Dynamic Control of Product Innovation, Advertising Effort, and Strategic Transfer-Pricing in a Marketing-Operations Interface

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We develop a dynamic control model of a monopolist composed of two profit centers, e.g., an operations department in charge of the product innovation and a marketing department controlling advertising effort as well as the retail price. Meanwhile, knowledge accumulating in product innovation and advertising effort which lead to reducing the corresponding investment cost is considered. The customer inverse demand function depends jointly on the quality level as well as the product goodwill which can be improved by product innovation and advertising efforts. Our results show that the learning rates of product innovation and advertising effort affect the product innovation and advertising effort investments level. In addition, compared with the administered transfer-pricing, the negotiation between the two departments results in a lower transfer price as well as a higher retail price. In the meantime, the advertising effort is lower while the quality improvement effort is higher. What is more, higher profits to both departments and the firm can be brought about by the negotiation means.

1. Introduction

Nowadays, more and more firms have been building divisional organizations to utilize the distinctive advantages of decentralized decision-making. In these organizations, one striking feature is that goods and (or) services are every now and then exchanged according to given transfer price between divisional centers within a firm. Therefore, to determine the transfer price is critical for decentralized decision-making [1].

There are two types of transfer-pricing in the modern business world, e.g., administered and negotiated transfer-pricing. When the transfer price is set by a central administrator monopoly, this refers to an administered transfer-pricing, while the negotiated transfer-pricing denotes the case that the price is shaped by the divisional centers through negotiating freely between them [2]. Which of the two transfer-pricing types is better? This is an essential problem for a firm who determines to achieve decentralized management.

In the administered transfer-pricing scenario, central management formulates transfer price from the perspective of overall profit maximization directly. However, it may be to some extent against the initial purpose of decentralizing management. If we apply the pattern of negotiated transfer-pricing, greater autonomy can be provided to divisional centers, but the outcomes of this pattern may be suboptimal because the negotiation between the divisional centers is to maximize their respective profits rather than to maximize the overall profits [3]. Hence, a choice must be drawn between the two types of transfer-pricing patterns.

In this paper, we consider a firm, which is composed of an operations department and a marketing department, intending to implement decentralized management mode. The operations department is in charge of improving the quality (product innovation) of the product through product innovation investment, and we assume that quality level affects goodwill and demand of the product positively. The marketing department is responsible for advertising effort and sells the product transferred from operations department to end customers at appropriate retail price which is decided by the marketing department independently. Particularly, the impacts of knowledge accumulation resulting from learning by doing are taken into account as they have important implications for decision-making. In the last few decades, the impacts of knowledge accumulation have been
investigated by many scholars ([4–8], etc.). But to the best of our knowledge, so far they have not been analyzed in transfer-pricing decision-making. In this paper, the effects of two kinds of knowledge accumulation on transfer-pricing decision-making are investigated, e.g., knowledge accumulation through learning by doing in product innovation carried out by the operations department and advertising effort activities performed by the marketing department.

Based on the above considerations, as a benchmark the centralized scenario in which central management makes all the decisions is investigated first. Then, we analyze the decentralized issue, in which we investigate the negotiated and administered transfer-pricing in turn. Our results show that the learning rates of product innovation and advertising effort affect the product innovation and advertising effort investments level, and compared with the administered transfer price, the negotiation between the two departments leads to a lower transfer price as well as a higher retail price. Furthermore, using the negotiation means will result in a lower advertising effort and higher quality improvement effort. What is more, the negotiated transfer-pricing leads to higher profits to both departments and the firm.

2. Literature Review

Our research is mainly the intersection between two streams of literature. The first stream involves the transfer-pricing issue for divisional centers within a firm, and the other stream refers to the issue of knowledge accumulation resulting from learning by doing.

In the past few decades, considerable effort has been focused on the transfer-pricing issue by many authors. In 1956, a seminal paper is provided by Hirshleifer, in which a transfer price that maximizes aggregate firm profit while inducing individual internal profit centers to maximize their own separate profits is analyzed. Hirshleifer [9] believes that when there is no market or the market is imperfectly competitive, the transfer price should be equal to the marginal cost. However, Holmstrom and Tirole [10] argue that it may be difficult to find out the supplying division’s "marginal cost". Therefore, Holmstrom and Tirole [10] consider that the optimal design of a transfer-pricing policy is a solution to the mechanism design problem. Baldenius et al. [11] examine transfer-pricing in multinational firms, where a firm decouples its internal transfer price from the arm’s length price used for tax purposes. They show that the optimal internal transfer price should be a weighted average of the pretax marginal cost and the most favorable arm’s length price. Vaysman [12] finds that when a firm designs a managerial-compensation schemes and bargaining infrastructures and uses it in the negotiated transfer-pricing structure, the upper bound on available profits can be reached. Pekgün et al. [13] find that, compared to the centralized scenario, decentralized scenario may lead to a larger total demand, a lower quoted price, and lower firms’ profits. Through a differential game model, Erickson [14] investigates the issue of the marketing-operations interface; it is the first look at the interface between functional areas within a firm as a noncooperative dynamic game. Extending Erickson’s work, Erickson [15] comes up with a transfer price mechanism, and subgame perfect feedback strategies are derived for price, advertising, and production as functions of the state variables. Dockner and Fruchter [1] study the decentralized strategic interactions of marketing and production issue. The results demonstrate that without coordination, the strategic interactions result in inefficiencies. However, if the two departments play a game with precommitment strategies, there exists a dynamic transfer price which results in Pareto-efficient company profits.

The above research mainly focuses on the administered or negotiated transfer-pricing decision-making and the internal coordination problem, but they cannot give an answer to a firm who wants to achieve decentralized management regarding which pattern of transfer-pricing should be adopted. Different from those research studies, Liu et al. [3] give a definite answer that the negotiated transfer-pricing is better than the administered transfer-pricing for a decentralized firm.

The second stream of literature that this paper involves, originates from the pioneering research by Arrow [4]. He accounts for the relationship between cost and cumulative quantity produced. Encouraged by the thinking of Arrow, a number of follow-up studies and extensions are presented in succession. Lucas [16] emphasizes the importance of on-the-job-training or learning by doing; he believes that they are at least as important as schooling in the formation of human capital. Hatch and Mowery [17] find that the learning by doing has a positive impact in terms of improving yields as well as reducing the cost. Recently, Pan and Li [7] and Li [8] insist that learning by doing influences firms’ corresponding investments. In addition to the above theoretical research, numerous empirical studies have shown that learning by doing has an important impact on the investment behavior of investors and results. For example, Klæsset al. [18] estimate a two-factor learning curve model and conclude that cost reduction depends on both cumulative production and R&D investment. Wang et al. [19] introduce a methodology to examine the effect of technological learning on industrial air pollutants intensity in China. They find that learning by doing can significantly reduce industrial air pollutants intensity through energy efficiency. Elshurafa et al. [20] estimate the learning curve of balance-of-system costs in photovoltaics for more than 20 countries via an extensive dataset. The results show learning by doing can bring down balance-of-system costs.

