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Research Article

A New Approach to Rough Set Based on Remote Neighborhood Systems

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The notion of neighborhood systems is abstracted from the geometric notion of "near", and it is primitive in the theory of topological spaces. Now, neighborhood systems have been applied in the study of rough set by many researches. The notion of remote neighborhood systems is initial in the theory of topological molecular lattice, and it is abstracted from the geometric notion of "remote". Therefore, the notion of remote neighborhood systems can be considered as the dual notion of neighborhood systems. In this paper, we develop a theory of rough set based on remote neighborhood systems. Precisely, we construct a pair of lower and upper approximation operators and discuss their basic properties. Furthermore, we use a set of axioms to describe the lower and upper approximation operators constructed from remote neighborhood systems.

1. Introduction and Background

Pawlak's rough set theory [1, 2] was based on equivalence for dealing with vagueness and granularity in information systems. It is well known that the basic notions in rough set theory are the lower and upper approximations [3, 4]. Generally, two different basic methods have been proposed to develop rough set theories, the constructive method and the axiomatic method [5–7]. In the constructive method, binary relations, coverings, and neighborhood systems on the universe of discourse are all primitive notions [8–10]. The lower and upper approximation operators are constructed by means of these notions [10–40].

Neighborhood system comes from topology. Let U be a universe of discourse and 2^U denotes the power set of U. A neighborhood system on U is a mapping $N: U \longrightarrow 2^{2^U}$, which is defined by assigning to each x of U a nonempty collection $N(x)$ of subset of U. Each member of $N(x)$ is called a neighborhood of x and represents the semantic of "near". The rough sets based on generalized neighborhood system attract the attention of many scholars. Lin and Michael [41–44] explored the approximations based on generalized neighborhood systems. Lin [45, 46] and Yao [47] discussed the approximation retrieval and information retrieval based on general neighborhood systems, respectively. Wang [48], Syau [49], and Zhang [50] investigated relationships between reflexive generalized neighborhood system-based rough sets and other rough sets. Quite recently, the second author and his coauthor [51] discussed the axiomatic characterization on approximation operators based on generalized neighborhood systems.

In [52], Wang studied the theory of topological molecular lattice by the tool of remote neighborhood systems [53]. A remote neighborhood system on a universe of discourse U is generally defined as a mapping $RN : U \longrightarrow 2^{2^U}$ with some additional condition. For any $x \in U$, each member of $RN(x)$ is called a remote neighborhood of x and represents the semantic of "remote". Therefore, the notion of remote neighborhood systems can be considered as the dual notion of neighborhood systems.

Now, the notion of neighborhood systems has been extensively used in rough sets. So, as the dual notion of neighborhood systems, can the remote neighborhood systems be used in the rough sets? In this paper, we shall develop a theory of rough sets base on general remote neighborhood systems.

The contents are arranged as follows. Section 2 recalls some notions and results about generalized neighborhood system-based rough sets as preliminary. Section 3 defines approximation operators based on generalized remote neighborhood system operator and discusses their basic properties. Section 4 presents the axiomatic characterization on newly defined approximation operators. Section 5 gives the conclusions and future works.

2. Preliminary

In this section, we will recall some basic conceptions of general rough set approximations. Let U be a nonempty set of all the objects under consideration, referred to as the universe. The power set of U denoted by 2^U is the collection of all subsets of U. Let R be an equivalence relation on U. We use U/R to denote the family of all equivalence classes of R and we use $[x]$ to denote an equivalence class in R containing an element $x \in U$. The pair (U, R) is called an approximation space. For any $X \subseteq U$ one can define the lower approximation and the upper approximation of X [1, 54] by $\underline{R}X = \{x \mid [x] \subseteq X\}$, $\overline{R}X = \{x \mid [x] \cap X \neq \emptyset\}$, respectively. The pair $RX = (\underline{R}X, \overline{R}X)$ is referred to as the rough set of X. The rough set RX denotes the description of X under the current scheme. If $\underline{R}X = \overline{R}X$, then X is a definable set of R, that is, X is an exact set of R. Next, we introduce some results about rough sets based on generalized neighborhood system [48, 50].

The notion of generalized neighborhood system is introduced by Sierpiński [55] and then discussed by Lin [41].

Definition 1 ([41, 55]). Let U be the universe of discourse and 2^U denote the power set of U. Then a function $N : U \longrightarrow 2^{2^U}$ is called a generalized neighborhood system operator on U if for any $x \in X$, $N(x)$ is nonempty. Usually, $N(x)$ is called generalized neighborhood system of x and each $K \in N(x)$ is called neighborhood of x.

The rough sets based on generalized neighborhood system is introduced by Lin-Yao [46] and then studied by Syau-Lin [48] and Zhang et al. [50].

