Research Article

New Constructions of Asymptotically Optimal Codebooks via Cyclotomic Classes of Order 8

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Asymptotically optimal codebooks are a family of codebooks that can approach an optimal codebook meeting the Welch bound when the lengths of codewords are large enough. They can be constructed easily and are a good alternative for optimal codebooks in many applications. In this paper, we construct a new class of asymptotically optimal codebooks by using the product of some special finite fields and almost difference sets, which are composed of cyclotomic classes of order eight.

1. Introduction

An (N, K) codebook $\mathcal{C}$ is a set $\{c_0, c_1, \dots, c_{N-1}\}$, where each codeword c_l , $0 \leq l \leq N-1$, is a unit norm complex vector in $\mathbb{C}^K$. The value of maximum cross-correlation amplitude of an (N, K) codebook $\mathcal{C}$ is defined by

$$I_{\max}(\mathcal{C}) = \max_{0 \leq i < j \leq N-1} |c_i c_j^H|, \quad (1)$$

where c^H denotes the conjugate transpose of a complex vector c . Codebooks with minimal possible $I_{\max}(\mathcal{C})$ are desirable in many applications. From a geometric point of view, the minimal possible $I_{\max}(\mathcal{C})$ means that the smallest angle between two different lines where the codewords are located is as large as possible. Welch [1] presented the following well-known lower bound for $I_{\max}(\mathcal{C})$.

Lemma 1 (see [1]). For any (N, K) codebook $\mathcal{C}$ with $N \geq K$,

$$I_{\max}(\mathcal{C}) \geq I_{\text{Welch}} = \sqrt{\frac{N-K}{(N-1)K}}. \quad (2)$$

Moreover, the equality holds if and only if, for all pairs of (i, j) with $i \neq j$, it holds that

$$|c_i c_j^H| = \sqrt{\frac{N-K}{(N-1)K}}. \quad (3)$$

A codebook $\mathcal{C}$ meeting the Welch bound is called a maximum-Welch-bound-equality (MWBE) codebook. The interests in MWBE codebooks have significantly increased because of their special properties. However, it is usually difficult to construct a family of MWBE codebooks [2, 3].

Most existing methods for constructing infinite families MWBE codebooks fall into three categories: strongly regular graph methods [4], difference sets methods [5], and Steiner systems methods [6].

It is known that the Welch bound is not tight when $N > K^2$. In fact, $N < K^2$ is only the necessary condition of the existence of an (N, K) MWBE codebook [2]. Even when $N < K^2$ holds, the corresponding (N, K) MWBE codebook may not exist [7]. In recent years, asymptotically optimal codebooks have attracted widespread attention, since they can be constructed easily and $I_{\max}(\mathcal{C})$ can approach the Welch bound for large enough N . Thus, they are good alternatives in many applications.

Many methods can be used to construct asymptotically optimal codebooks, such as almost difference sets [8–10], binary sequences [11], and character sums in finite fields [12, 13]. Hu and Wu [14] constructed some asymptotically optimal codebooks by using difference sets and the product of Abelian groups. In this paper, we generalize this method to the case of cyclotomic classes of order 8 in finite fields and propose new classes of codebooks asymptotically meeting the Welch bound.

This paper is organized as follows. In Section 2, we recall some basic theory and definitions that will be used in our discussion. In Section 3, we present our main results. In Section 4, we give conclusions and future work.

2. Preliminaries

Let p be a prime number; $q = p^n$, $n \geq 1$. For a finite field $\mathbb{F}_q$, let θ be its one fixed primitive element, and let $\mathbb{F}_q^*$ be its all nonzero elements.

Let $q = ef + 1$, $e \geq 2$, $f \geq 2$. The set $C_j^{(e,q)} := \{\theta^{ek+j}, k \in \mathbb{Z}\} = \{\theta^{ek+j}, k = 0, 1, \dots, f-1\}$, $j = 0, 1, \dots, e-1$ is called a cyclotomic class of order e . Clearly, $\mathbb{F}_q^* = \bigcup_{j=0}^{e-1} C_j^{(e,q)}$, and $|C_j^{(e,q)}| = f$, $j = 0, 1, \dots, e-1$.

The cyclotomic numbers of order e with respect to $\mathbb{F}_q$ are given by

$$(l, m)_e = \left| \left(C_l^{(e,q)} + 1 \right) \cap C_m^{(e,q)} \right|, \quad (4)$$

where $S + a := \{s + a : s \in S\}$. Cyclotomic number $(l, m)_e$ is the number of solutions to the equation $1 = y - x$, $x \in C_l^{(e,q)}$, $y \in C_m^{(e,q)}$.

Cyclotomic class is now widely used to construct almost difference sets. An almost difference set D of a finite field $\mathbb{F}_q$ is a subset with k elements satisfying special properties. Define

$$\Delta(x) = \left\{ (d, d') \in D^2 : d - d' = x \right\}. \quad (5)$$

D is called a (q, k, λ, t) almost difference set of $\mathbb{F}_q$ if $|\Delta(x)| = \lambda$ for t nonzero elements of $\mathbb{F}_q$, and $|\Delta(x)| = \lambda + 1$ for the remaining $q - 1 - t$ nonzero elements of $\mathbb{F}_q$. When $q \equiv 1 \pmod{4}$, $C_0^{(2,q)}$ is a $(q, (q-1)/2, (q-5)/4, (q-1)/2)$ almost difference set, which is also called the Paley partial difference set [15].