In this paper, we develop a dynamic product innovation, advertising effort, and transfer price decision model. To some extent, our research is an extension of Liu et al. [3]. But compared with the work of Liu et al. [3], the main features of this paper are as follows: (i) the operations and marketing department’s instantaneous cost functions of product innovation and advertising effort depend on the corresponding investments as well as the knowledge accumulations in product innovation and advertising effort activities, respectively; (ii) change rates of knowledge accumulations of product innovation and advertising effort are state variables; (iii) both quality level and product goodwill influence the
inverse demand function jointly. With this new model, we obtain a couple of really important results in which some are similar to Liu et al. [3]; for example, compared with the administered transfer price, the negotiation between the operations department and the marketing department leads to a higher retail price, lower advertising effort, and higher quality improvement effort. In particular, we find that if the knowledge accumulation resulting from learning by doing in product innovation and advertising effort is taken into account, negotiated transfer-pricing results in a lower transfer price; this result is opposite to Liu et al. [3]. The reason that our conclusion is different from Liu et al. [3] may be that Liu et al. [3] have not considered the knowledge accumulating from learning by doing. But in our model, the negotiated transfer-pricing leads to a higher quality improvement effort and more knowledge accumulating from learning by doing will be obtained, which significantly reduce investment costs in quality improvement effort and the total cost, so the operations department is willing to transfer its products at lower prices.

The paper proceeds as follows: Section 3 presents a differential game model involving an operations department and a marketing department within a firm. Section 4 investigates the centralized scenario. Section 5 studies the decentralized scenario. Section 6 is some numerical analysis. Section 7 concludes the paper.

3. The Basic Model

3.1. Parameters and Variables

**Variables**

- \(q(t)\): the quality of the product at time \(t\)
- \(k(t)\): the product innovation investment carried by the operations department at time \(t\)
- \(h(t)\): the advertising effort carried by the marketing department at time \(t\)
- \(g(t)\): the goodwill of the product at time \(t\)
- \(A_i(t)\): the accumulated experience (knowledge) in product innovation
- \(A_i(t)\): the accumulated experience (knowledge) in advertising effort
- \(C_i(k(t), A_i(t))\): the operations department's cost function carrying out product innovation
- \(C_i(h(t), A_i(t))\): the marketing department's cost function carrying out advertising effort
- \(p(t)\): the price level of product at time \(t\)
- \(D(q(t), g(t), p(t))\): the demand level of product at time \(t\)
- \(w(t)\): the transfer price at time \(t\)
- \(\pi_o(t), \pi_m(t), \pi_f(t)\): the instantaneous net profit rate of operations department, marketing department, and firm, respectively
- \(\lambda_i(t)\): the dynamic adjoint variables under the centralized scenario, \(i = 1, 2, 3, 4\)
- \(\delta_2(t)\) the operations department's dynamic adjoint variables under the decentralized scenario, \(i = 1, 2\)
- \(\delta_2(t)\) the marketing department’s dynamic adjoint variables under the decentralized scenario, \(i = 1, 2\).

**Parameters**

- \(C_0\): the constant marginal production cost, \(C_0 > 0\)
- \(\sigma\): the decay rate of quality level, \(\sigma > 0\)
- \(\alpha\): the inverse measure of product innovation efficiency, \(\alpha > 0\)
- \(\beta\): the inverse measure of advertising effort efficiency, \(\beta > 0\)
- \(\phi\): the influence coefficient of design quality on goodwill, \(\phi > 0\)
- \(\omega\): the decay rate of goodwill, \(\omega > 0\)
- \(\mu\): the growth rate of knowledge accumulation of product innovation
- \(\xi\): the growth rate of knowledge accumulation of advertising effort
- \(\gamma\): the decaying memory rate of knowledge accumulation in product innovation
- \(\theta\): the decaying memory rate of knowledge accumulation in advertising effort
- \(b_1\): the learning rates of knowledge accumulation in product innovation, \(b_1 > 0\)
- \(b_2\): the learning rates of knowledge accumulation in advertising effort, \(b_2 > 0\)
- \(q_0\): the initial quality level, \(q_0 = q(0) > 0\)
- \(g_0\): the initial goodwill, \(g(0) = g_0 > 0\)
- \(A_{10}\): the initial level of \(A_1(t)\)
- \(A_{20}\): the initial level of \(A_2(t)\)
- \(a_0\): a parameter of the inverse demand function, \(a_0 > 0\)
- \(a_1, a_2\): the constant parameters, respectively, \(a_1 > 0, a_2 > 0\)
- \(r\): the constant discount rate, \(r \in [0, 1]\).

3.2. The Model of Division of Product Innovation and Advertising Effort. In this paper, we investigate a dynamic control problem over continuous time \(t \in [0, \infty)\), where a firm \((F)\) consists of two departments, e.g., an operations department \((O)\) taking charge of improving the quality of a particular product (product innovation), and a marketing department \((M)\) in charge of advertising effort which buys the product from the operating department at a certain transfer price and sells this product to end customers. The product quality \(q(t)\) can be improved by the operations department's R&D investment \(k(t)\), and its time evolution is given as follows.

\[ \dot{q}(t) = k(t) - aq(t) \] (1)
In addition, we assume that the product goodwill \( g(t) \) can be improved by advertising effort as well as the improvement of design quality and it evolves following differential equation.

\[
\dot{g}(t) = h(t) + \phi q(t) - \omega g(t) \tag{2}
\]

Expression (2) shows that to promote the product goodwill can though carry advertising effort and (or) improve the design quality, but these two kinds of decision-making activities are attributable to operations department and marketing department, respectively, in the decentralized scenario, so in this scenario, to design an appropriate transfer price for coordinating the two departments is crucial issue.

Following El Ouardighi and Kogan [21], El Ouardighi [22], and Liu [3], we use (3) to describe the customer demand.

\[
D(q(t), g(t), p(t)) = (a_0 - a_3 p(t)) (a_1 q(t) + a_2 g(t)) \tag{3}
\]

From (3), we see that customer demand is negatively affected by price and positively affected by both the quality level and goodwill.