Definition 2 ([46, 48, 50]). Let $N : U \longrightarrow 2^{2^U}$ be a generalized neighborhood system operator. Then for each subset X of U, the upper and lower approximation operators $\underline{N}(X)$ and $\overline{N}(X)$ are defined as follows:

$$\begin{aligned} \underline{N}(X) &= \{x \in U \mid \exists K \in N(x),\ K \subseteq X\}, \\ \overline{N}(X) &= \{x \in U \mid \forall K \in N(x),\ K \cap X \neq \emptyset\}. \end{aligned} \tag{1}$$

The above two definitions are the text citation for [51].

In [50], it is observed that rough sets based on neighborhood operator can be regarded as special case of rough sets based on generalized neighborhood system operators.

3. Approximation Operators Based on Generalized Remote Neighborhood System

In this section, we will define a pair of approximation operators based on generalized remote neighborhood system operator and discuss their basic properties. In the following, we assume $X' = U - X$ be the complement of X, for any $X \subseteq U$.

For any $x \in U$ and $K \in 2^U$. It is easily seen that if K represents the nearness of x, then intuitively, K' represents the remoteness of x. Thus if $N : U \longrightarrow 2^{2^U}$ is a generalized neighborhood system operator on U, then intuitively, the function $RN : U \longrightarrow 2^{2^U}$ defined by

$$RN(x) = \left\{K' \mid K \in N(x)\right\}, \quad \forall x \in U \tag{2}$$

can be interpreted as a general remote system operator on U. Therefore, we give the following definition.

Definition 3. Let U be the universe of discourse. Then a function $RN : U \longrightarrow 2^{2^U}$ is called a generalized remote neighborhood system operator on U if for any $x \in X$, $RN(x)$ is nonempty. Usually, $RN(x)$ is called generalized remote neighborhood system of x and each $K \in RN(x)$ is called a remote neighborhood of x.

Remark 4. (1) The idea of remote neighborhood is initiated by Wang [52] in the context of topological modular lattice. The notion of generalized remote neighborhood systems is an extension of the notion of remote neighborhood systems of topological spaces in [56].

(2) Although both generalized neighborhood system and generalized remote neighborhood system are defined as functions from U to 2^{2^U}, their sematic meanings are different. The difference will be exhibited more obviously in the following study on serial condition, reflexive condition, transitive condition, etc.

Definition 5. Let $RN : U \longrightarrow 2^{2^U}$ be a generalized remote neighborhood system operator. Then for each subset X of U, the upper and lower approximation operators $\overline{RN}(X)$ and $\underline{RN}(X)$ are defined as follows:

$$\begin{aligned} \overline{RN}(X) &= \{x \in U \mid \forall K \in RN(x),\ X \not\subseteq K\}. \\ \underline{RN}(X) &= \left\{x \in U \mid \exists K \in RN(x),\ X' \subseteq K\right\}. \end{aligned} \tag{3}$$

X is called a definable set if $\underline{RN}(X) = \overline{RN}(X)$, and it is rough set otherwise.

Example 6. Let $U = \{a, b, c\}$ and $RN(a) = \{\emptyset, \{b\}, \{c\}\}$, $RN(b) = \{\emptyset, \{a, c\}, \{c\}\}$ $RN(c) = \{\emptyset, \{a\}, \{b\}\}$.

Next, we show $\underline{RN}(\{a, b\}) = \{a, b\}$ and $\overline{RN}(\{a, b\}) = \{a, b, c\}$ as example.

Let $X = \{a, b\}$. Then $X' = \{c\}$. By Definition 5 of $\underline{RN}(X)$, we have $X' = \{c\} \subseteq RN(a)$, $X' = \{c\} \subseteq RN(b)$, and $X' = \{c\} \nsubseteq RN(c)$. Therefore $\underline{RN}(\{a, b\}) = \{a, b\}$.

Since $X = \{a, b\}$, for any $K \in RN(a)$, we have $X \nsubseteq K$, so $a \in \overline{RN}(\{a, b\})$. The same can be verified $b, c \in \overline{RN}(\{a, b\})$. Thus $\overline{RN}(\{a, b\}) = \{a, b, c\}$.

Then

$$\begin{aligned}
\underline{RN}(\emptyset) &= \emptyset, \\
\underline{RN}(\{a\}) &= \emptyset, \\
\underline{RN}(\{b\}) &= \{b\}, \\
\underline{RN}(\{c\}) &= \emptyset, \\
\underline{RN}(\{a, b\}) &= \{a, b\}, \\
\underline{RN}(\{a, c\}) &= \{a, c\}, \\
\underline{RN}(\{b, c\}) &= \{b, c\}, \\
\underline{RN}(U) &= U. \\
\overline{RN}(\emptyset) &= \emptyset, \\
\overline{RN}(\{a\}) &= \{a\}, \\
\overline{RN}(\{b\}) &= \{b\}, \\
\overline{RN}(\{c\}) &= \{c\}, \\
\overline{RN}(\{a, b\}) &= \{a, b, c\}, \\
\overline{RN}(\{a, c\}) &= \{a, c\}, \\
\overline{RN}(\{b, c\}) &= \{a, b, c\}, \\
\overline{RN}(U) &= U.
\end{aligned} \tag{4}$$

Next, we give the basic properties of the defined approximation operators.