Let $q = 8f + 1$; Lehmer [16] presented the explicit form of the 64 cyclotomic numbers of order $e = 8$ when $q \equiv 9 \pmod{16}$, which are functions whose parameters are determined by the unique representation

$$q = x^2 + 4y^2 = a^2 + 2b^2, \quad (6)$$

$$x, y, a, b \in \mathbb{Z}, \quad x \equiv a \equiv 1 \pmod{4}.$$

The 64 cyclotomic numbers have two forms determined by whether 2 is a quartic residue in $\mathbb{F}_q$ or not. Let $D = C_0^{(8,q)} \cup C_1^{(8,q)} \cup C_2^{(8,q)} \cup C_5^{(8,q)}$, $\Delta_j = |(D + \theta^j) \cap D|$, $0 \leq j \leq 7$. When 2 is a quartic residue in $\mathbb{F}_q$, Ding et al. [17] calculated that

$$\begin{aligned} \Delta_0 = \Delta_4 &= \frac{16q - 48 + 8x - 8a - 48y}{64}, \\ \Delta_1 = \Delta_5 &= \frac{16q - 80 - 16x + 16a - 32y}{64}, \\ \Delta_2 = \Delta_6 &= \frac{16q - 48 + 8x - 8a - 16y}{64}, \\ \Delta_3 = \Delta_7 &= \frac{16q - 16}{64}. \end{aligned} \quad (7)$$

When 2 is not a quartic residue in $\mathbb{F}_q$, Zhang and Feng [10] calculated that

$$\begin{aligned} \Delta_0 = \Delta_4 &= \frac{16q - 48 + 8x - 8a + 16y}{64}, \\ \Delta_1 = \Delta_5 &= \frac{16q - 80}{64}, \\ \Delta_2 = \Delta_6 &= \frac{16q - 48 + 8x - 8a - 16y}{64}, \\ \Delta_3 = \Delta_7 &= \frac{16q - 16 - 16x + 16a}{64}. \end{aligned} \quad (8)$$

If l is a power of a prime and $l = t^2 + 2 \equiv 3 \pmod{8}$, then, for $q = l^2$, Ding [17] proved that $q \equiv 9 \pmod{16}$, $x = -l$, $y = 0$, $a = -l + 4$, $b = \pm 2t$, and 2 is a quartic residue in $\mathbb{F}_q^*$, and $D = C_0^{(8,q)} \cup C_1^{(8,q)} \cup C_2^{(8,q)} \cup C_5^{(8,q)}$ is a $(q, (q-1)/2, (q-5)/4, (q-1)/2)$ almost difference set of $\mathbb{F}_q$. Thus, $C_k^{(8,q)} \cup C_{1+k}^{(8,q)} \cup C_{2+k}^{(8,q)} \cup C_{5+k}^{(8,q)}$, $k = 0, 1, \dots, 7$ are also $(q, (q-1)/2, (q-5)/4, (q-1)/2)$ almost difference sets of $\mathbb{F}_q$.

An additive character of $\mathbb{F}_q$ with characteristic p is a homomorphism from $\mathbb{F}_q$ to the unit circle in the complex plane, which in fact has the following form [18]:

$$\chi_c(g) = \exp\left(\frac{2\pi\sqrt{-1}\text{Tr}(cg)}{p}\right), \quad \forall g \in \mathbb{F}_q, c \in \mathbb{F}_q, \quad (9)$$

where Tr is the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. Multiplicative characters of $\mathbb{F}_q$ are characters of the multiplicative group $\mathbb{F}_q^*$. Let θ be a fixed primitive element of $\mathbb{F}_q$. All multiplicative characters ψ_j of $\mathbb{F}_q$ have the following form [18]:

$$\psi_j(\theta^k) = \exp\left(\frac{2\pi\sqrt{-1}jk}{q-1}\right), \quad (10)$$

$$\text{for } k = 0, 1, \dots, q-2, \quad j = 0, 1, \dots, q-2.$$

The set of all additive characters of $\mathbb{F}_q$ is denoted by $\widehat{\mathbb{F}}_q$, and $|\widehat{\mathbb{F}}_q| = q$. When $c = 0$, χ_0 is trivial, which satisfies $\chi_0(g) = 1$, $\forall g \in \mathbb{F}_q$. For a subset D and an additive character χ of $\mathbb{F}_q$, $\chi(D)$ is defined to be $\chi(D) = \sum_{g \in D} \chi(g)$. Clearly, $\chi(\overline{D}) = -\chi(D)$.

For m additive characters $\chi_1, \chi_2, \dots, \chi_m$ of $\mathbb{F}_q$, we can define their product as

$$(\chi_1 \chi_2 \cdots \chi_m)(g) = \chi_1(g) \chi_2(g) \cdots \chi_m(g) \quad (11)$$

for any $g \in \mathbb{F}_q$.

The direct product of n finite fields $\mathbb{F}_{q_1}, \mathbb{F}_{q_2}, \dots, \mathbb{F}_{q_n}$ can be defined by

$$\begin{aligned} \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \cdots \times \mathbb{F}_{q_n} \\ = \{(g_1, g_2, \dots, g_n), g_i \in \mathbb{F}_{q_i}, i = 1, \dots, n\}. \end{aligned} \quad (12)$$

Correspondingly, an additive character $\chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n$ on the direct product is defined by

$$\begin{aligned} (\chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n)(g_1, g_2, \dots, g_n) \\ = \chi_1(g_1) \chi_2(g_2) \cdots \chi_n(g_n), \end{aligned} \quad (13)$$

where $(g_1, g_2, \dots, g_n) \in \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \dots \times \mathbb{F}_{q_n}$ and $\chi_1 \in \widehat{\mathbb{F}_{q_1}}, \chi_2 \in \widehat{\mathbb{F}_{q_2}}, \dots, \chi_n \in \widehat{\mathbb{F}_{q_n}}$.

