A New Model for Deriving the Priority Weights from Hesitant Triangular Fuzzy Preference Relations

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Fuzzy preference relation is a common tool to express the uncertain preference information of decision maker in the process of decision making. However, the traditional fuzzy preference relation will fail under hesitant fuzzy environment as the membership has a single value. In addition, it is very difficult to obtain the precise membership values. Therefore, a new model of fuzzy preference relation is proposed in this paper. Firstly, the concept of hesitant triangular fuzzy preference relation is defined and its properties are investigated based on the concepts of hesitant fuzzy set, hesitant triangular fuzzy set, fuzzy preference relation, and hesitant fuzzy preference relation. Then, the steps of applying this novel model are offered for the case of determining the weights of failure modes. Finally, an example is used to illustrate the proposed model.

1. Introduction

Having received extensive attention in the last few decades, preference relation is a widely used and effective tool to express the preference of decision makers over the alternatives in decision making [1, 2]. There are two kinds of preference relations, i.e., fuzzy preference relation [2] and multiplicative preference relation [3]. Saaty [4] originally proposed the multiplicative preference relation and used the 1/9-9 scale to measure the intensity of the pairwise comparison between two different alternatives. The fuzzy preference relation was proposed by Orlovsky [2], and the 0-1 scale was employed to describe the assessment information of decision maker by comparing the alternatives. Since the emergence of preference relation, extensive research has been conducted and some meaningful conclusions have been drawn, including the consensus models and methodologies [5, 6], the consistency methods [7, 8], the priority methodologies [9-11], and the incomplete values-determined methods [12, 13]. Previous researches have shown that fuzzy preference relation and multiplicative preference relation have been extended to more general forms of fuzzy set, such as interval-valued environment [11, 14] and linguistic environment [15]. To depict the intensities of both preferences and non-preferences, Xu [16] introduced the intuitionistic fuzzy values based on intuitionistic fuzzy set (IFS) [17], where intuitionistic fuzzy values are composed of a membership degree, a nonmembership degree, and a hesitancy degree [18]. The exact values of the three membership degrees in IFS are however difficult to be determined in some practical applications. Atanassov and Gargov [19] introduced the interval-valued intuitionistic fuzzy sets (IVIFSs) which permit decision makers to use interval values rather than point values to express their information. Despite the superiority of IVIFSs in describing fuzziness and uncertainty compared with other preference relations, IVIFSs are not applicable to the situation of several possible preference values commonly seen in many practical problems. For this limitation, Torra and Narukawa [20, 21] introduced the hesitant fuzzy sets (HFSs), a novel and recent extension of fuzzy sets. Because the membership function of HFSs involves a set of possible values, HFSs are considered as a more powerful tool to manage the imprecise and vague information of decision maker in the process of decision making.

Since the proposing of HFS, it has attracted much attention, and a lot of relevant studies have been conducted in the last few years. Torra [21] discussed the relationship between HFS and other kinds of fuzzy sets and showed that
the envelope of HFS was actually an IFS. Xia and Xu [22] gave the mathematical representation of HFS and defined some useful operations and aggregation operators for hesitant fuzzy information. Xu and Xia [23] proposed a variety of measures for hesitant fuzzy sets, e.g., distance, similarity, and correlation. Xu and Xia [24] also introduced the concepts of entropy and cross-entropy for hesitant fuzzy information and discussed the relationships among them. In order to cluster the massive evaluation information provided by different experts, Chen and Xu [25] proposed a series of correlation coefficient formulae for HFSs and applied them to calculating the degrees of correlation among HFSs. Rodriguez, R.M., et al. [26] made an overview on hesitant fuzzy sets including concept, extension, aggregation operators, measures, and applications like decision making, evaluation, and clustering. To deal with the case of quantitative settings, Zhao and Lin [27] introduced the concept of hesitant fuzzy linguistic term set (HFLTS). The theory of HFLTSs is very useful in objectively dealing with situations in which experts are hesitant in providing linguistic assessments. Recently, HFLTS has garnered considerable attention from researchers. Wei et al. [28] proposed a hesitant fuzzy LWA operator, a hesitant fuzzy LOWA operator, and comparison methods for the HFLTS. Liu and Rodriguez [29] presented a new representation of the HFLTS by means of a fuzzy envelope to carry out the computing with words processes. Liao et al. [28] investigated the correlation measures and correlation coefficients of HFLTSs and proposed several different types of correlation coefficients for HFLTSs such as intuitionistic fuzzy sets and hesitant fuzzy sets. To incorporate distribution information of hesitant fuzzy sets, Wu and Xu [30] defined the concept of possibility distribution for an HFLTS and proposed the corresponding operators and consensus measure. For further applications of HFLTSs to decision making, Zhu and Xu [31] developed a concept of hesitant fuzzy linguistic preference relations (HFLPRs) as a tool to collect and present the decision makers' preferences.