Proposition 7. *Let RN be a generalized remote neighborhood system operator on U. Then*

(1) $\overline{RN}(\emptyset) = \emptyset$, $\underline{RN}(U) = U$,

(2) for any $X, Y \subseteq U$ *and* $X \subseteq Y \Longrightarrow \underline{RN}(X) \subseteq \underline{RN}(Y)$, $\overline{RN}(X) \subseteq \overline{RN}(Y)$.

Proof. (1) Now we prove $\overline{RN}(\emptyset) = \emptyset$ and $\underline{RN}(U) = U$.

By Definition 5 of $\overline{RN}(\emptyset)$, for any $x \in U$ and $K \in RN(x)$, then $\emptyset \nsubseteq K$. It is contradictory for $\emptyset \subseteq K$. Therefore $\overline{RN}(\emptyset) = \emptyset$.

Because $\underline{RN}(U) = \{x \in U \mid \exists K \in RN(x),\ U' \subseteq K\}$ and for any $x \in U$, we have $U' = \emptyset \subseteq K$, so $\underline{RN}(U) = U$.

(2) Firstly, we prove that for any $X, Y \subseteq U$ and $X \subseteq Y \Longrightarrow \underline{RN}(X) \subseteq \underline{RN}(Y)$.

For any $x \in \underline{RN}(X)$, then there exists $K \in RN(x)$ such that $X' \subseteq K$. By $X \subseteq Y$, we have $Y' \subseteq X'$. Therefore $Y' \subseteq K$. Then $x \in \underline{RN}(Y)$. We obtain $\underline{RN}(X) \subseteq \underline{RN}(Y)$.

Secondly, we prove that for any $X, Y \subseteq U$ and $X \subseteq Y \Longrightarrow \overline{RN}(X) \subseteq \overline{RN}(Y)$.

For any $x \in \overline{RN}(X)$ and for any $K \in RN(x)$, we have $X \nsubseteq K$. By $X \subseteq Y$, we have $Y \nsubseteq K$. This tells us $x \in \overline{RN}(Y)$. Thus $\overline{RN}(X) \subseteq \overline{RN}(Y)$. □

Theorem 8 shows that $\overline{RN}$ and $\underline{RN}$ are dual.

Theorem 8. *Let RN be a generalized remote neighborhood system operator on U. Then*

(1) $(\overline{RN}(X'))' = \underline{RN}(X)$.

(2) $(\underline{RN}(X'))' = \overline{RN}(X)$.

Proof. We only prove (1) as an example; the proof of (2) is similar to (1).

On the one hand, for any $x \in (\overline{RN}(X'))'$, then $x \notin \overline{RN}(X')$. We have $\exists K \in RN(x)$ such that $X' \subseteq K$. Therefore $x \in \underline{RN}(X)$. Thus $(\overline{RN}(X'))' \subseteq \underline{RN}(X)$.

On the other hand, for any $x \in \underline{RN}(X)$, then there exists $K \in RN(x)$ such that $X' \subseteq K$. We have $x \notin \overline{RN}(X')$. Thus $x \in (\overline{RN}(X'))'$. Therefore $(\overline{RN}(X'))' \supseteq \underline{RN}(X)$. We obtain that $(\overline{RN}(X'))' = \underline{RN}(X)$. □

In the recent papers [48, 50], the researchers studied serial, reflexive, symmetric, transitive, weak-transitive, and weak-unary generalized neighborhood system and related rough sets.

Let N be a generalized neighborhood system operator on U. Then

(1) N is said to be serial, if for any $x \in U$ and $K \in N(x)$, $K \neq \emptyset$.

(2) N is said to be reflexive, if for any $x \in U$ and $K \in N(x)$, $x \in K$.

(3) N is said to be symmetric, if for any $x, y \in U$, $K \in N(x)$, and $V \in N(y)$, $x \in V \Longrightarrow y \in K$.

(4) N is said to be transitive, if for any $x, y, z \in U$, $K \in N(y)$, and $L \in N(z)$, $x \in K$ and $y \in L \Longrightarrow x \in L$.

(5) N is said to be weak-transitive, if for any $x \in U$ and $K \in N(x)$, there exists an $V \in N(x)$ such that for any $y \in V$ there exists an $V_y \in N(y)$ with $V_y \subseteq K$.

(6) N is said to be weak-unary, if for any $x \in U$ and $K, V \in N(x)$, there exists an $M \in N(x)$ such that $M \subseteq K \cap V$.

The above six concepts are the text citation for [51]

Now, for generalized remote neighborhood system operator we define the condition of serial, reflexive, symmetric, transitive, weak-transitive, and weak-unary, respectively.