3. Main Result

In this section, we will follow Hu's method [14] to construct asymptotically optimal codebooks. Hu's method [14] used the product of difference sets of finite Abelian groups of order N_i , where $N_i \equiv 3 \pmod{4}$. In our paper, we use two classes of $(q, (q-1)/2, (q-5)/4, (q-1)/2)$ almost difference sets in finite fields: one class is $C_k^{(8,q)} \cup C_{1+k}^{(8,q)} \cup C_{2+k}^{(8,q)} \cup C_{5+k}^{(8,q)}$, $k = 0, 1, \dots, 7$, $q \equiv 9 \pmod{16}$, and the other is $C_0^{(2,q)}$ and $C_1^{(2,q)}$, $q \equiv 1 \pmod{4}$.

For the case of $D = C_0^{(8,q)} \cup C_1^{(8,q)} \cup C_2^{(8,q)} \cup C_5^{(8,q)}$, we first estimate $\chi(D)$ for any nontrivial additive characters of a finite field. In fact, Zhang and Feng had given an upper bound of $\chi(D)$ in Theorem 1 of [10]. We rephrase it in our setting.

Lemma 2. Let $\mathbb{F}_q$ be a finite field, $q \equiv 9 \pmod{16}$, and x, y, a, b are determined by (6). $D = C_0^{(8,q)} \cup C_1^{(8,q)} \cup C_2^{(8,q)} \cup C_5^{(8,q)}$. $\chi(\cdot) = \chi_c(\cdot) = \exp(2\pi\sqrt{-1}\text{Tr}(c\cdot)/p)$ is a nontrivial additive character. If $\lim_{q \rightarrow \infty} |x - a|/\sqrt{q} = 0$ and $\lim_{q \rightarrow \infty} |y|/\sqrt{q} = 0$, then there exists a function $f(q)$ satisfying $\lim_{q \rightarrow \infty} f(q)/q = 0$ such that

$$\begin{aligned} |\chi(D)| &\leq \frac{\sqrt{q + f(q)}}{2}, \\ |\chi(\overline{D})| &\leq \frac{\sqrt{q + f(q)}}{2}. \end{aligned} \quad (14)$$

Proof. Let $\eta_j = \sum_{g \in C_j^{(8,q)}} \exp(2\pi\sqrt{-1}\text{Tr}(g)/p)$. η_j is a Gauss period of order 8. Since $c \neq 0$, let $c \in C_\lambda^{(8,q)}$. Then

$$\begin{aligned} |\chi(D)|^2 &= \sum_{x \in D} \sum_{y \in D} \exp\left(\frac{2\pi\sqrt{-1}\text{Tr}(c(x-y))}{p}\right) \\ &= \frac{q-1}{2} \\ &\quad + \sum_{x \in D} \sum_{y \neq x, y \in D} \exp\left(\frac{2\pi\sqrt{-1}\text{Tr}(c(x-y))}{p}\right) \\ &= \frac{q-1}{2} \\ &\quad + \sum_{j=0}^7 \sum_{x \in D} \sum_{y \in D, x-y \in C_j^{(8,q)}} \exp\left(\frac{2\pi\sqrt{-1}\text{Tr}(c(x-y))}{p}\right) \\ &= \frac{q-1}{2} + \sum_{j=0}^7 \eta_{j+\lambda} \Delta_j = \frac{q-1}{2} \end{aligned}$$

$$\begin{aligned} &+ \sum_{j=0}^7 \eta_{j+\lambda} \left(\Delta_j - \frac{q-1}{4} + \frac{q-1}{4} \right) = \frac{q-1}{4} \\ &+ \sum_{j=0}^7 \eta_{j+\lambda} \left(\Delta_j - \frac{q-1}{4} \right). \end{aligned} \quad (15)$$

Note that, whether 2 is a quartic residue in $\mathbb{F}_q$ or not, $\Delta_i = \Delta_{i+2}$, $i = 0, 1, 2, 3$. Thus

$$|\chi(D)|^2 = \frac{q-1}{4} + \sum_{j=0}^3 (\eta_{j+\lambda} + \eta_{j+4+\lambda}) \left(\Delta_j - \frac{q-1}{4} \right). \quad (16)$$

Let ψ be the multiplicative character satisfying $\psi(\theta) = \psi_{(q-1)/4}(\theta^1) = \exp(2\pi\sqrt{-1}/4)$. $\psi(g\theta^{-j}) = \psi(g) \exp(-2\pi\sqrt{-1}j/4)$. So if $g \in C_j^{(8,q)} \cup C_{j+4}^{(8,q)}$, we have $\sum_{i=0}^3 \psi^i(g\theta^{-j}) = 4$, or else $\sum_{i=0}^3 \psi^i(g\theta^{-j}) = 0$. Thus,

$$\begin{aligned} \eta_j + \eta_{j+4} &= \sum_{g \in C_j^{(8,q)} \cup C_{j+4}^{(8,q)}} \exp\left(\frac{2\pi\sqrt{-1}\text{Tr}(g)}{p}\right) \\ &= \frac{1}{4} \sum_{g \in \mathbb{F}_q^*} \exp\left(\frac{2\pi\sqrt{-1}\text{Tr}(g)}{p}\right) \sum_{i=0}^3 \psi^i(g\theta^{-j}) \\ &= \frac{1}{4} \sum_{i=0}^3 G(\psi^i) \exp\left(\frac{-2\pi\sqrt{-1}ij}{4}\right), \end{aligned} \quad (17)$$

where $G(\psi) = \sum_{g \in \mathbb{F}_q^*} \psi(g) \exp(2\pi\sqrt{-1}\text{Tr}(g)/p)$ is the Gaussian sum over $\mathbb{F}_q$ and $G(\psi^0) = -1$, $|G(\psi^i)| = \sqrt{q}$, $i = 1, 2, 3$ [18]. Thus, $|\eta_j + \eta_{j+4}| \leq (1/4)(1 + 3\sqrt{q})$.