Inspired by the superiority of FHS in expressing human's hesitancy and based on the concept of fuzzy preference relation, fuzzy preference relations [32] are the most common tools to express decision makers' preferences over alternatives in decision making. A lot of studies have been done about them, such as the consistency methods [8], the group consensus methods [33], the priority methods [34, 35], and the incomplete values-determined methods [12]. Although different kinds of preference relations have already been successfully applied for decision making under hesitant fuzzy information, there have been few papers which have considered hesitation in a linguistic environment. Zhu and Xu [31] defined the concept of hesitant fuzzy linguistic preference relation (HFLPR). Zhang and Wu [36] defined the multiplicative consistency of an HFLPR. Wang and Xu [37] presented some consistency measures for an extended HFLPR (EHFLPR). In group decision making (GDM), preference relations are popular and powerful techniques for decision maker preference modeling [38]. The use of hesitant information in pairwise comparisons enriches the flexibility of qualitative decision making, and hesitant fuzzy linguistic preference relations (HFLPR) are one of the most commonly used. Aiming to deal with the linguistic preferences given by the decision makers, various linguistic models have been presented, such as the 2-tuple linguistic model, the symbolic model, the fuzzy number based model, the type-2 fuzzy sets based model, and the granular method [39–43]. Taking into account that the supplied preferences should satisfy some transitive properties and a consensus reaching process, Wu and Xu [44] developed separate consistency and consensus processes to deal with HFLPR individual rationality and group rationality. Furthermore, in the context of probability hesitant fuzzy preference relations (PHFPR), Wu and Xu [45] proposed an optimization based consistency improvement process to deal with the inconsistencies in a given PHFPR. In the proposed approaches, consensus measures based on the distances between the individuals were computed on three levels: an alternative pair level, an alternatives level, and a preference relations level. Hesitant fuzzy linguistic term sets (HFLTSs) can represent a much broader range of linguistic data. Wu and Xu [46] developed compromise solutions for multiple-attribute group decision making (MAGDM) using HFLTS and proposed two models to derive a compromise solution for the MAGDM problems. One is based on the VIKOR method and the other is based on the TOPSIS method.

It is obviously that hesitant fuzzy preference relation can aggregate more useful information to describe the uncertain and valueless situations in a more deep and objective way than fuzzy preference relation. But the set of the possible values of hesitant fuzzy elements are exact and crisp. However, under many conditions, in the process of trying to determine the relative importance via pairwise comparison, it is inadequate or insufficient to use the membership with a set of possible exact and crisp values due to the complexities and uncertainties of real problems. In addition, the estimation is inaccurate due to the inherent vague thought of human. Besides, the hesitant fuzzy elements of the hesitant fuzzy set are still crisp numbers, which are not easy to obtain because of human's hesitancy in actual life. Fortunately, triangular fuzzy is a method of transforming vague and uncertain linguistic variables into definite values. Furthermore triangular fuzzy number can solve the contradiction that the performance of the evaluated object cannot be measured accurately but can only be evaluated by natural language. So, the membership of hesitant fuzzy preference relation with a set of possible triangular fuzzy values is more objective than that with a set of exact and crisp values. In this paper, hesitant triangular fuzzy preference relation is proposed to overcome the limitation of HFPR. The rest of this paper is organized as follows. In Section 2, an important algorithm of fuzzy AHP is introduced, along with some basic knowledge on hesitant fuzzy set, hesitant triangular fuzzy set, fuzzy preference relation, and hesitant fuzzy preference relation. In Section 3, the model of hesitant triangular preference relation is proposed and the steps for applying the model are offered. In Section 4, an example is used to illustrate the proposed model. And finally, conclusions are given.

2. Preliminaries

In this section, an important algorithm of fuzzy AHP is introduced, along with some basic knowledge on hesitant
fuzzy set (HFS), hesitant triangular fuzzy set (HTFS), fuzzy preference relation (FPR), and hesitant fuzzy preference relation (HFPR).

2.1. Fuzzy AHP. In the following, Chang’s extent analysis method [47] is explained. Let $X = (x_1, x_2, \ldots, x_m)$ be an object set and $U = (g_1, g_2, \ldots, g_n)$ be a goal set. According to Chang’s method, the object is considered one by one, and extent analysis is carried out for each goal. Therefore, there are $m$ extent analysis values for each object, shown as below:

$$\tilde{M}_i^1, \tilde{M}_i^2, \ldots, \tilde{M}_i^n, \quad i = 1, 2, \ldots, n,$$  \hspace{1cm} (1)

where $\tilde{M}_i^j$ ($i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$) are triangular fuzzy numbers.

Next, the steps of Chang’s extent analysis are demonstrated.

Step 1. The value of fuzzy synthetic extent with respect to the $i$th object is defined as

$$S_i = \sum_{j=1}^{m} \tilde{M}_i^j \otimes \left[ \sum_{j=1}^{m} \tilde{M}_i^j \right]^{-1}.$$  \hspace{1cm} (2)

For obtaining $\sum_{j=1}^{m} \tilde{M}_i^j$, the fuzzy addition operation of $m$ extent analysis values is performed on a particular matrix, i.e.,

$$\sum_{j=1}^{m} \tilde{M}_i^j = \left( \sum_{j=1}^{m} l_{ij}, \sum_{j=1}^{m} m_{ij}, \sum_{j=1}^{m} u_{ij} \right),$$  \hspace{1cm} (3)

and to obtain $\sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{M}_i^j$, the fuzzy addition operation of $\tilde{M}_i^j$ ($j = 1, 2, \ldots, m$) values is performed, i.e.,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{M}_i^j = \left( \sum_{i=1}^{n} l_{ij}, \sum_{i=1}^{n} m_{ij}, \sum_{i=1}^{n} u_{ij} \right)$$  \hspace{1cm} (4)

then, the inverse of above vector is calculated, i.e.,

$$\left[ \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{M}_i^j \right]^{-1} = \left( \frac{1}{\sum_{i=1}^{n} l_{ij}}, \frac{1}{\sum_{i=1}^{n} m_{ij}}, \frac{1}{\sum_{i=1}^{n} u_{ij}} \right).$$  \hspace{1cm} (5)