Let RN be a generalized remote neighborhood system operator on U. Then

(1) RN is said to be serial, if for any $x \in U$ and $K \in RN(x)$, $K \neq U$.

(2) RN is said to be reflexive, if for any $x \in U$ and $K \in RN(x)$, $x \notin K$.

(3) RN is said to be symmetric, if for any $x, y \in U$, $K \in RN(x)$, and $V \in RN(y)$, $x \notin V \Longrightarrow y \notin K$.

(4) RN is said to be transitive, if for any $x, y, z \in U$, $K \in RN(y)$, and $L \in RN(z)$, $x \notin K$ and $y \notin L \Longrightarrow x \notin L$.

(5) RN is said to be weak-transitive, if for any $x \in U$ and $K \in RN(x)$, there exists a $V \in RN(x)$ such that for any $y \notin V$ there exists a $V_y \in RN(y)$ with $K \subseteq V_y$.

(6) RN is said to be weak-unary, if for any $x \in U$ and $K, V \in RN(x)$, there exists an $M \in RN(x)$ such that $K \cup V \subseteq M$.

Theorem 9 ([50]). *Let N be a generalized neighborhood system operator on U. Then*

(1) N is serial $\Longleftrightarrow \underline{N}(\emptyset) = \emptyset \Longleftrightarrow \overline{N}(U) = U$.

(2) N is reflexive $\Longleftrightarrow \forall X \subseteq U$, $\underline{N}(X) \subseteq X \Longleftrightarrow \forall X \subseteq U$, $X \subseteq \overline{N}(X)$.

(3) N is weak-transitive $\Longrightarrow \forall X \subseteq U$, $\underline{N}(X) \subseteq \underline{N}(\underline{N}(X))$ and $\forall X \subseteq U$, $\overline{N}(X) \supseteq \overline{N}(\overline{N}(X))$.

(4) N is weak-unary $\Longleftrightarrow \forall X, Y \subseteq U$, $\underline{N}(X \cap Y) = \underline{N}(X) \cap \underline{N}(Y) \Longleftrightarrow \forall X, Y \subseteq U$, $\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)$.

The above theorem is the text citation for [51]

Proposition 10 ([50]). *Let N be a generalized neighborhood system operator on U. Then*

(1) If N is symmetric then $\forall X \subseteq U$, $X \subseteq \underline{N}(\overline{N}(X))$, and $X \supseteq \overline{N}(\underline{N}(X))$.

(2) If N is transitive then $\forall X \subseteq U$, $\underline{N}(X) \subseteq \underline{N}(\underline{N}(X))$, and $\overline{N}(X) \supseteq \overline{N}(\overline{N}(X))$.

The above preposition is the text citation for [51]

In [50] Zhang used the generalized neighborhood to get the conclusions above.

Now we use the remote neighborhood to get the following conclusions.

Proposition 11. *Let RN be a generalized remote neighborhood system operator on U. Then the following are equivalent:*

(1) RN is serial,

(2) $\underline{RN}(\emptyset) = \emptyset$,

(3) $\overline{RN}(U) = U$.

Proof. We only prove (1) $\Longleftrightarrow$ (2) as an example. (1) $\Longleftrightarrow$ (3) is similar to (1) $\Longleftrightarrow$ (2).

(1) $\Longrightarrow$ (2) If RN is serial, then $\forall x \in U$, $K \in RN(x)$, $K \neq U$. It is obvious that $\underline{RN}(\emptyset) = \emptyset$.

(2)$\Longrightarrow$ (1) Since $\underline{RN}(\emptyset) = \emptyset$. Then there is not any $K \in RN(x)$ such that $U \subseteq K$. Therefore $\forall K \in RN(x)$, $K \neq U$. So RN is serial. □

Proposition 12. *Let RN be a generalized remote neighborhood system operator on U. Then the following are equivalent:*

(1) RN is reflexive,

(2) $\forall X \subseteq U$, $\underline{RN}(X) \subseteq X$,

(3) $\forall X \subseteq U$, $X \subseteq \overline{RN}(X)$.

Proof. We only prove (1) $\Longleftrightarrow$ (2) as an example. (1) $\Longleftrightarrow$ (3) is similar to (1) $\Longleftrightarrow$ (2).

(1) $\Longrightarrow$ (2) For any $X \subseteq U$ and $x \notin X$, by RN is reflexive, we have $\forall K \in RN(x)$, $x \notin K$. Then $X' \not\subseteq K$; this tells us $x \notin \underline{RN}(X)$, so $\underline{RN}(X) \subseteq X$.