Then

$$\begin{aligned} |\chi(D)|^2 &= \frac{q-1}{4} + \sum_{j=0}^3 (\eta_{j+\lambda} + \eta_{j+4+\lambda}) \left(\Delta_j - \frac{q-1}{4} \right) \\ &\leq \frac{q-1}{4} + \frac{1}{4} (1 + 3\sqrt{q}) \sum_{j=0}^3 \left| \Delta_j - \frac{q-1}{4} \right|. \end{aligned} \quad (18)$$

When 2 is a quartic residue in $\mathbb{F}_q$, substituting (7) into (18) yields

$$|\chi(D)|^2 \leq \frac{q+1}{4} + \frac{3\sqrt{q}}{2} + \frac{1+3\sqrt{q}}{8} (|x-a| + 3|y|). \quad (19)$$

When 2 is a quartic nonresidue in $\mathbb{F}_q$, substituting (8) into (18) yields

$$|\chi(D)|^2 \leq \frac{q+1}{4} + \frac{3\sqrt{q}}{2} + \frac{1+3\sqrt{q}}{8} (|x-a| + |y|). \quad (20)$$

Denote $f(q) = 1/4 + 3\sqrt{q}/2 + ((1+3\sqrt{q})/8)(|x-a| + 3|y|)$. If $\lim_{q \rightarrow \infty} |x-a|/\sqrt{q} = 0$ and $\lim_{q \rightarrow \infty} |y|/\sqrt{q} = 0$, then $\lim_{q \rightarrow \infty} f(q)/q = 0$. Thus

$$|\chi(D)| \leq \frac{\sqrt{q + f(q)}}{2}. \quad (21)$$

Since $\chi(\bar{D}) = -\chi(D)$,

$$|\chi(\bar{D})| = |\chi(D)| \leq \frac{\sqrt{q+f(q)}}{2}. \quad (22)$$

Remark 3. There exist infinitely many x, y, a, b satisfying the assumptions in Lemma 2. Ding et al. [17] proved that if $q = l^2 \equiv 9 \pmod{16}$, $l = t^2 + 2 \equiv 3 \pmod{8}$, then $x = -l$, $y = 0$, $a = -l + 4$, $b = \pm 2t$, where t is an integer. In this case, we have $|x - a| = 4$, $|y| = 0$, D becomes a $(q, (q-1)/2, (q-5)/4, (q-1)/2)$ almost difference set, and $|\chi(D)| \leq \sqrt{q+12\sqrt{q}+3}/2$.

Remark 4. Lemma 2 still holds if we use $\bar{D} = C_l^{(8,q)} \cup C_{1+l}^{(8,q)} \cup C_{2+l}^{(8,q)} \cup C_{5+l}^{(8,q)}$, $l = 1, \dots, 7$ instead of $D = C_0^{(8,q)} \cup C_1^{(8,q)} \cup C_2^{(8,q)} \cup C_5^{(8,q)}$. The reason is that new $\bar{\Delta}_j = |(\bar{D} + \theta^j) \cap \bar{D}|$ is equal to Δ_{j-j} . Thus, it has no influence on estimating $|\chi(D)|$. However, if we use other combinations of four cyclotomic classes which do not correspond to a family of almost difference sets, there may not exist infinitely many x, y, a, b satisfying the assumptions in Lemma 2. The reason is that the eight $\bar{\Delta}_j$, $j = 0, \dots, 7$ will have more different values, which inevitably lead to more restrictions on x, y, a, b . Thus, it is more difficult to find infinitely many q such that $|\chi(D)|$ is not too large.

Remark 5. Since $D = C_0^{(8,q)} \cup C_2^{(8,q)} \cup C_4^{(8,q)} \cup C_6^{(8,q)}$ can be seen as $C_0^{(2,q)}$, we consider $C_0^{(2,q)}$ and $C_1^{(2,q)}$, where $q \equiv 1 \pmod{4}$. Ding [19] proved that $\chi(C_0^{(2,q)}) = (-1 \pm \sqrt{q})/2$ and $\chi(C_1^{(2,q)}) = (-1 \mp \sqrt{q})/2$. Thus, for $D = C_0^{(2,q)}$ or $D = C_1^{(2,q)}$, we have $\max|\chi(D)| = (\sqrt{q}+1)/2$.

Suppose that $\mathbb{F}_{q_1}, \mathbb{F}_{q_2}, \dots, \mathbb{F}_{q_n}$ are n finite fields. Let $\mathbb{F} = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \dots \times \mathbb{F}_{q_n}$. Then $\bar{\mathbb{F}} = \{\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n} \mid \chi_{i,j_i} \in \bar{\mathbb{F}}_{q_i}, 1 \leq i \leq n\}$.

Let $D_i = C_0^{(8,q_i)} \cup C_1^{(8,q_i)} \cup C_2^{(8,q_i)} \cup C_5^{(8,q_i)}$ be the union of the four cyclotomic classes of order 8 in $\mathbb{F}_{q_i}$, $q_i \equiv 9 \pmod{16}$, $i = 1, \dots, n$. Let $\bar{D}_i$ be the complement of D_i with respect to $\mathbb{F}_{q_i}$. Note that $|D_i| = (q_i - 1)/2$.