Step 2. Assuming that $\tilde{M}_1$ and $\tilde{M}_2$ are two triangular fuzzy numbers, the degree of possibility of $\tilde{M}_2 \geq \tilde{M}_1 = (l_1, m_1, u_1)$ is defined as

$$V(\tilde{M}_2 \geq \tilde{M}_1) = \sup \left[ \min \left( \tilde{M}_1(x), \tilde{M}_2(x) \right) \right].$$  \hspace{1cm} (6)

This expression can be equivalently expressed as follows:

$$V(\tilde{M}_2 \geq \tilde{M}_1) = \mu_{\tilde{h}_l}(\tilde{M}_2 \cap \tilde{M}_1) = \tilde{M}_2(d)$$

$$= \begin{cases} 1, & \text{if } m_2 \geq m_1, \\ 0, & \text{if } l_1 \geq u_2, \\ \frac{l_1 - u_2}{m_2 - u_2} - \frac{m_1 - l_1}{m_2 - u_2}, & \text{otherwise}. \end{cases}$$  \hspace{1cm} (7)

Here $d$ is the abscissa value of the highest intersection point $D$ between $\tilde{M}_1$ and $\tilde{M}_2$, as shown in Figure 1. In order to compare $\tilde{M}_1$ and $\tilde{M}_2$, both values of $V(\tilde{M}_1 \geq \tilde{M}_2)$ and $V(\tilde{M}_2 \geq \tilde{M}_1)$ are required.

Step 3. The degree of possibility for a convex fuzzy number to be greater than $k$ convex fuzzy numbers $\tilde{M}_i$ ($i = 1, 2, \ldots, k$) can be defined by

$$V(\tilde{M} \geq \tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_k) = \min_{i=1}^{k} \left( \tilde{M}_i \right),$$  \hspace{1cm} (8)

Step 4. $W = (\min V(S_1 \geq S_k), \min V(S_2 \geq S_k), \ldots, \min V(S_n \geq S_k))^T$ is the weight vector for $k = 1, 2, \ldots, n$.

2.2. Concepts of HFS, HTFS, and FPR

Definition 1 (see [20, 21]). Let $X$ be a fixed set; a hesitant fuzzy set on $X$ is in terms of a function $h$ that when applied to $X$ returns a subset of $[0, 1]$, which can be represented as the following mathematical symbol by Xia and Xu [22]:

$$E = \left( \langle x, h_E(x) \rangle \mid x \in X \right),$$  \hspace{1cm} (9)

where $h_E(x)$ is a set of values in $[0, 1]$, denoting the possible membership degree of the element $x \in X$ to the set $E$. For convenience, Xia and Xu [22] took $h = h_E(x)$ as a hesitant fuzzy element (HFE) and $H$ as the set of all HFE.

Definition 2 (see [22]). For a HFE $h$, $s(h) = (1/#h) \sum_{y \in h} y$ is called the score function of $h$, where $#h$ represents the number of elements in $h$. For two HFEs $h_1$ and $h_2$, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Definition 3 (see [27]). Let $X$ be a fixed set; a hesitant triangular fuzzy set (HTFS) on $X$ is in terms of a function that when applied to each $x \in X$ returns a subset of values in $[0, 1]$, which can be expressed by

$$E = \left( \langle x, \tilde{h}_{E(x)} \rangle \mid x \in X \right),$$  \hspace{1cm} (10)

where $\tilde{h}_{E(x)}$ is a set of possible triangular fuzzy values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set $E$. For convenience, Zhao and Lin [27] called
\( \tilde{h}_{R(X)} = \tilde{h} = (\gamma^L, \gamma^M, \gamma^R) \) as a hesitant triangular fuzzy element (HTFE).

Based on Definition 3 and the operational principle of HFS, Zhao and Lin [27] defined some new operations on HTFE \( \tilde{h} = (\gamma^L, \gamma^M, \gamma^R) \), \( \tilde{h}_1 = (\gamma^L_1, \gamma^M_1, \gamma^R_1) \), and \( \tilde{h}_2 = (\gamma^L_2, \gamma^M_2, \gamma^R_2) \):

1. \( \tilde{h}_1 + \tilde{h}_2 = \sum_{x \in X} \{(\gamma^L_1(x), \gamma^M_1(x), \gamma^R_1(x))\}
2. \( \lambda \tilde{h} = \sum_{x \in X} \{(1 - \gamma^L_1), 1 - (1 - \gamma^M_1), 1 - (1 - \gamma^R_1)\}
3. \( \tilde{h}_1 \oplus \tilde{h}_2 = \sum_{x \in X} \{(\gamma^L_1(x) + \gamma^L_2(x) - \gamma^L_1(x), \gamma^M_1(x) + \gamma^M_2(x) - \gamma^L_1(x), \gamma^R_1(x) + \gamma^R_2(x) - \gamma^L_1(x)\}
4. \( \tilde{h}_1 \odot \tilde{h}_2 = \sum_{x \in X} \{(\gamma^L_1(x), \gamma^M_1(x) \gamma^M_2(x), \gamma^R_1(x) \gamma^R_2(x))\}