(2) $\Longrightarrow$ (1) For any $x \in U$ and $K \in \underline{RN}(x)$, since $(K')' \subseteq K$, so $x \in \underline{RN}(K')$. By (2), we have $\underline{RN}(K') \subseteq K'$. Then $x \in K'$. Therefore $x \notin K$. This shows that RN is reflexive. □

Proposition 13. *Let RN be a generalized remote neighborhood system operator on U. Then the following are equivalent:*

(1) RN is weak-unary,

(2) $\forall X, Y \subseteq U$, $\underline{RN}(X \cap Y) = \underline{RN}(X) \cap \underline{RN}(Y)$,

(3) $\forall X, Y \subseteq U$, $\overline{RN}(X \cup Y) = \overline{RN}(X) \cup \overline{RN}(Y)$.

Proof. We only prove (1) $\Longleftrightarrow$ (2) as an example. (1) $\Longleftrightarrow$ (3) is similar to (1) $\Longleftrightarrow$ (2).

(1) $\Longrightarrow$ (2) $\forall X, Y \subseteq U$, we only need to prove $\underline{RN}(X \cap Y) \supseteq \underline{RN}(X) \cap \underline{RN}(Y)$.

$\forall x \in \underline{RN}(X) \cap \underline{RN}(Y)$. Then $x \in \underline{RN}(X)$ and $x \in \underline{RN}(Y)$. There exists $K, V \in RN(x)$ such that $X' \subseteq K$ and $Y' \subseteq V$. We have $X' \cup Y' \subseteq K \cup V$. Because RN is weak-unary, so for $K, V \in RN(x)$, there exists an $M \in RN(x)$ such that $K \cup V \subseteq M$. We have $X' \cup Y' \subseteq M$. Thus $(X \cap Y)' \subseteq M$. Hence $x \in \underline{RN}(X \cap Y)$. Then $\underline{RN}(X \cap Y) \supseteq \underline{RN}(X) \cap \underline{RN}(Y)$.

(2) $\Longrightarrow$ (1) For any $x \in U$ and $K, V \in RN(x)$, we prove that there exists an $M \in RN(x)$ such that $K \cup V \subseteq M$.

For any $x \in U$ and $K, V \in RN(x)$, by $(K')' \subseteq K$, we have $x \in \underline{RN}(K')$. In the same way, we obtain $x \in \underline{RN}(V')$. Therefore $x \in \underline{RN}(K') \cap \underline{RN}(V')$=$\underline{RN}(K' \cap V')$. Then there exists $M \in RN(x)$ such that $(K' \cap V')' \subseteq M$. Thus $K \cup V \subseteq M$. Then RN is weak-unary. □

Proposition 14. *Let RN be a generalized remote neighborhood system operator on U. Then the following are equivalent:*

(1) RN is weak-transitive,

(2) $\forall X \subseteq U$, $\underline{RN}(X) \subseteq \underline{RN}(\underline{RN}(X))$,

(3) $\forall X \subseteq U$, $\overline{RN}(X) \supseteq \overline{RN}(\overline{RN}(X))$.

Proof. We only prove (1) $\Longleftrightarrow$ (2) as an example. (1) $\Longleftrightarrow$ (3) is similar to (1) $\Longleftrightarrow$ (2).

(1)$\Longrightarrow$(2) For any $x \in \underline{RN}(X)$, there exists a $K \in RN(x)$ such that $X' \subseteq K$. Since RN is weak-transition, for the $K \in RN(x)$, there exists a $V \in RN(x)$ such that for any $y \notin V$, there exists a $V_y \in RN(y)$ with $K \subseteq V_y$, so $X' \subseteq V_y$. Hence $y \in \underline{RN}(X)$. We have $V' \subseteq \underline{RN}(X)$, i.e., $(\underline{RN}(X))' \subseteq V$. Then $x \in \underline{RN}(\underline{RN}(X))$. Therefore $\underline{RN}(X) \supseteq \underline{RN}(\underline{RN}(X))$.

(2)$\Longrightarrow$(1) For any $x \in U$ and $\forall K \in RN(x)$. Then $x \in \underline{RN}(K') \subseteq \underline{RN}(\underline{RN}(K'))$. There exists $V \in RN(x)$ such that $(\underline{RN}(K'))' \subseteq V$, i.e., $V' \subseteq \underline{RN}(K')$. For any $y \notin V$, we have $y \in \underline{RN}(K')$, then there exists $V_y \in RN(y)$ such that $K = (K')' \subseteq V_y$. Thus RN is weak-transitive. □

Proposition 15. *Let RN be a generalized remote neighborhood system operator on U. If RN is symmetric then $\forall X \subseteq U$, $X \subseteq \underline{RN}(\overline{RN}(X))$ and $X \supseteq \overline{RN}(\underline{RN}(X))$.*

Proof. We only prove that if RN is symmetric then $\forall X \subseteq U$, $X \supseteq \overline{RN}(\underline{RN}(X))$ as an example.