Next, let us inspect the time evolution of accumulated experience (knowledge). Following Arrow [4], Dorroh et al. [23, 24], Aghion and Howitt [25], and Li [8], we assume that the function of the accumulated experience (knowledge) in product innovation or advertising effort is an exponential smoothing process of the corresponding historical investment, which is introduced, respectively, as follows.

\[
A_1(t) = e^{-\gamma t} \left[ A_{10} + \mu \int_0^t e^{\gamma \tau} k(\tau) d\tau \right] \tag{4}
\]

\[
A_2(t) = e^{-\phi t} \left[ A_{20} + \xi \int_0^t e^{\phi \tau} h(\tau) d\tau \right] \tag{5}
\]

Differentiating (4) and (5) with respect to time \( t \), one can obtain the following.

\[
\dot{A}_1(t) = \mu k(t) - \gamma A_1(t) \tag{6}
\]

\[
\dot{A}_2(t) = \xi h(t) - \theta A_2(t) \tag{7}
\]

Now, we investigate the cost functional forms of product innovation and advertising effort. We assume that the cost functions of product innovation and advertising effort are given by the following.

\[
C_1(k(t), A_1(t)) = \alpha k^2(t) - b_1 (A_1(t) - A_{10}) \tag{8}
\]

\[
C_2(h(t), A_2(t)) = \beta h^2(t) - b_2 (A_2(t) - A_{20}) \tag{9}
\]

Such widely used quadratic cost functions, \( \alpha k^2(t) \) and \( \beta h^2(t) \), imply increasing marginal costs of corresponding investment [26, 27]. Meanwhile, following theoretical and empirical research ([4, 8, 18, 19], etc.), investment costs decrease with the increasing of the accumulated experience (knowledge), so for simplicity we suppose that \( C_1(\cdot) \) and \( C_2(\cdot) \) decrease linearly with the accumulated experience (knowledge). Furthermore, for simplicity, let us denote the constant production marginal cost with \( C_0 \), \( C_0 > 0 \), and assume no fixed cost.

Now, using (1)–(9), we obtain the profit functions of the two departments, respectively.

\[
\pi_o(t) = [w(t) - C_0] D(q(t), g(t), p(t)) - C_1(k(t), A_1(t)) \tag{10}
\]

\[
\pi_m(t) = [p(t) - w(t)] D(q(t), g(t), p(t)) - C_2(h(t), A_2(t)) \tag{11}
\]

The corresponding profit function of the firm can be expressed as follows.

\[
\pi_f(t) = [p(t) - C_0] D(q(t), g(t), p(t)) - C_1(k(t), A_1(t)) - C_2(h(t), A_2(t)) \tag{12}
\]

Now, the operations department’s objective is to find the optimal investment levels in product innovation \( k(t) \) and marketing department’s objective is to find the optimal advertising effort levels \( h(t) \) as well as the optimal retail price level \( p(t) \) over continuous time \( t \in [0, \infty) \), such that the discounted profit flow is maximized, respectively.

\[
\prod_o = \max_{k(t)} \int_0^\infty e^{-r t} \{ [w(t) - C_0] D(q(t), g(t), p(t)) - C_1(k(t), A_1(t)) \} \, dt
\]

\[
\prod_m = \max_{p(t), h(t)} \int_0^\infty e^{-r t} \{ [p(t) - w(t)] D(q(t), g(t), p(t)) - C_2(h(t), A_2(t)) \} \, dt
\]

\[
\text{S.t. } \dot{q}(t) = k(t) - \sigma q(t) \hspace{1cm} \dot{g}(t) = h(t) + \phi q(t) - \omega g(t) \hspace{1cm} \dot{A}_1(t) = \mu k(t) - \gamma A_1(t) \hspace{1cm} \dot{A}_2(t) = \xi h(t) - \theta A_2(t) \tag{13}
\]
There are four control variables in the differential game problem (13), e.g., \( k(t), h(t), p(t), \) and \( w(t) \), and four state variables, \( q(t), g(t), A_1(t), \) and \( A_2(t) \). Following Eliasberg and Chatterjee [28] and Dockner and Jørgensen [29], we suppose that the players use open-loop equilibrium strategies.

\[
\prod_f = \max_{p,k,h} \int_0^\infty \left\{ \left[ p(t) - C_0 \right] D(q(t), g(t), p(t)) - C_1 (k(t), A_1(t)) - C_2 (h(t), A_2(t)) \right\} dt
\]

S.t. \( q(t) = k(t) - \sigma q(t) \)
\( g(t) = h(t) + \phi q(t) - \omega g(t) \)
\( \dot{A}_1(t) = \mu k(t) - \gamma A_1(t) \)
\( \dot{A}_2(t) = \xi h(t) - \theta A_2(t) \)

The corresponding current value Hamiltonian function is written as follows.

\[
H = \left[ p(t) - C_0 \right] \left[ a_1 q(t) + a_2 g(t) \right] \left[ a_0 - a_3 p(t) \right]
- \left[ a k^2(t) - b_1 (A_1(t) - A_{10}) \right]
- \left[ b h^2(t) - b_2 (A_2(t) - A_{20}) \right]
+ \lambda_1(t) \left[ k(t) - \sigma q(t) \right]
+ \lambda_2(t) \left[ h(t) + \phi q(t) - \omega g(t) \right]
+ \lambda_3(t) \left[ \mu k(t) - \gamma A_1(t) \right]
+ \lambda_4(t) \left[ \xi h(t) - \theta A_2(t) \right]
\]

The necessary conditions for the maximization of the Hamiltonian function (15) are written as follows.

\[
\frac{\partial H}{\partial k(t)} = -2ak(t) + \lambda_1(t) + \mu \lambda_3(t) = 0 \quad (16)
\]
\[
\frac{\partial H}{\partial h(t)} = -2bh(t) + \lambda_2(t) + \xi \lambda_4(t) = 0 \quad (17)
\]
\[
\frac{\partial H}{\partial p(t)} = \left[ a_1 q(t) + a_2 g(t) \right] \left[ a_0 - a_3 p(t) \right] = 0 \quad (18)
\]

The corresponding costate equations can be given by

\[
\dot{\lambda}_1(t) = r \lambda_1(t) - \frac{\partial H}{\partial q(t)}
= (r + \sigma) \lambda_1(t) - a_1 \left[ p(t) - C_0 \right] \left[ a_0 - a_3 p(t) \right]
- \phi \lambda_2(t)
\]

\[
\dot{\lambda}_2(t) = r \lambda_2(t) - \frac{\partial H}{\partial g(t)}
= (r + \omega) \lambda_2(t) - a_2 \left[ p(t) - C_0 \right] \left[ a_0 - a_3 p(t) \right]
\]

\[
\dot{\lambda}_3(t) = r \lambda_3(t) - \frac{\partial H}{\partial A_1(t)}
= (r + \gamma) \lambda_3(t) - b_1
\]

\[
\dot{\lambda}_4(t) = r \lambda_4(t) - \frac{\partial H}{\partial A_2(t)}
= (r + \theta) \lambda_4(t) - b_2
\]

The corresponding transversality condition can be given by

\[
\lim_{t \rightarrow +\infty} \frac{\partial H}{\partial q} = 0, \quad \lim_{t \rightarrow +\infty} \frac{\partial H}{\partial g} = 0
\]

\[
\lim_{t \rightarrow +\infty} \frac{\partial H}{\partial A_1} = 0, \quad \lim_{t \rightarrow +\infty} \frac{\partial H}{\partial A_2} = 0
\]

4. The Firm Optimum under Centralized Scenario (“C”)

4.1. The Optimal Conditions and Characteristics.

In this section, we investigate the firm’s optimization behavior under centralized scenario and we seize this scenario as a benchmark. The corresponding optimization problem of the centralized managers can be given as follows.

\[
\begin{align*}
\dot{A}_1(t) &= \mu k(t) - \gamma A_1(t) \\
\dot{A}_2(t) &= \xi h(t) - \theta A_2(t)
\end{align*}
\]

with boundary conditions \( \lim_{t \rightarrow +\infty} e^{-rt} \lambda_i(t) = 0, \ i \in \{1,2,3,4\} \). Accordingly, the transversality condition can be given as \( \lim_{t \rightarrow +\infty} e^{-rt} k(t) = 0 \) and \( \lim_{t \rightarrow +\infty} e^{-rt} h(t) = 0 \).