Based on the above definition, Zhao and Lin [27] also defined the score function of \( \tilde{h} \):

\[
s(\tilde{h}) = \frac{1}{\#(\tilde{h})} \sum_{x \in \tilde{h}} \left( \frac{1}{\#(\tilde{h})} \sum_{x \in \tilde{h}} \gamma^L(x), \frac{1}{\#(\tilde{h})} \sum_{x \in \tilde{h}} \gamma^M(x), \frac{1}{\#(\tilde{h})} \sum_{x \in \tilde{h}} \gamma^R(x) \right),
\]

where \( \#(\tilde{h}) \) is the number of triangular fuzzy values in \( \tilde{h} \), and \( s(\tilde{h}) \) is a triangular fuzzy value in \([0, 1]\). For two HTFEs \( \tilde{h}_1 \) and \( \tilde{h}_2 \), if \( s(\tilde{h}_1) \geq s(\tilde{h}_2) \), then \( \tilde{h}_1 \geq \tilde{h}_2 \).

**Definition 4 (see [27])**. Let \( \tilde{h}_j \) \((j = 1, 2, \ldots, n)\) be a collection of HTFEs. The hesitant triangular fuzzy weighted averaging (HTFWA) operator is a mapping \( \tilde{H}^n \rightarrow \tilde{H} \).

\[
HTFWA(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \bigoplus_{j=1}^{n} (\omega_j \tilde{h}_j)
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) represents the weight vector of \( \tilde{h}_j \) \((j = 1, 2, \ldots, n)\), and \( \omega_j > 0 \). \( \sum_{j=1}^{n} \omega_j = 1 \). In particular, if \( \omega = (1/n, 1/n, \ldots, 1/n)^T \), then the HTFWA operator degrades to the hesitant triangular fuzzy averaging (HTFA) operator.

\[
HTFA(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \bigoplus_{j=1}^{n} \left( \frac{1}{n} \tilde{h}_j \right)
\]

**Definition 5 (see [27])**. Let \( \tilde{h}_j \) \((j = 1, 2, \ldots, n)\) be a collection of HTFEs. The hesitant triangular fuzzy weighted geometric (HTFWG) operator is a mapping \( \tilde{H}^n \rightarrow \tilde{H} \).

\[
HTFWG(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \bigoplus_{j=1}^{n} \left( \prod_{j=1}^{n} (\frac{\tilde{h}_j}{\prod_{j=1}^{n} \tilde{h}_j})^{-1/n} \right)
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) represents the weight vector of \( \tilde{h}_j \) \((j = 1, 2, \ldots, n)\), and \( \omega_j > 0 \). \( \sum_{j=1}^{n} \omega_j = 1 \). In particular, if \( \omega = (1/n, 1/n, \ldots, 1/n)^T \), then the HTFWG operator degrades to the hesitant triangular fuzzy geometric (HTFG) operator.

**Definition 6 (see [48])**. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set of alternatives; then \( R = (r_{ij})_{n \times n} \) is called an additive FPRR on the product set \( X \times X \) with the membership function \( u_{R}: X \times X \rightarrow [0, 1] \), \( u_{R}(x_i, x_j) = r_{ij} \), satisfying \( r_{ij} + r_{ji} = 1 \), \( i, j = 1, 2, \ldots, n \).

Usually, a matrix \( R = (r_{ij})_{n \times n} \) denotes the preference relation between two alternatives, and \( r_{ij} \) denotes the preference degree of \( x_i \) over \( x_j \). In particular, \( r_{ij} = 0 \) implies that \( x_j \) is totally preferred to \( x_i \), and \( r_{ij} = 0.5 \) indicates that there is no difference between \( x_i \) and \( x_j \).

2.3. Concepts of Hesitant Fuzzy Preference Relations. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a fixed set of alternatives; then \( A = (a_{ij})_{n \times n} \) is called the fuzzy preference relation matrix [2] on \( X \times X \) with \( a_{ij} \geq 0, a_{ij} + a_{ji} = 1 \), \( i, j = 1, 2, \ldots, n \), where \( a_{ij} \) denotes the degree that the alternative \( x_i \) is prior to \( x_j \). In particular, \( a_{ij} = 0.5 \) means the same importance of \( x_i \) and \( x_j \), which can be expressed by \( x_i = x_j \); \( 0 \leq a_{ij} < 0.5 \) means that \( x_j \) is more important than \( x_i \), expressed by \( x_i < x_j \), and the larger the value of \( a_{ij} \), the more important \( x_j \) to \( x_i \); on the contrary, \( 0.5 < a_{ij} \leq 1 \) means that \( x_j \) is more important than \( x_i \), denoted by \( x_j > x_i \), and the larger the value of \( a_{ij} \), the more important \( x_i \) to \( x_j \). Obviously, the value of \( a_{ij} \) indicates that fuzzy preference relation is a certain value between 0 and 1. If a decision group containing several
Definition 7 (see [49, 50]). Let \( X = \{x_1, x_2, \cdots, x_n\} \) be a fixed set; then a hesitant fuzzy preference relation \( A \) on \( X \) is expressed by a matrix \( A = (a_{ij})_{n \times n} \subset X \times X \), where \( a_{ij} = \{a_{ij}^r, s = 1, 2, \cdots, I_{ij}\} \) is a HFE denoting all possible degrees to which \( x_i \) is preferred to \( x_j \). Additionally, \( a_{ij} \) should satisfy the following requirements:

\[
\begin{align*}
&\quad a_{ij}^\sigma(s) + a_{ji}^\sigma(s^{-1}) = 1, \quad i, j = 1, 2, \cdots, n, \quad (16) \\
&\quad a_i = \{0.5\}, \quad i = 1, 2, \cdots, n, \quad (17) \\
&\quad I_{ij} = I_{ji}, \quad i, j = 1, 2, \cdots, n, \quad (18)
\end{align*}
\]

where the values in \( a_i \) are assumed to be arranged in an increasing order, \( a_{ij}^\sigma(s) \) denotes the \( s \)th largest value in \( a_{ij} \), and \( I_{ij} \) denotes the number of values in \( a_{ij} \). Obviously, (19) is the reciprocal condition, which means that if \( a_{ij} \) is a HFE, then \( a_{ji} \) can be obtained easily by (19). (20) implies that there is no difference between \( x_i \) and \( x_j \). If there is a single value in \( a_{ij} \), \( i, j = 1, 2, \cdots, n \), then hesitant triangular fuzzy preference relation degrades to the usual triangular fuzzy ones.

3. Proposed Methodology

3.1. Hesitant Triangular Fuzzy Preference Relations (HTFPR) Model. In many practical situations, due to the incompleteness and uncertainties of information as well as the hesitancy of humans, it is not easy for decision makers to express their preferences between two alternatives with exact and crisp values in traditional HFS. Hence, triangular fuzzy set can model real-life decision problems in a more suitable and sufficient way than real numbers.

Definition 8. Let \( X = \{x_1, x_2, \cdots, x_n\} \) be a fixed set; then an additive hesitant triangular fuzzy preference relation \( \tilde{H} \) on \( X \) is expressed by a matrix \( \tilde{H} = (\tilde{h}_{ij})_{n \times n} \subset X \times X \), where \( \tilde{h}_{ij} = \{(y_{ij}^L, y_{ij}^M, y_{ij}^R), s = 1, 2, \cdots, I_{ij}\} \) is a hesitant triangular fuzzy element (HTFE) denoting all possible degrees to which \( x_i \) is preferred to \( x_j \). Moreover, \( \tilde{h}_{ij} \) satisfies the following requirements.

\[
\begin{align*}
&\quad (y_j^L, y_j^M, y_j^R)^\sigma(s) + (y_i^L, y_i^M, y_i^R)^\sigma(s^{-1}) = 1, \quad i, j = 1, 2, \cdots, n; \\
&\quad (y_j^L, y_j^M, y_j^R)^\sigma(s) + (y_i^L, y_i^M, y_i^R)^\sigma(s^{-1}) = 1, \quad i, j = 1, 2, \cdots, n;
\end{align*}
\]

Example 9. Let \( X = \{x_1, x_2, x_3\} \); a decision group containing several experts is authorized to provide the preference values of \( x_1 \) over \( x_2 \). There are several possible results among the experts. Some experts provide \( (0.2, 0.4, 0.5) \), while others provide \( (0.3, 0.4, 0.5) \). Then according to Definition 6, the degree of \( x_2 \) is preferred to \( x_1 \), which should be \( (1-0.5 = 0.5, 1-0.4 = 0.6, 1-0.2 = 0.8) \) and \( (1-0.4 = 0.6, 1-0.3 = 0.7, 1-0.2 = 0.8) \), respectively. In this case, \( \tilde{h}_{12} = \{(0.2, 0.3, 0.4), (0.3, 0.4, 0.5)\} \) is called a HTFE, denoting the preference information about \( x_1 \) preference to \( x_2 \). The preference information about \( x_2 \) over \( x_1 \) is denoted by a HTFE \( \tilde{h}_{21} = \{(0.5, 0.6, 0.7), (0.6, 0.7, 0.8)\} \). Similarly, let \( \tilde{h}_{13} = \{(0.5, 0.6, 0.7)\} \) and \( \tilde{h}_{31} = \{(0.2, 0.4, 0.5), (0.5, 0.6, 0.7), (0.5, 0.7, 0.8)\} \); then, we can correspondingly obtain \( \tilde{h}_{32} = \{(0.3, 0.4, 0.5)\} \) and \( \tilde{h}_{32} = \{(0.2, 0.3, 0.5), (0.3, 0.4, 0.5)\} \). Based on above analysis, the hesitant triangular fuzzy preference relation model can be used to determine
### Table 2: The hesitant triangular fuzzy preference relation \( \tilde{H} \).

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>{0.5}</td>
<td>{(0.5,0.6,0.7),(0.6,0.7,0.8)}</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>{0.5}</td>
<td>{(0.5,0.6,0.7)}</td>
</tr>
<tr>
<td>( x_3 )</td>
<td></td>
<td>{0.5,0.6,0.7}</td>
</tr>
</tbody>
</table>

The weights of failure modes of a work system. The steps are shown below.

**Step 1.** A committee comprised of several experts is set up to identify the potential failure modes of a work system and to provide their preference information via pairwise comparison. The assessment preference values are denoted by hesitant triangular fuzzy elements \( h_{ij} \) \( (i, j = 1, 2, \cdots, n) \), and the HTFPR matrix \( \tilde{H} = (\tilde{h}_{ij})_{n \times n} \) is constructed.