For any $x \in \overline{RN}(\underline{RN}(X))$ and for any $K \in RN(x)$, we have $\underline{RN}(X) \not\subseteq K$. Then there exists a $y \in \underline{RN}(X)$ such that $y \notin K$. By $y \in \underline{RN}(X)$, there exists $V \in RN(y)$ such that $X' \subseteq V$. Because RN is symmetric, so for $V \in RN(y)$, we have $x \notin V$. Thus $x \notin X'$. So $x \in X$. Therefore $X \supseteq \overline{RN}(\underline{RN}(X))$. □

The converse of the above statements is not true; next Example 16 shows this.

Example 16. Let $U = \{a, b\}$, $RN(a) = \{\{b\}, \emptyset\}$, $RN(b) = \{\{a\}, \emptyset\}$. Then $\overline{RN}(\underline{RN}(\emptyset)) = \emptyset$,

$$\begin{aligned} \overline{RN}\left(\underline{RN}\left(\{a\}\right)\right) &= \{a\}, \\ \overline{RN}\left(\underline{RN}\left(\{b\}\right)\right) &= \{b\}, \\ \overline{RN}\left(\underline{RN}\left(U\right)\right) &= U. \end{aligned} \tag{5}$$

Therefore $X \supseteq \overline{RN}(\underline{RN}(X))$ for all $X \subseteq U$.

Since $a \notin \emptyset \in RN(b)$ and $b \in \{b\} \subseteq RN(a)$, we conclude that RN is not symmetric.

Proposition 17. *Let RN be a generalized remote neighborhood system operator on U. If RN is transitive then $\forall X \subseteq U$, $\underline{RN}(X) \subseteq \underline{RN}(\underline{RN}(X))$ and $\overline{RN}(X) \supseteq \overline{RN}(\overline{RN}(X))$.*

Proof. We only prove $\overline{RN}(X) \supseteq \overline{RN}(\overline{RN}(X))$ as an example. $\forall x \in \overline{RN}(\overline{RN}(X))$ and $K \in RN(x)$, we have $\overline{RN}(X) \nsubseteq K$. Then there exists a $y \in \overline{RN}(X)$ such that $y \notin K$. By $y \in \overline{RN}(X)$, we have for all $M \in RN(y)$, $X \nsubseteq M$, then there exists $z \in X$ but $z \notin M$. Then $z \notin K$. We obtain $X \nsubseteq K$. Thus $x \in \overline{RN}(X)$. Hence $\overline{RN}(X) \supseteq \overline{RN}(\overline{RN}(X))$. □

The converse of the Proposition 17 is not true.

Example 18. Let $U = \{a, b, c\}$ and $RN(a) = \emptyset$, $RN(b) = \emptyset$, $RN(c) = \{\emptyset, \{a\}, \{a, c\}, \{b, c\}, \{a, b\}\}$. Then

$$\begin{aligned} \underline{RN}\left(\emptyset\right) &= \emptyset = \underline{RN}\left(\underline{RN}\left(\emptyset\right)\right), \\ \underline{RN}\left(\{a\}\right) &= \{c\} = \underline{RN}\left(\underline{RN}\left(a\right)\right), \\ \underline{RN}\left(\{b\}\right) &= \{c\} = \underline{RN}\left(\underline{RN}\left(b\right)\right), \\ \underline{RN}\left(\{c\}\right) &= \{c\} = \underline{RN}\left(\underline{RN}\left(c\right)\right), \\ \underline{RN}\left(\{a, b\}\right) &= \{c\} = \underline{RN}\left(\underline{RN}\left(a, b\right)\right), \\ \underline{RN}\left(\{a, c\}\right) &= \{c\} = \underline{RN}\left(\underline{RN}\left(a, c\right)\right), \\ \underline{RN}\left(\{b, c\}\right) &= \{c\} = \underline{RN}\left(\underline{RN}\left(b, c\right)\right), \\ \underline{RN}\left(U\right) &= U = \underline{RN}\left(\underline{RN}\left(U\right)\right). \end{aligned} \tag{6}$$

Hence $\underline{RN}(X) \subseteq \underline{RN}(\underline{RN}(X))$ for all $X \subseteq U$. Since $a \notin \emptyset = RN(b)$, $b \notin \{a, c\} \subseteq RN(c)$, but $a \in \{a, c\}$. Hence $RN(x)$ is not transitive.

4. Axiomatic Characterization on Approximation Operators Based on General Remote Neighborhood Systems

In this section, we will give an axiomatic characterization on approximation operators based on general remote neighborhood systems.

4.1. On Upper Approximation Operator

Theorem 19. *Let $f : 2^U \longrightarrow 2^U$ be an operator. Then there exists a generalized remote neighborhood system operator RN such that $f = \overline{RN}$ if and only if f satisfies*

(T1): $f(\emptyset) = \emptyset$; (T2): $A \subseteq B \Longrightarrow f(A) \subseteq f(B)$.

Proof.

($\Longrightarrow$). It follows immediately from Proposition 7.

