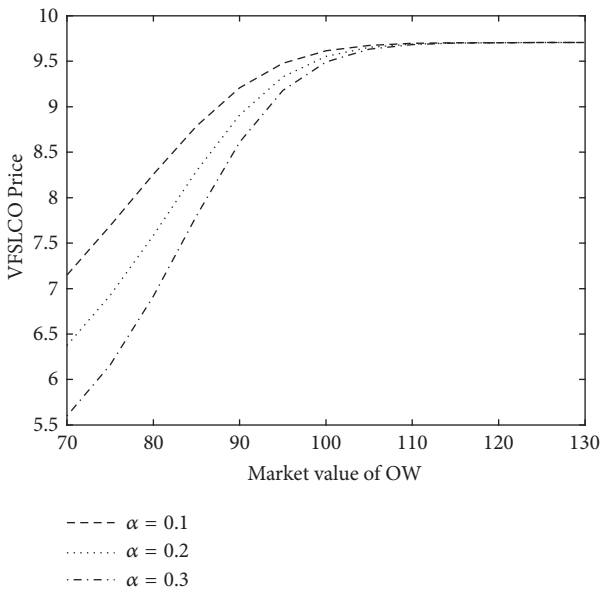
Figure 2: Value of VFSLCO against market value of option writer (OW).

correctness of the formulas of the vulnerable fixed-strike lookback option.

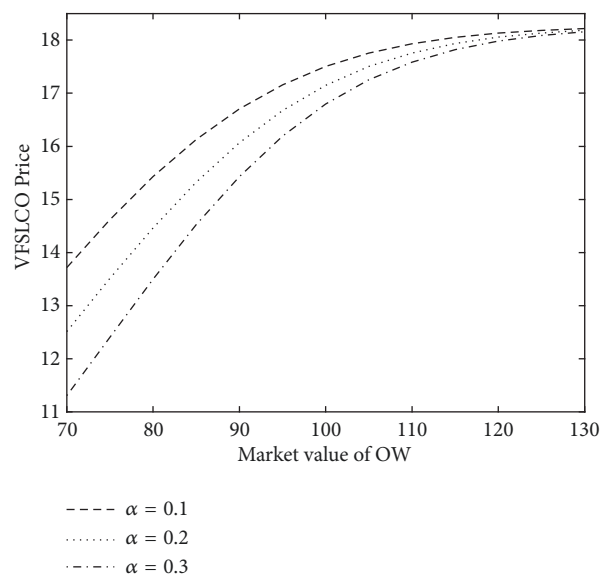
5. Discussion: The Pricing of Fixed-Lookback Options Based on the Black-Cox Model

Continuous default monitoring under the credit risk model of Black-Cox is more realistic than the model of Merton (Black-Scholes). Generally, dealing with the option pricing on Black and Cox model that the default occurs prior to the maturity T is more complicated than that on the Merton default model that the default only takes place at the terminal time. In case of Black-Cox model, the default may occur at any time before or on the expiration date; we have to consider the first-pass time that the firm value of the option's writer reaches the critical level D for the first time. In this case, to handle the pricing of vulnerable fixed-strike lookback option in the setting of the Black-Cox model, (4) mentioned in our paper will become more complex. It will be represented by

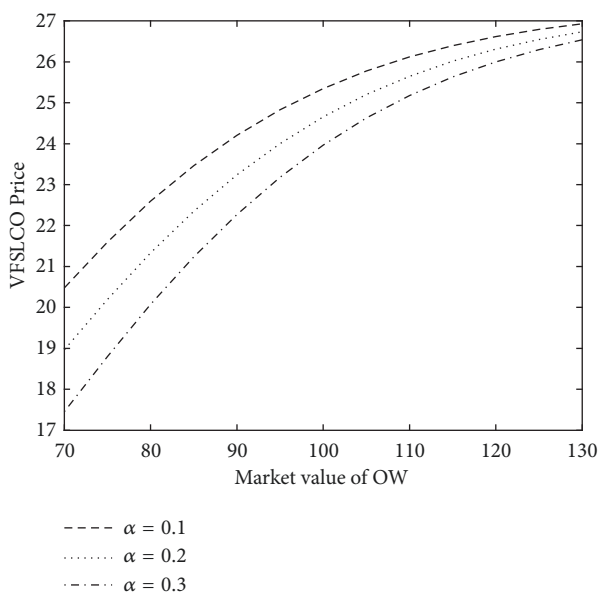
$$VC_{fix}(t, s, v, m) = e^{-r(T-t)}\mathbb{E}^{Q}\left[\left(M_T^s - K\right)^+ \mathbf{1}_{\{\tau>T\}}\right.$$



(a) Value of VFSLCO in the market with $\alpha = 0.1, 0.2, 0.3$: $\sigma_s = \sigma_v = 0.1$



(b) Value of VFSLCO in the market with $\alpha = 0.1, 0.2, 0.3$: $\sigma_s = \sigma_v = 0.2$



(c) Value of VFSLCO in the market with $\alpha = 0.1, 0.2, 0.3$: $\sigma_s = \sigma_v = 0.3$

Figure 3: Value of VFSLCO against market value of option writer (OW).

$$+ \frac{(1-\alpha) V_T}{D} \left(M_T^s - K\right)^+ \mathbf{1}_{\{\tau > T\}} \mid S_t = s,\ V_t = v,\ M_t^s = m \Bigg], \tag{14}$$

where $\tau = \inf\{t > 0 \mid V_t = D\}$ and $(\cdot)^+ \triangleq \max\{\cdot, 0\}$. However, it will be very difficult for us to solve this problem because, compared to the problem of the fixed-strike lookback options under Merton default model mentioned in (4), we have to change (14) more several times by using the mathematical skills, and then, to solve the solution, we should use the methods of images one more time on the transformed partial differential equation. As seen in Kim and Jeon [25], they proposed vulnerable options whose payoffs allow for the credit risk prior to maturity of the options in the Black-Cox frameworks. We will take into consideration the pricing of the fixed-strike lookback options based on Black-Cox model as a future work.