**Step 2.** HTFWA (or HTFWG) operator is used to aggregate all \( h_{ij} \) \( (j = 1, 2, \cdots, n) \) which correspond to the alternative \( x_i \).

**Step 3.** The score values \( s(\tilde{h}_i) \) of \( x_i \) \( (i = 1, 2, \cdots, n) \) are calculated using (11).

**Step 4.** Chang’s fuzzy AHP method is used to obtain the weights of the failure modes. Based on the fuzzy synthetic extent proposed by Chang [31], \( \sum_{j=1}^{m} \tilde{M}_{x_i}^{j} \) can be treated as \( s(\tilde{h}_i) \), and \( \sum_{j=1}^{n} \sum_{j=1}^{m} \tilde{M}_{x_i}^{j} \) can be treated as \( \sum_{i=1}^{n} s(\tilde{h}_i) \). In this case, the value of HTF synthetic extent with respect to the \( i \)-th alternative can be expressed by

\[
S_i = s(\tilde{h}_i) \otimes \left[ \sum_{i=1}^{n} s(\tilde{h}_i) \right]^{-1},
\]

and

\[
V(s(\tilde{h}_2) \geq s(\tilde{h}_1)) = \sup \left[ \min \left( s(\tilde{h}_1(x)), s(\tilde{h}_2(x)) \right) \right],
\]

\[
V(s(\tilde{h}_2) \geq s(\tilde{h}_1)) = \text{hgt} \left( s(\tilde{h}_1) \cap s(\tilde{h}_2) \right) = s(\tilde{h}_2(d))
\]

\[
V(s(\tilde{h}_2) \geq s(\tilde{h}_1), s(\tilde{h}_2), \cdots, s(\tilde{h}_k)) = \min V(s(\tilde{h}) \geq s(\tilde{h}_1)) , \ i = 1,2,\cdots, k
\]

thus, \( W = (\min V(S'_i \geq S'_k), \min V(S'_2 \geq S'_k), \cdots, \min V(S'_i \geq S'_k))^T \) is the weight vector for \( k = 1,2,\cdots, n \) of failure modes.

4. Case Illustration

This section provides an example of a practical case involving the assessment of the potential failure modes of a kind of typical amusement rides-roller coaster, to illustrate the proposed model. Then a comparative analysis is conducted between the proposed model and the hesitant fuzzy preference relation [49] to show its superiority.
Amusement rides are a very popular recreational activity and take many forms in structure and movement. For example, a device or vehicle may be driven or guided by a power equipment or operator on a track or slide, and sometimes it walks through or bounces on an air bounce, funhouse, or maze [51]. Usually, amusement rides are classified into two categories, i.e., “fixed site ride” and “mobile ride” [51, 52] which are operated in fixed site and mobile forms, respectively [53]. Generally, the injuries and failure of amusement ride are considered to be infrequent by public, but are noteworthy when they occur. Carrying out quantitative and qualitative safety assessment of amusement ride draws much attention of the public and is important for continuous improvement [54–56]. As a kind of typical ride, roller coaster is very popular among young people and is located at outdoor midways generally, as well as big amusement parks or theme parks. It is classified as ‘fixed site ride’, and its safe and reliable operation is also worthy of researching. To find out the major potential failure modes and derive the priority of them, it is important to conduct the safety management of roller coasters.

4.1. The Proposed Method. In the following, the proposed method is used to obtain the metal-frame major potential failure modes and its priority of roller coaster.

Step 1. An expert team consisting of three cross-functional members was organized to identify the failure modes of a roller coaster and to prioritize them via pairwise comparison. According to the expert team, the major potential failure modes were identified as weld metal cracking, corrosion, joint angle loosening, deformation, and wear denoted by \( x_i \) (i = 1, 2, 3, 4, 5). Because of the team members’ limited expertise in the problem domain, lack of knowledge or data, and other reasons, it is difficult for experts to reach a consensus to evaluate these five potential failure modes and their relative importance weights precisely. So, the values of preference relations taking the form of HTFE by experts are more reasonable for real practice. Then, the expert team compare these five potential failure modes and provide the preference values denoted by \( \tilde{h} = (\tilde{h}_{ij})_{5 \times 5} \), as expressed in Table 3, where \( \tilde{h}_{ij} \) (i, j = 1, 2, 3, 4, 5) are in the form of HTFPRs.

Step 2. The preference relation information given in matrix \( \tilde{h} \) and the HTFWA (or HTFWG) operator are used to aggregate all \( \tilde{h}_{ij} \) (j = 1, 2, …, n) which correspond to the failure modes \( x_i \). Taking the failure modes weld metal cracking \( x_1 \) as an example, since \( \omega = (1/n, 1/n, \ldots, 1/n)^T \), we have

\[
\tilde{h}_1 = \text{HTFWA} = \text{HTFA} (\tilde{h}_{11}, \tilde{h}_{12}, \tilde{h}_{13}, \tilde{h}_{14}, \tilde{h}_{15})
\]

\[
= \bigoplus_{j=1}^{5} \left( \frac{1}{5} \tilde{h}_{1j} \right) = \bigcup_{j=1}^{5} \left\{ \left( \begin{array}{c}
1 \\
\frac{1}{5} \tilde{h}_{ij}
\end{array} \right) \right\}
\]

\[
- \prod_{j=1}^{5} \left( 1 - y_{1j} \right)^{1/5}, 1 - \frac{1}{5} \prod_{j=1}^{n} \left( 1 - y_{1j} \right)^{1/5}, 1
\]