The Hamiltonian function (15) can be thought of as a dynamic profit rate with five separate terms: (i) the instantaneous net profit rate \( \pi_f \); (ii) the value \( \lambda_1(t) q(t) \) of the quality improvement generated by product innovation at rate \( k(t) \); (iii) the value \( \lambda_2(t) g(t) \) of the new goodwill come into being by advertising effort at rate \( h(t) \) and quality improvement at rate \( q(t) \); (iv) the new accumulated knowledge of the product quality improvement generated in product innovation at rate \( \mu k(t) \); and (v) the value \( \lambda_4(t) A_2(t) = \lambda_4(t) \left[ \xi h(t) - \theta A_2(t) \right] \) of the new accumulated knowledge of advertising effort created by doing in the advertising effort at rate \( \xi h(t) \).

Solving the differential (21) and (22) and using boundary conditions \( \lim_{t \rightarrow +\infty} e^{-rt} k(t) = 0 \) and \( \lim_{t \rightarrow +\infty} e^{-rt} h(t) = 0 \), one gets the following.

\[
\begin{align*}
\lambda_3(t) &= \frac{b_1}{r + \gamma} \\
\lambda_4(t) &= \frac{b_2}{r + \theta}
\end{align*}
\]

From (10) and (11), we find the shadow price \( \lambda_3(t) \) or \( \lambda_4(t) \) is a positive constant.
Solving (18), we obtain the following.
\[ \rho(t) = \frac{a_0 + a_1 C_0}{2a_3} \]  
\[ (25) \]

Substituting (23) and (24) into (16) and (17), we have the following.
\[ \lambda_1(t) = 2\alpha k(t) - \frac{\mu b_1}{r + \gamma} \]
\[ \lambda_2(t) = 2\beta h(t) - \frac{\xi b_2}{r + \theta} \]  
\[ (26) \]
\[ (27) \]

From (19), (20), (25), (26), and (27), the following differential equation can be obtained.
\[ \dot{k}(t) = (r + \sigma) k(t) - \frac{\beta \omega}{\alpha} h(t) \]
\[ - \frac{1}{2\alpha} \left[ \frac{\mu b_1 (r + \sigma)}{r + \gamma} + \frac{\xi b_2 \phi}{r + \theta} - \frac{a_1 (a_0 - a_3 C_0)^2}{4a_3} \right] \]
\[ (28) \]
\[ \dot{h}(t) = (r + \omega) h(t) \]
\[ - \frac{1}{2\beta} \left[ \frac{\xi b_2 (r + \omega)}{r + \theta} + \frac{a_2 (a_0 - a_3 C_0)^2}{4a_3} \right] \]
\[ (29) \]

Inspecting (28) and (29), we get the following intuitive proposition.

**Proposition 1.** Under the centralized firm’s optimum, \( \forall t \in [0, \infty), \) there are (i) \( \partial k(t)/\partial b_1 < 0; \partial h(t)/\partial b_2 < 0; \) (ii) \( \partial k(t)/\partial h(t) < 0; \partial h(t)/\partial k(t) = 0. \)

**Proof.** See Appendix A. \( \square \)

Proposition 1–(i) shows that the change rates of investments in product innovation \( k(t) \) and advertising effort \( h(t) \) are decreasing with the learning rates \( (b_1 \text{ and } b_2) \) of knowledge accumulation in product innovation and advertising effort, respectively.

Proposition 1–(ii) shows that the change rates of investments in product innovation \( k(t) \) is decreasing with the investments in process innovation \( h(t) \), while the change rates of investments in advertising effort \( h(t) \) with the investments in product innovation \( k(t) \) are changeless.

### 4.2. Steady State Equilibrium and General Solutions

Together with (1), (2), (28), and (29), the following dynamic system of linear differential equations is given.

\[ \dot{k}(t) = (r + \sigma) k(t) - \frac{\beta \phi}{\alpha} h(t) \]
\[ \dot{h}(t) = (r + \omega) h(t) \]
\[ - \frac{1}{2\alpha} \left[ \frac{(r + \sigma) \mu b_1}{r + \gamma} - \frac{\xi \phi b_2}{r + \theta} + \frac{a_1 (a_0 - a_3 C_0)^2}{4a_3} \right] \]
\[ \dot{q}(t) = k(t) - a q(t) \]
\[ \dot{g}(t) = h(t) + \theta q(t) - \omega g(t) \]
\[ (30) \]

There are four differential equations in dynamic system (30). In general, very often it is very difficult to find explicitly or implicitly the solutions of a differential equations system involving \( n \geq 4 \) variables. However, we can investigate the steady state of this differential equations system and it can be stated in Proposition 2.

**Proposition 2.** In the centralized scenario, the steady state equilibrium denoted by superscript “*\&*” is given by the following.

\[ \bar{k} = \frac{1}{2\alpha} \left[ \frac{(a_0 - a_3 C_0)^2}{4a_3} \left( \frac{a_3 \phi}{r + \omega} + a_1 \right) \right. \]
\[ + \left. \frac{\mu b_1 (r + \sigma)}{r + \gamma} \right] \]
\[ (31) \]
\[ \bar{h} = \frac{1}{2\beta} \left[ \frac{\xi b_2 (r + \omega)}{r + \theta} + \frac{a_2 (a_0 - a_3 C_0)^2}{4a_3} \right] \]
\[ (32) \]
\[ \bar{q} = \frac{1}{2\alpha \sigma} \left[ \frac{(a_0 - a_3 C_0)^2}{4a_3} \left( \frac{a_2 \phi}{r + \omega} + a_1 \right) \right. \]
\[ + \left. \frac{\mu b_1 (r + \sigma)}{r + \gamma} \right] \]
\[ (33) \]
\[ \bar{g} = \frac{1}{2\omega} \left[ \frac{\xi b_2}{\beta (r + \theta)} + \frac{a_2 (a_0 - a_3 C_0)^2}{4a_3} \right. \]
\[ + \left. \frac{\theta (a_0 - a_3 C_0)^2}{4a_3 \alpha (r + \gamma)} \left( \frac{a_3 \phi}{r + \omega} + a_1 \right) \right] \]
\[ (34) \]

Proposition 2 suggests that all the strategies under steady state equilibrium are constants over time. Further, from (31)–(34), we derive Corollary 3.