Let $E_1 = D_1$. For any $1 \leq i \leq n-1$, put

$$E_{i+1} = (E_i \times \bar{D}_{i+1}) \cup (\bar{E}_i \times D_{i+1}). \quad (23)$$

Finally, let $D = E_n$, and $K = |D|$. Let $N = |\mathbb{F}|$; then $N = q_1 q_2 \dots q_n$.

We can define similar sets in the case of $\mathbb{F}_{q_i}$, $q_i \equiv 1 \pmod{4}$, $i = 1, \dots, n$. More specifically, let $D_i = C_0^{(2,q_i)} \subset \mathbb{F}_{q_i}$, $q_i \equiv 1 \pmod{4}$, $i = 1, \dots, n$. Then we can define $\bar{D}_i$, $i = 1, \dots, n$ in a similar way.

For both of the two cases, we have the following lemma.

Lemma 6. $|E_i| = (q_1 q_2 \dots q_i - 1)/2$ for any $1 \leq i \leq n$. In particular, $K = (N - 1)/2$.

Proof. For $i = 1$, we have $|E_1| = (q_1 - 1)/2$. For $i > 1$, suppose that $|E_i| = (q_1 q_2 \dots q_i - 1)/2$. Then, it follows that

$$\begin{aligned} |E_{i+1}| &= \frac{q_1 q_2 \dots q_i - 1}{2} \cdot \frac{q_{i+1} + 1}{2} + \frac{q_1 q_2 \dots q_i + 1}{2} \\ &\quad \cdot \frac{q_{i+1} - 1}{2} = \frac{(q_1 q_2 \dots q_{i+1} - 1)}{2}. \end{aligned} \quad (24)$$

Therefore, $|E_n| = (q_1 q_2 \dots q_n - 1)/2 = (N - 1)/2$. $\square$

We define an (N, K) codebook by

$$\begin{aligned} \mathcal{C}_D &= \{C_{j_1, j_2, \dots, j_n} \mid j_i = 0, 1, \dots, q_i - 1, i = 1, 2, \dots, n\}. \end{aligned} \quad (25)$$

The vector $C_{j_1, j_2, \dots, j_n} \in C^K$ is defined by

$$\begin{aligned} C_{j_1, j_2, \dots, j_n} &= \frac{1}{\sqrt{K}} (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \\ &\quad \otimes \chi_{n,j_n}) (g_{i,1}, g_{i,2}, \dots, g_{i,n})_{i=1}^K \\ &= \frac{1}{\sqrt{K}} (\chi_{1,j_1} (g_{i,1}) \chi_{2,j_2} (g_{i,2}) \dots \chi_{n,j_n} (g_{i,n}))_{i=1}^K, \end{aligned} \quad (26)$$

where $\chi_{i,j_i} \in \bar{\mathbb{F}}_{q_i}$, $(g_{i,1}, g_{i,2}, \dots, g_{i,n}) \in D$ and $\chi_{i,0}$ is the corresponding trivial character. It is obvious that $|\mathcal{C}_D| = N$, and the length of the vector $C_{j_1, j_2, \dots, j_n}$ is $|D| = K$.

Next we will investigate $I_{\max}(\mathcal{C}_D)$. For the case of $D_i = C_0^{(8,q_i)} \cup C_1^{(8,q_i)} \cup C_2^{(8,q_i)} \cup C_5^{(8,q_i)} \subset \mathbb{F}_{q_i}$, $q_i \equiv 9 \pmod{16}$, we need the following lemma.

Lemma 7. With the above notations, for any character $\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n}$ of $\mathbb{F}_q$, $j_i = 0, 1, \dots, q_i - 1$, $q_i \equiv 9 \pmod{16}$, $i = 1, 2, \dots, n$. If $(j_1, j_2, \dots, j_n) = (0, 0, \dots, 0)$, we have

$$|(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n})(D)| = K. \quad (27)$$

If $(j_1, j_2, \dots, j_n) \neq (0, 0, \dots, 0)$,

$$\begin{aligned} &|(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n})(D)| \\ &\leq \frac{\sqrt{q_1 + f(q_1)} \dots \sqrt{q_n + f(q_n)}}{2}. \end{aligned} \quad (28)$$

Proof. If $n = 1$, when $j_1 = 0$, χ_{1,j_1} is the trivial character; thus, $\chi_{1,j_1}(D) = |D| = (q_1 - 1)/2 = K$. When $j_1 \neq 0$, χ_{1,j_1} is a nontrivial character. Thus, by Lemma 2, $|\chi_{1,j_1}(D)| \leq \sqrt{q_1 + f(q_1)}/2$.