Step 3. The score values \( s(\tilde{h}_i) \) (i = 1, 2, 3, 4, 5) of the overall hesitant triangular fuzzy preference relation values \( \tilde{h}_i \) (i = 1, 2, 3, 4, 5) are calculated:

\[
s(\tilde{h}_1) = \frac{1}{\#h_{1j} \neq h_i} \sum \gamma_i
\]

\[
= \left( \frac{1}{\#h_{1j} \neq h_i} \sum \gamma_i^{L}, \frac{1}{\#h_{1j} \neq h_i} \sum \gamma_i^{M}, \frac{1}{\#h_{1j} \neq h_i} \sum \gamma_i^{R} \right)
\]

\[
= (0.6518, 0.7461, 0.9149),
\]

Step 4. The weights of the failure modes are obtained.

Firstly, by applying (22), we have

\[
S_1' = s(\tilde{h}_1) \otimes \left[ \sum_{i=1}^{5} s(\tilde{h}_i) \right]^{-1}
\]

\[
= (0.6518, 0.7461, 0.9149)
\]

\[
\otimes \left( \frac{1}{3.2612}, \frac{1}{2.707}, \frac{1}{2.2963} \right)
\]

\[
= (0.1999, 0.2756, 0.3984).
\]

\[
S_2' = (0.3471, 0.4344, 0.5174)
\]

\[
\otimes \left( \frac{1}{3.2612}, \frac{1}{2.707}, \frac{1}{2.2963} \right)
\]

\[
= (0.1064, 0.1605, 0.2253).
\]

\[
S_3' = (0.4902, 0.5589, 0.6591)
\]

\[
\otimes \left( \frac{1}{3.2612}, \frac{1}{2.707}, \frac{1}{2.2963} \right)
\]

\[
= (0.1503, 0.2065, 0.287).
\]
Table 3: The hesitant triangular fuzzy preference relation matrix $\tilde{H}$ of failure modes.

<table>
<thead>
<tr>
<th></th>
<th>weld metal cracking</th>
<th>corrosion</th>
<th>joint angle loosening</th>
<th>deformation</th>
<th>wear</th>
</tr>
</thead>
<tbody>
<tr>
<td>weld metal cracking</td>
<td>[0.5]</td>
<td>(0.5, 0.7, 0.8), (0.6, 0.8, 0.9)</td>
<td>(0.7, 0.8, 0.9), (0.6, 0.9, 1.0)</td>
<td>(0.6, 0.7, 0.8), (0.7, 0.8, 0.9)</td>
<td>(0.6, 0.8, 0.9)</td>
</tr>
<tr>
<td>corrosion</td>
<td>(0.1, 0.2, 0.4), (0.2, 0.3, 0.5)</td>
<td>[0.5]</td>
<td>(0.3, 0.4, 0.5), (0.6, 0.7, 0.8)</td>
<td>(0.3, 0.4, 0.5)</td>
<td>(0.2, 0.4, 0.5), (0.3, 0.4, 0.5)</td>
</tr>
<tr>
<td>fatigue</td>
<td>(0.0, 0.1, 0.4), (0.1, 0.2, 0.3)</td>
<td>(0.2, 0.3, 0.4), (0.5, 0.6, 0.7)</td>
<td>[0.5]</td>
<td>(0.6, 0.7, 0.8), (0.7, 0.8, 0.9)</td>
<td>(0.6, 0.7, 0.8)</td>
</tr>
<tr>
<td>deformation</td>
<td>(0.1, 0.2, 0.3), (0.2, 0.3, 0.4)</td>
<td>(0.5, 0.6, 0.7)</td>
<td>(0.1, 0.2, 0.3), (0.2, 0.3, 0.4)</td>
<td>[0.5]</td>
<td>(0.2, 0.3, 0.5), (0.4, 0.5, 0.6)</td>
</tr>
<tr>
<td>wear</td>
<td>(0.1, 0.2, 0.4)</td>
<td>(0.5, 0.6, 0.7), (0.5, 0.6, 0.8)</td>
<td>(0.2, 0.3, 0.4)</td>
<td>(0.4, 0.5, 0.6), (0.5, 0.7, 0.8)</td>
<td>[0.5]</td>
</tr>
</tbody>
</table>
Then, using (25) and (26), we have

\[
\begin{align*}
V (S'_1 \geq S'_2) &= 1, \\
V (S'_1 \geq S'_3) &= 1, \\
V (S'_1 \geq S'_4) &= 1, \\
V (S'_1 \geq S'_5) &= 1, \\
V (S'_2 \geq S'_1) &= 0.26, \\
V (S'_2 \geq S'_3) &= 0.62, \\
V (S'_2 \geq S'_4) &= 1, \\
V (S'_2 \geq S'_5) &= 0.68, \\
V (S'_3 \geq S'_1) &= 0.19, \\
V (S'_3 \geq S'_2) &= 1, \\
V (S'_3 \geq S'_4) &= 1, \\
V (S'_3 \geq S'_5) &= 1, \\
V (S'_4 \geq S'_1) &= 0.21, \\
V (S'_4 \geq S'_2) &= 0.98, \\
V (S'_4 \geq S'_3) &= 0.61, \\
V (S'_4 \geq S'_5) &= 0.67, \\
V (S'_5 \geq S'_1) &= 0.37, \\
V (S'_5 \geq S'_2) &= 1, \\
V (S'_5 \geq S'_3) &= 0.95, \\
V (S'_5 \geq S'_4) &= 1.
\end{align*}
\]