**Corollary 3.** In the centralized scenario, we have (i) \( \partial \bar{k}/\partial b_1 > 0, \partial \bar{k}/\partial b_2 > 0; \) (ii) \( \partial \bar{h}/\partial b_1 = 0, \partial \bar{h}/\partial b_2 > 0; \) (iii) \( \partial \bar{q}/\partial b_1 > 0, \partial \bar{q}/\partial b_2 > 0; \) (iv) \( \partial \bar{g}/\partial b_1 > 0, \partial \bar{g}/\partial b_2 > 0. \)

It is shown from Corollary 3–(i) and (ii) that the investment in product innovation under steady state equilibrium increases with \( b_1 \) and \( b_2, \) respectively; the investment in
advertising under steady state equilibrium increases with the learning rate of knowledge accumulation in advertising effort \(b_2\), while the learning rate of knowledge accumulation in product innovation \(b_1\) has no effect on investment in advertising under steady state equilibrium. From Corollary 3-(iii) and (iv) we see that the investments in product innovation and advertising under steady state equilibrium are increasing with \(b_1\) and \(b_2\), respectively.

Substituting (31) and (32) into (5) and (19), the steady state solutions of accumulated knowledge are obtained.

\[
\begin{align*}
\hat{A}_1 &= \frac{\mu}{2\alpha\gamma (r + \sigma)} \left[ \frac{(a_0 - a_1 C_0)^2}{4a_3} \frac{a_2 \phi}{r + \omega} + a_1 \right] + \frac{\mu b_1 (r + \sigma)}{r + \gamma} \\
\hat{A}_2 &= \frac{\xi}{2\beta\theta (r + \omega)} \left[ \frac{\xi b_2 (r + \omega)}{r + \theta} + \frac{a_2 (a_0 - a_1 C_0)^2}{4a_3} \right] + \frac{\mu b_1 (r + \sigma)}{r + \gamma} \frac{1}{r + \gamma} \\
&+ b_1 \left[ \frac{\mu}{2\alpha\gamma (r + \sigma)} \left( \frac{(a_0 - a_1 C_0)^2}{4a_3} \frac{a_2 \phi}{r + \omega} + a_1 \right) + \frac{\mu b_1 (r + \sigma)}{r + \gamma} \right] - A \omega \right]
\end{align*}
\]

Now, let us use Proposition 4 to investigate the stability properties of the steady state equilibrium.

**Proposition 4.** In the centralized scenario, there exist admissible parameter constellations such that the steady state equilibrium \([\hat{k}, \hat{h}, \hat{q}, \hat{g}]\) is a saddle point.

**Proof.** See Appendix B.

Having obtained the monopolist’s steady state equilibrium \([\hat{k}, \hat{h}, \hat{q}, \hat{g}]\) in the centralized scenario, we can explore the general solutions of the differential equation system (30). In order to obtain the general solutions, we linearize system (30) in the neighborhood of steady state equilibrium \([\hat{k}, \hat{h}, \hat{q}, \hat{g}]\). Let \(\{\phi_1, \phi_2, \phi_3, \phi_4\}\) and \(E\) denote the eigenvalues and unit matrix of the Jacobian matrix (B.1), respectively, and \(\{v_1, v_2, v_3, v_4\}\) denote the eigenvectors corresponding to eigenvalues \(\phi_1, \phi_2, \phi_3, \phi_4\), where \(v_i = \{v_{i1}, v_{i2}, v_{i3}, v_{i4}\}^{-1}\), \(\tau = 1, 2, 3, 4\). Further, the eigenvectors \(v_i\) are given by \(\Sigma - \phi_i E v_i = 0\), \(\tau = 1, 2, 3, 4\). Thus, the general solutions of system (30) in the neighborhood of steady state equilibrium \([\hat{k}, \hat{h}, \hat{q}, \hat{g}]\) can be expressed as a linear combination of the eigenvalues and the eigenvectors (here we use symbol \(n\) to represent the system’s general solutions). That is,

\[
\begin{align*}
\{k^n(t), h^n(t), q^n(t), g^n(t)\} &= \left\{ \hat{k} + \sum_{\tau=1}^{4} z_{\tau} v_{\tau 1} e^{\phi_1 t}, \right. \\
&\left. \hat{h} + \sum_{\tau=1}^{4} z_{\tau} v_{\tau 2} e^{\phi_2 t}, \right. \\
&\left. \hat{q} + \sum_{\tau=1}^{4} z_{\tau} v_{\tau 3} e^{\phi_3 t}, \right. \\
&\left. \hat{g} + \sum_{\tau=1}^{4} z_{\tau} v_{\tau 4} e^{\phi_4 t} \right\}
\end{align*}
\]

where \(z_{\tau}\) is constant coefficients, \(\tau = 1, 2, 3, 4\), which can be determined by using initial conditions.

Now, we are in a position to solve the equilibrium profit of both departments and the firm. Substituting the steady state equilibrium \([\hat{k}, \hat{h}, \hat{q}, \hat{g}]\) into (II) and (II), the equilibrium profits of the operations department and the marketing department under centralized scenario are obtained, respectively.

\[
\pi^o_C = \frac{(a_0 - a_1 C_0) w}{2} \left[ \frac{a_1}{2\alpha \gamma (r + \sigma)} \left( \frac{(a_0 - a_1 C_0)^2}{4a_3} \frac{a_2 \phi}{r + \omega} + a_1 \right) + \frac{\mu b_1 (r + \sigma)}{r + \gamma} \right] \\
+ b_1 \left[ \frac{\mu}{2\alpha \gamma (r + \sigma)} \left( \frac{(a_0 - a_1 C_0)^2}{4a_3} \frac{a_2 \phi}{r + \omega} + a_1 \right) + \frac{\mu b_1 (r + \sigma)}{r + \gamma} \right] - A \omega \right]
\]

The firm’s equilibrium profit can be given by \(\pi^C + \pi^m_C\).

5. Decentralized Scenario (“D”)

In this section, we investigate both negotiated and administered transfer-pricing in turn. The backward induction method will be used to solve the two-stage game.

Now we investigate the equilibrium strategies of both departments under a given constant transfer price. From (26), we can write the current value Hamiltonian function for the two departments, respectively,

\[
H_o = [w(t) - C_0] \left[ a_1 q(t) + a_2 g(t) \right] \left[ a_0 - a_0 p(t) \right] - \left[ \alpha k^2(t) - b_1 (A_1(t) - A_{10}) \right]
\]
with the dynamic costate variables $\delta_i(t)$ and $\theta_i(t)$, $\tau = 1, 2$.