($\Longleftarrow$). Let $f : 2^U \longrightarrow 2^U$ be an operator satisfying (T1) and (T2). Then we define an operator $RN_f : U \longrightarrow 2^{2^U}$ as follows:

$\forall x \in U, A \in 2^U, A \in RN_f(x) \Longleftrightarrow \exists B \in 2^U, A \subseteq B$ and $x \notin f(B)$.

By (T1), it is clear that for any $x \in U$, $\emptyset \in RN_f(x)$ and so $RN_f(x)$ is nonempty. Hence RN_f is a generalized remote neighborhood system operator. Next, we prove that $\overline{RN_f} = f$. Obviously, we need only check that $\forall A \in 2^U$, $x \notin \overline{RN_f}(A) \Longleftrightarrow x \notin f(A)$.

Let $x \notin \overline{RN_f}(A)$. Then there exists a $B \in RN_f(x)$ such that $A \subseteq B$. So $A \in RN_f(x)$. By the definition of RN_f, we have $x \notin f(B)$. By (T2) we have $f(A) \subseteq f(B)$. Therefore $x \notin f(A)$, as desired.

Conversely, let $x \notin f(A)$. Then $A \in RN_f(x)$. By the definition of $\overline{RN_f}(A)$, $\forall B \in RN_f(x)$, $A \nsubseteq B$. Expressly, for $B = A$. Then $A \subseteq B$. Thus $x \notin \overline{RN_f}(A)$. □

Theorem 20. *Let $f : 2^U \longrightarrow 2^U$ be an operator. Then there exists a serial generalized remote neighborhood system operator RN such that $f = \overline{RN}$ if and only if f satisfies (T1), (T2), and (T3): $f(U) = U$.*

Proof.

Necessity. Let RN be a serial generalized remote neighborhood system operator on U and $f = \overline{RN}$. By Propositions 7 and 11, we have $f = \overline{RN}$ satisfying (T1), (T2), and (T3).

Sufficiency. Let f be an operator satisfying (T1), (T2), and (T3) and let RN_f be defined as that in Theorem 19. It is easily seen that we need only check that (T3) implies the serial condition. Indeed, for each $x \in U$, by $f(U) = U$ we have $x \in f(U)$; it follows that $U \notin RN_f(x)$. Therefore RN is serial. □

Theorem 21. *Let $f : 2^U \longrightarrow 2^U$ be an operator. Then there exists a reflexive generalized remote neighborhood system operator RN such that $f = \overline{RN}$ if and only if f satisfies (T1), (T2), and (T4): $X \subseteq f(X)$, for all $X \in 2^U$.*

Proof.

Necessity. Let RN be a reflexive generalized remote neighborhood system operator on U and $f = \overline{RN}$. Then it follows by Propositions 7 and 12 that $f = \overline{RN}_f$ satisfies (T1), (T2), and (T4).

Sufficiency. Let f be an operator satisfying (T1), (T2), and (T4). Let RN_f be defined as that in Theorem 19. It is easily seen that we need only to check that (T4) implies the reflexive condition. Indeed, $\forall x \in U$, let $A \in RN_f(x)$, by the definition of RN_f, then there exists an $B \in 2^U$ such that $A \subseteq B$ and $x \notin f(B)$. By (T4), we have $B \subseteq f(B)$. Thus $x \notin B$, we have $x \notin A$. Hence RN_f is reflexive. □

Theorem 22. *Let $f : 2^U \longrightarrow 2^U$ be an operator. Then there exists a weak-transitive generalized remote neighborhood system operator RN such that $f = \overline{RN}$ if and only if f satisfies (T1), (T2), and (T5): $f(X) \supseteq f(f(X))$, for all $X \in U$.*

Proof.

Necessity. Let RN be a weak-transitive generalized remote neighborhood system operator on U and $f = \overline{RN}$. Then it follows by Propositions 7 and 14 that $f = \overline{RN}_f$ satisfies (T1), (T2), and (T5).

Sufficiency. Let f be an operator satisfying (T1), (T2), and (T5) and let RN_f be defined as that in Theorem 19. It is easily seen that we need only check that (T5) implies the weak-transitive condition. $\forall x \in U$, let $A \in RN_f(x)$. For $x \notin f(A) = \overline{RN}_f(A)$, by (T5), we obtain $x \notin \overline{RN}_f(\overline{RN}_f(A))$; by Definition 5, there exists $B \in RN_f(x)$ such that $\overline{RN}_f(A) \subseteq B$. $\forall y \notin B$, then $y \notin \overline{RN}_f(A)$. Therefore there exists $V_y \in RN_f(y)$ such that $A \subseteq V_y$. Thus RN_f is weak-transitive. □

Theorem 23. *Let $f : 2^U \longrightarrow 2^U$ be an operator. Then there exists a weak-unary generalized remote neighborhood system operator RN such that $f = \overline{RN}$ if and only if f satisfies (T1), (T2), and (T6): $f(X \cup Y) = f(X) \cup f(Y)$, for all $X, Y \in 2^U$.*

Proof.