If $n = 2$, we need to consider three cases:

(1) $j_1 = j_2 = 0$

$$|(\chi_{1,j_1} \otimes \chi_{2,j_2})(D)| = |D| = |E_2| = \frac{q_1 q_2 - 1}{2} = K. \quad (29)$$

(2) $j_1 = 0, j_2 \neq 0$

$$\begin{aligned}
 (\chi_{1,j_1} \otimes \chi_{2,j_2})(D) &= \chi_{1,j_1}(D_1) \chi_{2,j_2}(\overline{D}_2) \\
 &\quad + \chi_{1,j_1}(\overline{D}_1) \chi_{2,j_2}(D_2) \\
 &= |D_1| \chi_{2,j_2}(\overline{D}_2) + |\overline{D}_1| \chi_{2,j_2}(D_2) \quad (30) \\
 &= (|D_1| - |\overline{D}_1|) \chi_{2,j_2}(\overline{D}_2) \\
 &\quad + |\overline{D}_1| \chi_{2,j_2}(\mathbb{F}_{q_2}).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 |(\chi_{1,j_1} \otimes \chi_{2,j_2})(D)| &= (|D_1| - |\overline{D}_1|) \chi_{2,j_2}(\overline{D}_2) \\
 &\leq ||D_1| - |\overline{D}_1|| \frac{\sqrt{q_2 + f(q_2)}}{2} \\
 &= \frac{\sqrt{q_2 + f(q_2)}}{2} \quad (31) \\
 &< \frac{\sqrt{q_1 + f(q_1)} \sqrt{q_2 + f(q_2)}}{2}.
 \end{aligned}$$

In a similar way, if $j_1 \neq 0, j_2 = 0$,

$$\begin{aligned}
 |(\chi_{1,j_1} \otimes \chi_{2,j_2})(D)| &\leq \frac{\sqrt{q_1 + f(q_1)}}{2} \\
 &< \frac{\sqrt{q_1 + f(q_1)} \sqrt{q_2 + f(q_2)}}{2}. \quad (32)
 \end{aligned}$$

(3) $j_1 \neq 0, j_2 \neq 0$

$$\begin{aligned}
 |(\chi_{1,j_1} \otimes \chi_{2,j_2})(D)| &= |\chi_{1,j_1}(D_1) \chi_{2,j_2}(\overline{D}_2) + \chi_{1,j_1}(\overline{D}_1) \chi_{2,j_2}(D_2)| \\
 &= 2 |\chi_{1,j_1}(D_1) \chi_{2,j_2}(D_2)| \quad (33) \\
 &\leq \frac{\sqrt{q_1 + f(q_1)} \cdot \sqrt{q_2 + f(q_2)}}{2}.
 \end{aligned}$$

Now we assume that the formula holds for $n - 1$; we will prove it for n .

If $(j_1, j_2, \dots, j_n) = (0, 0, \dots, 0)$, then the character $\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n}$ is trivial. Hence, the result is correct. In the following, we consider the case of $n > 2$ and $(j_1, j_2, \dots, j_n) \neq (0, 0, \dots, 0)$. The proof can also be divided into three cases:

Case (1) $((j_1, j_2, \dots, j_{n-1}) = (0, 0, \dots, 0), j_n \neq 0)$. We compute

$$\begin{aligned}
 &(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n})(D) \\
 &= (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n}) \\
 &\cdot ((E_{n-1} \times \overline{D}_n) \cup (\overline{E}_{n-1} \times D_n)) = |E_{n-1}| \chi_{n,j_n}(\overline{D}_n) \quad (34) \\
 &+ |\overline{E}_{n-1}| \chi_{n,j_n}(D_n) = -|E_{n-1}| \chi_{n,j_n}(D_n) + |\overline{E}_{n-1}| \\
 &\cdot \chi_{n,j_n}(D_n).
 \end{aligned}$$

By Lemma 6, $|\overline{E}_{n-1}| - |E_{n-1}| = 1$. So we have

$$\begin{aligned}
 |(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n})(D)| &= |\chi_{n,j_n}(D_n)| \\
 &\leq \frac{\sqrt{q_n + f(q_n)}}{2} < \frac{\sqrt{q_1 + f(q_1)} \dots \sqrt{q_n + f(q_n)}}{2}. \quad (35)
 \end{aligned}$$

Case (2) $((j_1, j_2, \dots, j_{n-1}) \neq (0, 0, \dots, 0), j_n = 0)$. In this case, we get

$$\begin{aligned}
 &(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n})(D) \\
 &= (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n}) \\
 &\cdot ((E_{n-1} \times \overline{D}_n) \cup (\overline{E}_{n-1} \times D_n)) \\
 &= (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n-1,j_{n-1}})(E_{n-1}) |\overline{D}_n| \quad (36) \\
 &+ (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n-1,j_{n-1}})(\overline{E}_{n-1}) |D_n| \\
 &= (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n-1,j_{n-1}})(E_{n-1}) |\overline{D}_n| \\
 &- (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n-1,j_{n-1}})(E_{n-1}) |D_n| \\
 &= (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n-1,j_{n-1}})(E_{n-1}).
 \end{aligned}$$

Hence, by the assumption,

$$\begin{aligned}
 |(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n})(D)| &= |(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n-1,j_{n-1}})(E_{n-1})| \\
 &\leq \frac{\sqrt{q_1 + f(q_1)} \dots \sqrt{q_{n-1} + f(q_{n-1})}}{2} \quad (37) \\
 &< \frac{\sqrt{q_1 + f(q_1)} \dots \sqrt{q_n + f(q_n)}}{2}.
 \end{aligned}$$

Case (3) $((j_1, j_2, \dots, j_{n-1}) \neq (0, 0, \dots, 0), j_n \neq 0)$. In this case, we get

$$\begin{aligned}
 &(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n})(D) \\
 &= (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \dots \otimes \chi_{n,j_n})
 \end{aligned}$$

$$\begin{aligned}
& \cdot ((E_{n-1} \times \overline{D}_n) \cup (\overline{E}_{n-1} \times D_n)) \\
& = (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \cdots \otimes \chi_{n-1,j_{n-1}})(E_{n-1}) \cdot \chi_{n,j_n}(\overline{D}_n) \\
& + (\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \cdots \otimes \chi_{n-1,j_{n-1}})(\overline{E}_{n-1}) \cdot \chi_{n,j_n}(D_n) \\
& = -2(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \cdots \otimes \chi_{n-1,j_{n-1}})(E_{n-1}) \\
& \cdot \chi_{n,j_n}(D_n).
\end{aligned} \tag{38}$$