6. Concluding Remarks

In this paper, we examined the closed-form formula of a fixed-strike lookback option (FSLO) with default risk of

Table 2: Change in the relative errors for the Monte-Carlo simulation and the numerical solution by Finite Difference Method (FDM) against closed-form solution of VFSLCO in terms of the underlying asset price. The parameter values are given by $T = 1$, $K = 100$, $M = 100$, $V = 100$, $r = 0.03$, $\rho = 0.3$, $D = 90$, $\alpha = 0.5$, $D^* = 90$, and $\sigma_s = \sigma_v = 0.2$.

S(Underlying asset price)	Monte-Carlo simulation	Closed-form solution	Relative error 1
90	7.7215	7.8128	0.01160
100	15.9428	16.0846	0.00881
110	25.7262	25.8752	0.00575
120	35.5096	35.6658	0.00438
130	45.2930	45.4563	0.00359
S(Underlying asset price)	Numerical Solution by FDM	Closed-form solution	Relative error 2
90	8.1840	7.8128	0.04751
100	16.1701	16.0846	0.00531
110	25.9693	25.8752	0.00363
120	35.7684	35.6658	0.00287
130	45.5675	45.4563	0.00244

Table 3: Change in the relative errors for the Monte-Carlo simulation and the numerical solution by Finite Difference Method (FDM) against closed-form solution of VFSLCO with regard to the market value of the option's writer. The parameter values chosen are $T = 1$, $S = 100$, $K = 100$, $M = 100$, $r = 0.03$, $\rho = 0.3$, $D = 90$, $\alpha = 0.5$, $D^* = 90$, and $\sigma_s = \sigma_v = 0.2$.

V(Market value of option writer)	Monte-Carlo simulation	Closed-form solution	Relative error 1
90	14.0193	14.1606	0.00997
100	15.9428	16.0846	0.00881
110	17.0710	17.2383	0.00970
120	17.6482	17.8269	0.01002
V(Market value of option writer)	Numerical Solution by FDM	Closed-form solution	Relative error 2
90	14.2143	14.1606	0.00379
100	16.1701	16.0846	0.00531
110	17.0127	17.2383	0.01309
120	17.4634	17.8269	0.02039

Table 4: Change in the relative errors for the Monte-Carlo simulation and the numerical solution by Finite Difference Method (FDM) against closed-form solution of VFSLCO with respect to the maximum value. The parameter values chosen are $T = 1$, $S = 100$, $K = 100$, $V = 100$, $r = 0.03$, $\rho = 0.3$, $D = 90$, $\alpha = 0.5$, $D^* = 90$, and $\sigma_s = \sigma_v = 0.2$.

M(Maximum value)	Monte-Carlo simulation	Closed-form solution	Relative error 1
105	16.3114	16.4313	0.00729
110	17.3581	17.4566	0.00564
115	19.0328	19.1099	0.00403
120	21.2578	21.3139	0.00263
M(Maximum value)	Numerical Solution by FDM	Closed-form solution	Relative error 2
105	16.4541	16.4313	0.00138
110	17.4680	17.4566	0.00065
115	19.2008	19.1099	0.00475
120	21.4209	21.3139	0.00502

the counterparty and verified the correctness of the result from a Monte-Carlo simulation and a Finite Difference Method (FDM). First, we obtained the PDE with three variables for VFSLO and derived the closed-form formula by taking advantage of the method of images and the double Mellin transform techniques. Second, we investigated the price behavior of VFSLO in terms of the model parameters based on our explicit integral solution. In particular, the changes in the underlying asset price and firm's value have a significant effect on the pricing of VFSLO. Third, we noted that the Monte-Carlo simulation and the FDM are efficient approaches for demonstrating the accuracy of the explicit closed-form solution for VFSLO. That is, by verifying that the closed-form formula described in Theorem 2 approaches the numerical solution given by the Monte-Carlo simulation for a sufficiently large number of simulations or the numerical

solution obtained by Implicit-FDM, which becomes the best approximation to a real solution in the financial market, this implies that our solution provided by Theorem 2 is an accurate VFSLO solution.

In practice, our results can be used for ensuring credit quality of insurance companies. By the Solvency II regulation, insurance companies are required to hold a certain amount of capital to guarantee the obligation towards policyholders. Therefore, they have been trying to manage their credit risk and we mentioned that the lookback option is related to the dynamic protection fund. Since the dynamic protection fund can ensure that the fund value is upgraded if it ever falls below a certain threshold level, our results about the valuation of the lookback option have considerable features for insurance companies.

Finally, through further research, our chosen VFSLO can be extended to FSLOs using other types of models. In particular, we will study American vulnerable path-dependent options which have two or more underlying assets as a future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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