The necessary conditions and costate equations are

\[
\frac{\partial H}{\partial k}(t) = -2\alpha k(t) + \delta_1(t) + \mu \delta_2(t) = 0 \implies \delta_1(t) = 2\alpha k(t) - \mu \delta_2(t) \tag{42}
\]

\[
\frac{\partial H_m}{\partial h}(t) = -2\beta h(t) + \delta_1(t) + \xi \delta_2(t) = 0 \implies \delta_1(t) = 2\beta h(t) - \xi \delta_2(t) \tag{43}
\]

\[
\frac{\partial H_m}{\partial p}(t) = [a_1 q(t) + a_2 g(t)] [a_0 + a_3 w(t) - 2a_3 p(t)] = 0 \tag{44}
\]

\[
\dot{\delta}_1(t) = r \delta_1(t) - \frac{\partial H_0}{\partial q(t)} = (r + \sigma) \delta_1(t) - a_1 [w(t) - C_0] [a_0 - a_3 p(t)] \tag{45}
\]

\[
\dot{\delta}_2(t) = r \delta_2(t) - \frac{\partial H_0}{\partial A_1(t)} = (r + \gamma) \delta_2(t) - b_1 \tag{46}
\]

\[
\dot{\theta}_1(t) = r \theta_1(t) - \frac{\partial H_m}{\partial g(t)} = (r + \omega) \theta_1(t) - a_2 [p(t) - w(t)] [a_0 - a_3 p(t)] \tag{47}
\]

\[
\dot{\theta}_2(t) = r \theta_2(t) - \frac{\partial H_m}{\partial A_2(t)} = (r + \theta) \theta_2(t) - b_2 \tag{48}
\]

Now, solving the differential equations (46) and (48) and using the terminal boundary conditions $\lim_{t \to +\infty} \delta_1(t) = 0$, $\lim_{t \to +\infty} b_2(t) = 0$, one yields the following.

\[
\begin{align*}
\delta_1(t) &= \frac{b_1}{r + \gamma} \tag{50} \\
\delta_2(t) &= \frac{b_2}{r + \theta} \tag{51}
\end{align*}
\]

Substituting (42) and (43) into (45) and (47), respectively, using (49)–(51), we obtain following differential equations.

\[
\begin{align*}
\dot{k}(t) &= (r + \sigma) k(t) - \frac{\mu b_1 (r + \sigma)}{2\alpha(r + \gamma)} \frac{a_1 [w(t) - C_0]}{[a_0 - a_3 w(t)]} \tag{52} \\
\dot{h}(t) &= (r + \omega) h(t) - \frac{\xi b_2 (r + \omega)}{2\beta(r + \theta)} \frac{a_2 [a_0 - a_3 w(t)]}{2a_3 r + \omega} \\
\dot{\gamma}(t) &= (r + \omega) \gamma(t) - \frac{\xi b_2 (r + \omega)}{2\beta(r + \theta)} \frac{a_2 [a_0 - a_3 w(t)]}{2a_3 r + \omega} + \frac{\mu b_1}{2a_3 \sigma(r + \sigma)} \frac{a_1 [w(t) - C_0]}{[a_0 - a_3 w(t)]} \tag{53}
\end{align*}
\]

The solutions of differential equation system (1), (2), (52), and (53) under the steady state conditions $q(t) = \gamma(t) = \dot{k}(t) = h(t) = 0$ can be demonstrated by Proposition 5.

**Proposition 5.** In the decentralized scenario, when transfer price is a constant, the steady state equilibrium is given by, respectively, the following.

\[
\begin{align*}
\bar{k} &= \frac{b_1}{2\alpha(r + \gamma)} + \frac{a_1 (w - C_0) (a_0 - a_3 w)}{4\alpha(r + \sigma)} \tag{54} \\
\bar{h} &= \frac{\xi b_2}{2\beta(r + \theta)} + \frac{a_1 (a_0 - a_3 w)^2}{8\beta a_3 (r + \omega)} \tag{55} \\
\bar{\gamma} &= \frac{b_1}{2\alpha \sigma(r + \gamma)} + \frac{a_1 (w - C_0) (a_0 - a_3 w)}{4\alpha \sigma(r + \sigma)} + \frac{\mu b_1}{2a_3 \sigma \alpha \sigma(r + \sigma)} \tag{56} \\
\bar{\bar{\gamma}} &= \frac{\xi b_2}{2\beta \omega (r + \theta)} + \frac{a_1 (a_0 - a_3 w)^2}{8\beta a_3 \alpha \sigma (r + \omega)} + \frac{\mu b_1}{2a_3 \sigma \omega (r + \sigma)} \tag{57}
\end{align*}
\]

Now, we use Proposition 5 to analyze the stability properties of the steady state equilibrium $[\bar{k}, \bar{h}, \bar{\gamma}, \bar{\bar{\gamma}}]$.

**Proposition 6.** In the decentralized scenario, there exist admissible parameter constellations such that the steady state equilibrium $[\bar{k}, \bar{h}, \bar{\gamma}, \bar{\bar{\gamma}}]$ is a saddle point. 

**Proof.** See Appendix C. \qed

From Proposition 6, Corollary 7 can be acquired.

**Corollary 7.** At steady state, the advertising effort increases when transfer price $w$ decreasing.
Corollary 8. At steady state, when \( w = \frac{a_0}{2a_3} \), the best quality level can be caused; when \( w \in \left[ 0, \frac{a_0}{2a_3} \right] \), the quality improvement effort and quality level are increasing with \( w \); when \( w \in \left[ \frac{a_0}{2a_3}, \frac{a_0}{a_3} \right] \), the quality improvement effort and quality level are decreasing with \( w \).

Substituting (54) and (55) into (5) and (6), the steady state solutions of accumulated knowledge under decentralized scenario can be obtained.

\[
\overline{A}_1 = \frac{\mu^2 b_1}{2\alpha(r + \gamma)} + \frac{a_1 \mu}{4\alpha(r + \sigma)} (w - C_0)(a_0 - a_3)w
\]

\[
\overline{A}_2 = \frac{\xi^2 b_2}{2\beta(r + \theta)} + \frac{a_2 \xi}{8\beta a_3}(a_0 - a_3)w^2
\]

Now we are in the position to solve the profit of both departments and the firm in the centralized scenario. Substituting the equilibrium strategies (54)–(5.20 into (11) and (12), we obtain the steady state equilibrium profit levels of both departments and the firm under decentralized scenario.

\[
\pi_o^D (w) = \frac{w - C_0}{2} (a_0 - a_3)w + \frac{a_0}{4\alpha(r + \sigma)} (a_0 - a_3)w
\]

\[
\pi_m^D (w) = \frac{a_0}{4\alpha(r + \sigma)} (a_0 - a_3)w^2 + \frac{a_2}{8\beta a_3}(a_0 - a_3)w^2
\]