Finally, using (27), the weight vector of these five potential failure modes can be obtained, i.e.,

\[
W_{FM} = (0.49, 0.13, 0.09, 0.11, 0.18)^T.
\]

Thus, the priority ranking of the five potential failure modes follows the order of

\[
FM_{x_1} > FM_{x_5} > FM_{x_2} > FM_{x_4} > FM_{x_3},
\]

as shown in Figure 2. The result indicates that the most important failure mode is the weld metal cracking, followed by wear, corrosion, deformation, and joint angle loosening, and the weld metal cracking should be given the top priority for correction.

4.2. The Hesitant Fuzzy Preference Relation Method. In the following, the hesitant fuzzy preference relation method [49] is used to obtain the metal-frame major potential failure modes and its priority for the same case as above. Based on Table 3, the same expert team gave the hesitant fuzzy preference relation matrix $H$ of failure modes as shown in Table 4. And using the hesitant fuzzy preference relation method, we can obtain the weight vector of these five potential failure modes as $W_{FM} = (0.42, 0.36, 0.16, 0.21, 0.38)^T$ (the detailed solution process can be referred to in [49]). Thus, the priority ranking of the five potential failure modes follows the order of

\[
FM_{x_1} > FM_{x_3} > FM_{x_2} > FM_{x_4} > FM_{x_5},
\]

as shown in Figure 2.
Table 4: The hesitant fuzzy preference relation matrix $H$ of failure modes.

<table>
<thead>
<tr>
<th></th>
<th>weld metal cracking</th>
<th>corrosion</th>
<th>joint angle loosening</th>
<th>deformation</th>
<th>wear</th>
</tr>
</thead>
<tbody>
<tr>
<td>weld metal cracking</td>
<td>[0.5]</td>
<td>[0.5, 0.6, 0.7, 0.8, 0.9]</td>
<td>[0.6, 0.7, 0.8, 0.9]</td>
<td>[0.6, 0.7, 0.8, 0.9]</td>
<td>[0.6, 0.8, 0.9]</td>
</tr>
<tr>
<td>corrosion</td>
<td>[0.1, 0.2, 0.3, 0.4, 0.5]</td>
<td>[0.5]</td>
<td>[0.4, 0.5, 0.6, 0.7, 0.8]</td>
<td>[0.3, 0.4, 0.5]</td>
<td>[0.2, 0.3, 0.4, 0.5]</td>
</tr>
<tr>
<td>joint angle loosening</td>
<td>[0.1, 0.2, 0.3, 0.4, 0.5]</td>
<td>[0.2, 0.3, 0.4, 0.5, 0.6]</td>
<td>[0.5]</td>
<td>[0.6, 0.7, 0.8, 0.9]</td>
<td>[0.6, 0.7, 0.8]</td>
</tr>
<tr>
<td>deformation</td>
<td>[0.1, 0.2, 0.3, 0.4]</td>
<td>[0.5, 0.6, 0.7]</td>
<td>[0.1, 0.2, 0.3, 0.4]</td>
<td>[0.5]</td>
<td>[0.2, 0.3, 0.4, 0.5, 0.6]</td>
</tr>
<tr>
<td>wear</td>
<td>[0.1, 0.2, 0.4]</td>
<td>[0.5, 0.6, 0.7, 0.8]</td>
<td>[0.2, 0.3, 0.4]</td>
<td>[0.4, 0.5, 0.6, 0.7, 0.8]</td>
<td>[0.5]</td>
</tr>
</tbody>
</table>
4.3. Comparative Analysis of the Two Methods. According to the inspection reports of failure modes of roller coasters coverage for the latest five-year period by the China Special Equipment Inspection Institute, the normalized proportions of these five failure modes are 0.46, 0.14, 0.11, 0.10, and 0.20, which means the priority ranking of them follows the order of $FM_{x_1} > FM_{x_2} > FM_{x_3} > FM_{x_4} > FM_{x_5}$, as shown in Figure 2. And Figure 2 shows that the results of the new method and Xia’s method are pretty similar to those of the inspection reports, except for the priority ranking of $FM_{x_3}$ and $FM_{x_4}$. The difference of weight values (ratio) of $FM_{x_3}$ and $FM_{x_4}$ is very slight, which has no practical significance and can be neglected from the point of view of engineering. Furthermore, it is obvious from Figure 2 that the gaps between the new method and the inspection reports are smaller than that between the Xia’s method and the inspection reports. Through the above comparative analysis, we can see that the new method is even more reasonable for reality.

5. Conclusion

In this paper, hesitant triangular fuzzy preference relation is proposed and is then applied to determine the priority of potential failure models of a work system. The known hesitant triangular fuzzy weighted averaging aggregation operators are adopted to aggregate the individual preferences and the score function is applied to calculate the score values of each alternative, based on which, the weights of each alternative are obtained using Chang’s fuzzy AHP method. Therefore, the priority of potential failure models in both qualitative and quantitative settings is obtained. Finally, an illustrative example is provided to demonstrate the proposed method. In the future, we shall focus on the application of the proposed model in decision making to other areas.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References