Necessity. Let RN be a weak-unary generalized remote neighborhood system operator on U and $f = \overline{RN}$. Then it follows by Propositions 7 and 13 that $f = \overline{RN}_f$ satisfies (T1), (T2), and (T6).

Sufficiency. Let f be an operator satisfying (T1), (T2), and (T6) and RN_f be defined as that in Theorem 19. It is easy to see that we need only check that (T6) implies the weak-unary condition. Indeed, $\forall x \in U$, let $K, V \in RN_f(x)$, $x \notin f(K)$, and $x \notin f(V)$. By (T6), we have $x \notin (f(K) \cup f(V)) = f(K \cup V)$. Then there exists $M \in RN_f(x)$ such that $K \cup V \subseteq M$. Therefore RN_f is weak-unary. □

4.2. On Lower Approximation Operators. Similar to the axiomatic characterizations on upper approximation operators, we can obtain the axiomatic characterizations on lower approximation operators.

Theorem 24. *Let $f : 2^U \longrightarrow 2^U$ be an operator. Then there exists a generalized remote neighborhood system operator RN such that $f = \underline{RN}$ if and only if f satisfies*

(R1): $f(U) = U$; (R2): $A \subseteq B \Longrightarrow f(A) \subseteq f(B)$.

Proof.

Necessity. It follows immediately from Proposition 7.

Sufficiency. Let $f : 2^U \longrightarrow 2^U$ be an operator satisfying $R1$ and $R2$. Then we define an operator $RN_f : U \longrightarrow 2^{2^U}$ as follows:

$\forall x \in U$, $A \in 2^U$, $A \in RN_f(x) \Longleftrightarrow$ there exist $B \in 2^U$, $A \subseteq B$, and $x \in f(B')$.

Next, we prove that $\underline{RN}_f = f$. Indeed for any $A \in 2^U$ and $x \in \underline{RN}_f(A)$. Then there exists $B \in RN_f(x)$ such that $A' \subseteq B$. So $A' \in RN_f(x)$. By the definition of RN_f, we have $x \in f(B')$. Since $A' \subseteq B$, so $B' \subseteq A$. By (R2), we have $f(B') \subseteq f(A)$ and $x \in f(A)$. Thus $\forall A \in 2^U$, $\underline{RN}_f(A) \subseteq f(A)$.

Conversely, let $x \in f(A)$, $A \subseteq U$. By the definition of RN_f, there exist $K \in RN_f(x)$ and $K \subseteq A'$. By the definition of $\underline{RN}_f$, we have $x \in \underline{RN}_f(A)$. Therefore $f(A) \subseteq \underline{RN}_f(A)$. □

The proof of the following theorem is similar to that of the upper approximation operator; we omit the proofs.

Theorem 25. *Let $f : 2^U \longrightarrow 2^U$ be an operator. Then*

(1) There exists a serial generalized remote neighborhood system operator RN such that $f = \underline{RN}$ if and only if f satisfies (R1), (R2), and (R3): $f(\emptyset) = \emptyset$.

(2) There exists a reflexive generalized remote neighborhood system operator RN such that $f = \underline{RN}$ if and only if f satisfies (R1), (R2), and (R4): $f(X) \subseteq X$, for all $X \in 2^U$.

(3) There exists a weak-transitive generalized remote neighborhood system operator RN such that $f = \underline{RN}$ if and only if f satisfies (R1), (R2), and (R5): $f(X) \subseteq f(f(X))$, for all $X \in 2^U$.

(4) There exists a weak-unary generalized remote neighborhood system operator RN such that $f = \underline{RN}$ if and only if f satisfies (R1), (R2), and (R6): $f(X \cap Y) = f(X) \cap f(Y)$, for all $X, Y \in 2^U$.

5. Conclusions

In this paper, we construct a pair of approximation operators based on general remote neighborhood systems. Then we discuss the basic properties and axiomatic characterization on this pair of approximation operators. It is well known that reduction theory is one of the most important contents in rough set. It is the basis of the application of rough set theory. In [57], the second author and his coauthor have established reduction theory of neighborhood systems-based rough sets, which can be regarded as a natural extension of reduction theory of covering-based rough set discussed in [58]. In the future work, we shall establish a reduction theory of remote neighborhood-based rough sets. Quite recently, the second author and his coauthor also fuzzify the notion of generalized neighborhood systems and then develop a theory of fuzzy rough sets based on fuzzy general neighborhood systems [59]. In [60], the author and his coauthor discussed the dual matroids and spanning; we shall also consider a theory of fuzzy rough sets and fuzzy matroids based on fuzzy general remote systems, which played an important role in the theory of fuzzy topological spaces.

Data Availability

No data were used to support this study.

Disclosure

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that there are no conflicts of interests.

Authors' Contributions

The first author contributed to the approximation operators and their properties. The second author contributed to axiom of approximate operators. The third author contributed to modification of the text.

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