5.1. Negotiated Transfer Price ("DN"). Following Liu et al. [3], when the transfer price is determined by both departments through negotiation, the solution can utilize the classic Nash bargaining model to derive, e.g., solving the following problem.

\[
\max_w \pi_{o, m}^D (w) \pi_{o, m}^D (w)
\]

Substituting (11), (12), (49), and (54)–(61) into (63) and using the first-order conditions of (37), we get the optimal negotiated transfer price as follows:

\[
w_{DN} = m - b - \sqrt{S + T}
\]

where

\[
a = B_4 - B_9,
\]

\[
b = B_5 - B_{10} + \frac{B_1}{2},
\]

\[
m = \sqrt{b - \frac{8ac}{3} + 2\left(\sqrt{3} - 1\right) aU + \left(\sqrt{3} - 1\right)^2 aV},
\]

\[
S = 2b - \frac{16ac}{3} - 2\left(\sqrt{3} - 1\right) aU - \left(\sqrt{3} - 1\right)^2 aV,
\]

\[
T = \frac{8abc - 16a^2 d - 2b^3}{m},
\]

\[
c = B_6 - B_{11} + \frac{B_1 (a_0 - a_3 C_0)}{2a_3} + \frac{B_2}{2},
\]

\[
d = B_7 - B_{12} + \frac{B_3 (a_0 - a_3 C_0)}{2a_3} + \frac{B_5 (a_0 - a_3 C_0)}{2a_3} - 1,
\]

\[
e = B_8 - B_{13} + \frac{B_3 (a_0 - a_3 C_0)}{2a_3},
\]

\[
Q = \frac{27ad^2 + 2c^3 + 27b^2 c - 72ace - 9bcd}{54},
\]

\[
P = \frac{c^2 + 12ae - 3bd}{9},
\]

\[
D = \sqrt{Q^2 + P^3},
\]

\[
U = \sqrt{Q + D},
\]
5.2. Administered Transfer Price ("DA"). Now we are in the position to inspect the administered transfer price. In this case, the transfer price is decided by the firm through maximizing its profit; i.e., one can obtain the solution by solving following equation.

$$\max_w \pi^{DA} = n_o^D (w) + n_m^D (w)$$  \hspace{1cm} (66)$$

Substituting (11), (12), (49), and (54)–(61) into (66) and applying the first-order conditions of (66), the optimal administered transfer price is given as

$$w^{DA} = \sqrt[3]{\frac{D_2 - 2D_1D_3D_4 + 4D_1^2D_3}{32D_1^3} + \sqrt{\frac{D_2^3 - 2D_1D_2D_3 + 4D_1^2D_4}{32D_1^3} \left( \frac{D_2^3 - 2D_1D_2D_3 + 4D_1^2D_4}{32D_1^3} \right)^2 + \left( \frac{8D_1D_3 - 3D_2^2}{48D_1^2} \right)^3}}$$  \hspace{1cm} (67)$$

where

$$D_1 = \frac{a_i^2a_s^2}{16a \sigma (r + \sigma)} - \frac{a_i^2a_s^2}{32\beta \omega (r + \omega)} + \frac{a_i a_s a_i^3}{2a \sigma (r + \sigma)} + \frac{a_i a_s}{16a \omega (r + \sigma)}$$

$$D_2 = \left( \frac{a_i - a_i C_0}{4a \sigma (r + \sigma)} \right)^2 - \frac{a_i a_s}{2a \sigma (r + \sigma)} + \frac{a_i^2}{16\beta \omega (r + \omega)}$$

$$D_3 = \frac{a_i a_s b_i}{8a \sigma (r + \sigma)}$$

$$D_4 = \frac{a_i a_s b_i^3}{64a \beta (r + \omega)}$$

$$D_5 = \frac{a_i a_s b_i^3}{48a \beta (r + \omega)}$$

$$D_6 = \frac{a_i a_s b_i^3}{2a \sigma (r + \sigma)}$$

$$D_7 = \frac{a_i a_s}{2a \sigma (r + \sigma)}$$

$$D_8 = \frac{a_i a_s}{16a \omega (r + \omega)}$$

$$D_9 = \frac{a_i a_s}{4a \sigma (r + \sigma)}$$

$$D_{10} = \frac{a_i^2a_i^2}{16\beta (r + \omega)^2}$$

$$D_{11} = \frac{3a_i^2(a_i - a_i C_0)^2}{32\beta (r + \omega)^2} - \frac{a_i a_s b_i^3}{8a \beta (r + \omega)}$$
6. Numerical Examples

We are now in the position to carry out some numerical analysis for obtaining more managerial insights by investigating the following questions: (i) Are there some differences between the two optimal transfer prices? (ii) What are the features of the two optimal transfer prices and the corresponding demands as well as profits? (iii) How large can the efficiencies be for two optimal transfer prices? To answer these questions, we perform a series of sensitivity analyses on some key system parameters $a_0, a_1, a_2, a_3, b_1, b_2, r, \sigma, \phi, \omega$. The parameter values of benchmarks are set in Table 1. We carry out the sensitivity analysis by changing each value of the parameters by -50%, -25%, +25%, and +50% in turn, while keeping the remaining parameter values unchanged. The intuitive sensitivity analysis results are shown in Figure 1 and Table 2.

Observe Figure 1, we come up with Propositions 6 and 9.

**Proposition 9.** Under administered transfer-pricing, for any given set of parameters, there is a unique transfer price which maximizes the firm’s profit $\pi^D_A(w)$, and there exists $w^D_{DA*} \in (0, a_0/2a_3) < w^D_{DA*} < w^D_{DA*} \in (a_0/2a_3, a_0/a_3); \text{ e.g., }$ the marketing department’s optimal transfer price is below the firm’s optimal transfer price, while the firm’s optimal transfer price is below the operations department’s optimal transfer price.

**Proposition 10.** For any given set of parameters, there exists $w^D_{DA*} \in (0, a_0/2a_3) < w^D_{DA*} \in (a_0/2a_3, a_0/a_3); \text{ e.g., }$ administered transfer price is larger negotiated transfer price.

With Table 2, we put forward Propositions 11–15.

**Proposition 11.** For any given set of parameters, there exist $q^D_{DA} \in (0, a_0/2a_3) < q^D_{DA} \in (a_0/2a_3, a_0/a_3); \text{ e.g., }$ negotiated transfer-pricing to a higher quality improvement effort and a higher quality of the product.

**Proposition 12.** For any given set of parameters, there are $q^D_{DA} > q^D_{DA}$ and $h^D_{DA} > h^D_{DA}, \text{ e.g., }$ negotiated transfer-pricing to a lower advertising effort and a lower goodwill of the product.