Hence, by the assumption and Lemma 2,

$$\begin{aligned}
& |(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \cdots \otimes \chi_{n,j_n})(D)| \\
& = |-2(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \cdots \otimes \chi_{n-1,j_{n-1}})(E_{n-1}) \\
& \cdot \chi_{n,j_n}(D_n)| \leq \frac{\sqrt{q_1 + f(q_1)} \cdots \sqrt{q_n + f(q_n)}}{2}.
\end{aligned} \tag{39}$$

The proof is finished. $\square$

Theorem 8. Let $\mathcal{C}_D$ be a codebook defined by (25) in the case of $D_i = C_0^{(8,q_i)} \cup C_1^{(8,q_i)} \cup C_2^{(8,q_i)} \cup C_5^{(8,q_i)} \subset \mathbb{F}_{q_i}$, $q_i \equiv 9 \pmod{16}$, and q_i satisfies the assumptions in Lemma 2. Then $\mathcal{C}_D$ is an $(N, K) = (q_1 q_2 \cdots q_n, (q_1 q_2 \cdots q_n - 1)/2)$ codebook with

$$I_{\max}(\mathcal{C}_D) \leq \frac{\sqrt{q_1 + f(q_1)} \cdots \sqrt{q_n + f(q_n)}}{q_1 \cdots q_n - 1}, \tag{40}$$

and $I_{\max}(\mathcal{C}_D)$ can reach the Welch bound if $q_i \rightarrow \infty$ for any $1 \leq i \leq n$.

Proof. Let $C_{j_1, j_2, \dots, j_n}$ and $C_{j'_1, j'_2, \dots, j'_n}$ be two different codewords in $\mathcal{C}_D$. Then we have

$$\begin{aligned}
& |C_{j_1, j_2, \dots, j_n} C_{j'_1, j'_2, \dots, j'_n}^H| = \frac{1}{K} \left| \sum_{i=1}^K \chi_{1,j_1}(g_{i,1}) \chi_{2,j_2}(g_{i,2}) \right. \\
& \left. \cdots \chi_{n,j_n}(g_{i,n}) \cdot \overline{\chi}_{1,j'_1}(g_{i,1}) \overline{\chi}_{2,j'_2}(g_{i,2}) \cdots \overline{\chi}_{n,j'_n}(g_{i,n}) \right| \\
& = \frac{1}{K} \left| \sum_{i=1}^K (\chi_{1,j_1} \overline{\chi}_{1,j'_1})(g_{i,1}) (\chi_{2,j_2} \overline{\chi}_{2,j'_2})(g_{i,2}) \right. \\
& \left. \cdots (\chi_{n,j_n} \overline{\chi}_{n,j'_n})(g_{i,n}) \right| = \frac{1}{K} \left| ((\chi_{1,j_1} \overline{\chi}_{1,j'_1}) \right. \\
& \left. \otimes (\chi_{2,j_2} \overline{\chi}_{2,j'_2}) \otimes \cdots \otimes (\chi_{n,j_n} \overline{\chi}_{n,j'_n}))(D) \right|.
\end{aligned} \tag{41}$$

Since $C_{j_1, j_2, \dots, j_n}$ and $C_{j'_1, j'_2, \dots, j'_n}$ are different, $(\chi_{1,j_1} \overline{\chi}_{1,j'_1}) \otimes (\chi_{2,j_2} \overline{\chi}_{2,j'_2}) \otimes \cdots \otimes (\chi_{n,j_n} \overline{\chi}_{n,j'_n})$ is a nontrivial character of $\mathbb{F}_q$. So Lemma 7 yields

$$\begin{aligned}
I_{\max}(\mathcal{C}_D) & \leq \frac{1}{K} \cdot \frac{\sqrt{q_1 + f(q_1)} \cdots \sqrt{q_n + f(q_n)}}{2} \\
& = \frac{\sqrt{q_1 + f(q_1)} \cdots \sqrt{q_n + f(q_n)}}{q_1 \cdots q_n - 1}.
\end{aligned} \tag{42}$$

In this case, the Welch bound I_{Welch} for the codebook $\mathcal{C}_D$ is

$$I_{\text{Welch}} = \sqrt{\frac{N - K}{(N - 1)K}} = \frac{\sqrt{q_1 q_2 \cdots q_n + 1}}{q_1 q_2 \cdots q_n - 1}. \tag{43}$$

Hence,

$$\begin{aligned}
1 & \leq \frac{I_{\max}(\mathcal{C}_D)}{I_{\text{Welch}}} \leq \frac{\sqrt{q_1 + f(q_1)} \cdots \sqrt{q_n + f(q_n)}}{\sqrt{q_1 q_2 \cdots q_n + 1}} \\
& \rightarrow 1
\end{aligned} \tag{44}$$

if $q_i \rightarrow +\infty$ for any $1 \leq i \leq n$. $\square$

Corollary 9. With the above notations, we assume that all $\mathbb{F}_{q_i}$ are the same; that is, $q_i = q$, $i = 1, 2, \dots, n$. In this case, the codebook $\mathcal{C}_D$ is a $(q^n, (q^n - 1)/2)$ codebook satisfying

$$I_{\max}(\mathcal{C}_D) \leq \frac{(\sqrt{q + f(q)})^n}{q^n - 1}. \tag{45}$$

In this case, $I_{\text{Welch}} = \sqrt{q^n + 1}/(q^n - 1)$. So

$$1 \leq \frac{I_{\max}(\mathcal{C}_D)}{I_{\text{Welch}}} \leq \frac{(\sqrt{q + f(q)})^n}{\sqrt{q^n + 1}} \rightarrow 1 \tag{46}$$

if $q \rightarrow +\infty$.