**Proposition 13.** For any given set of parameters, there are $\pi^D_{DA} > \pi^D_{DA}$, $\pi^D_{DA} > \pi^D_{DA}$, and $\pi^D_{DA} > \pi^D_{DA}$, e.g., under negotiated transfer-pricing both the departments’ and the firm’s profits are larger than those under administered transfer-pricing.

**Proposition 14.** The quality of the product, goodwill, and the profits of both the departments and the firm rise with the rising of learning rates of product innovation and advertising effort knowledge accumulation resulting from learning by doing.

**Proposition 15.** For any given set of parameters, there is $\pi^D_{DA} < \pi^D_{DA}$; e.g., under centralized scenario the firm’s profit is larger than that under decentralized scenario.
Table 2: Variation in the quality levels, goodwill, and profits.

<table>
<thead>
<tr>
<th>parameter</th>
<th>(q^C)</th>
<th>(q^F)</th>
<th>(q^A)</th>
<th>(q^D)</th>
<th>(q^N)</th>
<th>(q^A)</th>
<th>(q^D)</th>
<th>(q^C)</th>
<th>(q^F)</th>
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<td>40.56</td>
<td>29.02</td>
<td>37.73</td>
<td>24.34</td>
<td>17.12</td>
<td>38.66</td>
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<tr>
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<td>15.36</td>
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<td>3.75</td>
<td>6.95</td>
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<td>24.23</td>
<td>13.01</td>
</tr>
<tr>
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<td>10.01</td>
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<td>8.09</td>
<td>11.96</td>
<td>40.55</td>
<td>29.09</td>
</tr>
<tr>
<td>(b_1)</td>
<td>0.5</td>
<td>9.77</td>
<td>0.15</td>
<td>10.89</td>
<td>7.03</td>
<td>8.79</td>
<td>8.15</td>
<td>11.02</td>
<td>10.97</td>
</tr>
<tr>
<td>(b_2)</td>
<td>0.25</td>
<td>17.36</td>
<td>0.15</td>
<td>10.89</td>
<td>7.03</td>
<td>8.79</td>
<td>8.15</td>
<td>11.02</td>
<td>10.97</td>
</tr>
<tr>
<td>(r)</td>
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<td>11.98</td>
<td>0.15</td>
<td>10.89</td>
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<td>8.79</td>
<td>8.15</td>
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<td>10.97</td>
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<td>(\omega)</td>
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7. Conclusions

In this paper, our main issue is to investigate the strategic transfer-pricing in a marketing-operations interface with knowledge accumulation resulting from learning by doing for a decentralized decision-making monopolist. In order to investigate this problem, a differential game model has been developed, in which a monopolist is composed of two profit centers, e.g., an operations department taking charge of the product innovation and a marketing department controlling the advertising effort as well as retail price. Both the product innovation level (design quality) and advertising effort impact on product goodwill positively, and the transfer price plays an important role in directing decision-making as well as coordinating profits for each center and the firm. As a benchmark the centralized scenario is investigated first, and then we focus on the analysis of decentralized scenario in which the negotiated and administered transfer-pricing
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Appendix

A. Proof of Proposition 1

From (28), we have \( \frac{\partial k(t)}{\partial b_1} = -\mu(r + \sigma)/2\alpha(r + \gamma) < 0 \) and \( \frac{\partial k(t)}{\partial h(t)} = -\beta \phi / \alpha < 0 \). From (29), one is easy to get 
\[
\frac{\partial h(t)}{\partial b_2} = -\xi(r + \omega)/2\beta(r + \theta) < 0 \quad \text{and} \quad \frac{\partial h(t)}{\partial k(t)} = 0.
\]
This finishes the proof.

B. Proof of Proposition 2

Let us replace \( k, h, q, \) and \( g \) with \( \chi_1, \chi_2, \chi_3, \) and \( \chi_4 \). So differential equation system (30) can be written as
\[
d\chi_i/dt = f(\chi_i, t), \quad \text{where} \quad i = 1, 2, 3, 4.
\]
We use the linearization method to prove the stability of differential equations \( \{\dot{\chi}_1, \dot{\chi}_2, \dot{\chi}_3, \dot{\chi}_4\} \).

Make \( \tilde{\chi}_i + \zeta_i = \{\tilde{\chi}_1 + \zeta_1, \tilde{\chi}_2 + \zeta_2, \tilde{\chi}_3 + \zeta_3, \tilde{\chi}_4 + \zeta_4\} \) as the disturbed solution of differential equation system (30), where \( \zeta_i \) is a small perturbation term.

Because \( (d/dt)(\chi_i + \zeta_i) = f(\chi_i + \zeta_i, t) \), making the differential equations a linear power expansion, one obtains
\[
f_j(\chi_i + \zeta_i, t) = f_j(\chi_i, t) + \sum_{i=1}^4(\partial f_j/\partial \chi_i)\zeta_i + o(\chi_i^2) \quad \text{where} \quad j = 1, 2, 3, 4.
\]

Therefore, we have \( dc_j/dt = Jc \), where
\[
J = \frac{\partial f_j}{\partial \chi_i} = \begin{bmatrix}
r + \sigma & \beta \phi & 0 & 0 \\
r + \omega & 0 & 0 & 0 \\
1 & 0 & -\sigma & 0 \\
0 & 1 & \theta & -\omega
\end{bmatrix}.
\]

Make \( J = (a_{ij})_{4 \times 4} \). System (B.1) is the linear perturbation equation of the nonlinear differential equations with variable coefficients of the original equations. From (B.1), the eigenvalues of the real matrix \( J \) are \( \lambda_j = r + \sigma, \lambda_{j1} = r + \omega, \lambda_{j2} = -\sigma, \) and \( \lambda_{j3} = -\omega, \) the trace of this matrix is \( trJ = 2r > 0 \). Hence the eigenvalues of the matrix should satisfy the following conditions so as to guarantee the stability of differential equation system:
\[
|\lambda_j| \leq \frac{1}{n} \left[ trJ + \sqrt{(n-1)(trJ^T - (trJ)^2)} \right].
\]

By some calculation we have the following.
\[
trJ^T = (r + \sigma)^2 + (r + \omega)^2 + \sigma^2 + \omega^2 + \left(\frac{\beta \phi}{\alpha}\right)^2 + 2 \quad (B.4)
\]
\[
trJ = 2r \quad (B.5)
\]

Substituting expressions (B.4) and (B.5) into the first equation of (B.3), one gets the following.
\[
3 \left[ 4trJ^T - (trJ)^2 \right] - (4r + 4\sigma - trJ)^2 > 0 \quad (B.6)
\]

In the same way, we can prove that the second equation of (B.3) is also true. Now, it can be concluded that \( \{\tilde{\chi}, \tilde{h}, \tilde{q}, \tilde{g}\} \) are the stable solution of the differential equation system (30).

This finishes the proof.
Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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