For the case $D_i = C_0^{(2,q_i)} \subset \mathbb{F}_{q_i}$, $q_i \equiv 1 \pmod{4}$, we have the following lemma.

Lemma 10. For any character $\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \cdots \otimes \chi_{n,j_n}$ of $\mathbb{F}_q$, $j_i = 0, 1, \dots, q_i - 1$, $q_i \equiv 1 \pmod{4}$, $i = 1, 2, \dots, n$, if $(j_1, j_2, \dots, j_n) = (0, 0, \dots, 0)$, we have

$$|(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \cdots \otimes \chi_{n,j_n})(D)| = K. \tag{47}$$

If $(j_1, j_2, \dots, j_n) \neq (0, 0, \dots, 0)$,

$$\begin{aligned}
& \max |(\chi_{1,j_1} \otimes \chi_{2,j_2} \otimes \cdots \otimes \chi_{n,j_n})(D)| \\
& = \frac{(\sqrt{q_1} + 1) \cdots (\sqrt{q_n} + 1)}{2}.
\end{aligned} \tag{48}$$

Proof. The proof is very similar to the proof of Lemma 7. We omit it here. $\square$

Theorem 11. Let $\mathcal{C}_D$ be a codebook defined by (25) in the case of $D_i = C_0^{(2,q_i)} \subset \mathbb{F}_{q_i}$, $q_i \equiv 1 \pmod{4}$. Then $\mathcal{C}_D$ is an $(N, K) = (q_1 q_2 \dots q_n, (q_1 q_2 \dots q_n - 1)/2)$ codebook with

$$I_{\max}(\mathcal{C}_D) = \frac{(\sqrt{q_1} + 1) \dots (\sqrt{q_n} + 1)}{q_1 \dots q_n - 1}, \quad (49)$$

and $I_{\max}(\mathcal{C}_D)$ can reach the Welch bound if $q_i \rightarrow \infty$ for any $1 \leq i \leq n$.

Proof. The proof is very similar to the proof of Theorem 8. We omit it here. $\square$

Remark 12. The result of Theorem 11 is similar to that of [14]. The difference between our result and that of [14] is that Hu's method is based on the product of difference sets in finite Abelian group G_i of order N_i , where $N_i \equiv 3 \pmod{4}$, whereas our method is based on the product of almost difference sets in finite field $\mathbb{F}_{q_i}$, where $q_i \equiv 1 \pmod{4}$.

Corollary 13. With the above notations, we assume that all $\mathbb{F}_{q_i}$ are the same; that is, $q_i = q \equiv 1 \pmod{4}$, $i = 1, 2, \dots, n$. In this case, the codebook $\mathcal{C}_D$ is a $(q^n, (q^n - 1)/2)$ codebook satisfying

$$I_{\max}(\mathcal{C}_D) = \frac{(\sqrt{q} + 1)^n}{q^n - 1}. \quad (50)$$

In this case, $I_{\text{Welch}} = \sqrt{q^n + 1}/(q^n - 1)$. So

$$1 \leq \frac{I_{\max}(\mathcal{C}_D)}{I_{\text{Welch}}} = \frac{(\sqrt{q} + 1)^n}{\sqrt{q^n + 1}} \rightarrow 1 \quad (51)$$

if $q \rightarrow +\infty$.

Remark 14. If $n = 1$, our codebooks (25) in the case of $D_i = C_0^{(8,q_i)} \cup C_1^{(8,q_i)} \cup C_2^{(8,q_i)} \cup C_5^{(8,q_i)} \subset \mathbb{F}_{q_i}$, $q_i \equiv 9 \pmod{16}$, turn into the asymptotically optimal codebooks in [10], and our condition is milder, since [10] requires that $|x - a|$ and $|y|$ are bounded, whereas we require $\lim_{q \rightarrow \infty} |x - a|/q = 0$ and $\lim_{q \rightarrow \infty} |y|/q = 0$. If $n = 1$, in the case of $D_i = C_0^{(2,q_i)} \subset \mathbb{F}_{q_i}$, $q_i \equiv 1 \pmod{4}$, codebooks (25) turn into the asymptotically optimal codebooks in [19].

Remark 15. By Remark 4 and Lemma 7, Theorem 8 still holds if we use $D_i = C_l^{(8,q_i)} \cup C_{1+l}^{(8,q_i)} \cup C_{2+l}^{(8,q_i)} \cup C_{5+l}^{(8,q_i)}$ instead of $D_i = C_0^{(8,q_i)} \cup C_1^{(8,q_i)} \cup C_2^{(8,q_i)} \cup C_5^{(8,q_i)}$, $l = 1, \dots, 7$. Similarly, by Remark 5 and Lemma 10, Theorem 11 still holds if we use $D_i = C_1^{(2,q_i)}$ instead of $D_i = C_0^{(2,q_i)}$.

4. Conclusions

In this paper, we construct a new class of asymptotically optimal codebooks based on the product of two classes of almost difference sets in finite fields. Future research is required to investigate whether this strategy works for any almost difference set of $(q, (q - 1)/2, \lambda, t)$ type. It is also interesting to modify the strategy such that we can start with MWBE codebooks and then derive new MWBE codebooks

via a similar direct product. Another interesting problem is whether any complement of asymptotically codebooks similar to the Naimark complement of MWBE codebooks exists.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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