Stochastic Dynamics of Discrete-Time Fuzzy Random BAM Neural Networks with Time Delays

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1. Introduction

In [1, 2], Kosko introduced the bidirectional associative memory (BAM) neural networks, which have been widely applied in psychophysics, parallel computing, perception, robotics, adaptive pattern recognition, associative memory, image processing pattern recognition, combinatorial optimization, and so on. All of these applications heavily depend on the (almost) periodicity and global exponential stability [3–14]. In the past twenty years, a large, fast growing body of investigations focused on the existence and global exponential stability of the equilibrium point, periodic, and almost periodic solutions of BAM neural networks with time delays in literatures [15–27].

Fuzzy theory was conceived in the 1960s by L. A. Zadeh, and it took about 20 years until the broader use of this theory in practice. Fuzzy technology joined forces with artificial neural networks and genetic algorithms under the title of computational intelligence or soft computing. In recent years, the research on the dynamical behaviours of fuzzy neural networks has attracted much attention (see [28–33]). On the other hand, uncertain models described by the stochastic differential equations have caused great concern since uncertain models have widely applications in practice such as engineering, physics, chemistry, and biology [34–38]. In the actual situations, random factors have consequences on the performance of the neural networks. In neural networks, the connection weights of the neurons depend on certain resistance and capacitance values that include modeling errors or uncertainties during the parameter identification process. Therefore, it is worth studying the dynamical behaviours of fuzzy stochastic neural networks.

The discrete-time neural networks become more important than the continuous-time counterparts when implementing the neural networks in a digital way. In order to investigate the dynamical characteristics with respect to digital signal transmission, it is essential to formulate the discrete analogue of neural networks. Mohamad and Gopalsamy [39, 40] proposed a novel method (i.e.,
semidiscretization technique) in formulating a discrete-time analogue of the continuous-time neural networks, which faithfully preserved the characteristics of their continuous-time counterparts. With the help of the semidiscretization technique [39], many scholars obtained the semidiscrete analogue of the continuous-time neural networks and some meaningful results were gained for the dynamical behaviors of the semidiscrete neural networks, such as periodic solutions, almost periodic solutions and global exponential stability (see [41–47]).

For instance, Huang et al. [44] discussed the almost periodic dynamics of the following semidiscrete cellular neural networks:

\[ x_i(k+1) = e^{-a_i(k)}x_i(k) + \frac{1 - e^{-a_i(k)}}{a_i(k)} \sum_{j=1}^{n} b_{ij}(k) f_j(x_j(k)) + I_i(k), \]  

where \( k \in \mathbb{Z} \), in which \( \mathbb{Z} \) denotes the set of integers, and \( i = 1, 2, \ldots, n \).

In [45], Huang et al. considered the following semidiscrete models for a class of general neural networks:

\[ x_i(k+1) = e^{-a_i(k)}x_i(k) + \frac{1 - e^{-a_i(k)}}{a_i(k)} \left[ \sum_{j=1}^{n} \sum_{l=1}^{m} b_{ijl}(k) f_j(x_j(k - \tau_{ijl})) + I_i(k) \right], \]  

where \( k \in \mathbb{Z} \) and \( i = 1, 2, \ldots, n \). The authors [45] derived the existence of locally exponentially convergent \( 2^N \) almost periodic sequence solutions of system (2).

By careful observation, it easily discovers that the disquisitive models in literatures [41–47] are deterministic, such as (1) and (2). Stimulated by this point, it is necessary to consider random factors in the determinant models. Therefore, this paper considers the semidiscrete models for the following stochastic fuzzy BAM neural networks:

\[
\begin{align*}
\dot{x}_i(t) &= \left[ -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}(t)f_j(y_j(t - \tau_{ij}(t))) + \sum_{j=1}^{m} \alpha_{ij}f_j(y_j(t - \tau_{ij}(t))) \right] dt + \sum_{j=1}^{m} d_{ij}(t)\sigma_j(y_j(t - \eta_{ij}(t)))dw_j(t),
\end{align*}
\]

and

\[
\begin{align*}
\dot{y}_j(t) &= \left[ -\bar{a}_j(t)y_j(t) + \sum_{i=1}^{n} \bar{b}_{ji}(t)f_i(x_i(t)) + \sum_{i=1}^{n} \bar{c}_{ji}(t)f_i(x_i(t - \bar{\tau}_{ji}(t))) + \bar{\alpha}_{ji}f_i(x_i(t - \bar{\tau}_{ji}(t))) \right] dt + \sum_{i=1}^{n} \bar{d}_{ji}(t)\bar{\sigma}_i(x_i(t - \bar{\eta}_{ji}(t)))d\bar{w}_i(t).
\end{align*}
\]

System (3) is composed of two layers, that is, X-layer and Y-layer. \( x_i(t) \) denotes the membrane potentials of the set of \( n \) neurons in X-layer, and \( y_j(t) \) denotes the membrane potentials of the set of \( m \) neurons in Y-layer at time \( t \); \( f_j \) and \( f_i \) represent the measures of activation to its incoming potentials of the unit \( j \) from Y-layer and the unit \( i \) from X-layer, respectively; \( b_{ij} \) corresponds to the synaptic connection weight of the unit \( j \) on the unit \( i \), and \( \bar{b}_{ji} \) corresponds to the synaptic connection weight of the unit \( i \) on the unit \( j \); \( I_i \) and \( I_j \) signify the bounded external bias or input; \( \alpha_i \) and \( \bar{\alpha}_j \) denote rate with which the \( i \)th unit and \( j \)th unit will reset their potentials to the resting state in isolation when it is disconnected from the network and external inputs, respectively; \( \alpha_{ij}, \beta_{ji}, T_{ij}, S_{ij}, \bar{\alpha}_{ji}, \bar{\beta}_{ji}, \bar{T}_{ji}, \bar{S}_{ji} \) are elements of fuzzy feedback MIN, MAX template, fuzzy feed forward MIN, and MAX template, respectively; \( \land \) and \( \lor \) denote the fuzzy AND and fuzzy OR operation, respectively; \( \omega_j \) and \( \bar{\omega}_i \) are the standard Brownian motions, where \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \). The other coefficients in system (3) are similarly specified.

The main aim of this paper is to investigate the dynamics of the semidiscrete analogue of system (3) by using semidiscretization technique [39] and stochastic theory. The main contributions of this paper are summed up as follows: (1) the semidiscrete analogue is established for stochastic fuzzy BAM neural networks (3); (2) a Volterra additive equation is derived for the solution of the semidiscrete model; (3) the existence of \( 2p \)-th moment global exponential stability; and (5) the methods used in this article can be applied to study the dynamics of other discrete stochastic fuzzy models.

The work of this paper is a continuation of literatures [44–47], and the results in this paper complement the corresponding results in [44–47]. The paper is organized as follows. In Section 2, the discrete analogue of system (3) is established and some useful lemmas are given. In Section 3, we employ Krasnoselskii’s fixed point theorem to obtain sufficient conditions for the existence of at least one \( 2p \)-th mean almost periodic sequence solution. In Section 4, we
consider the 2p-th moment global exponential stability. Two illustrative examples and computer simulations are given in Section 5. In Section 6, the conclusions and future works of this paper are presented.

Throughout this paper, we use the following notations. Let \( \mathbb{R} \) denote the set of real numbers. \( \mathbb{R}^n \) denotes the \( n \)-dimensional real vector space. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. Denote by \( BC(Z, L^p(\Omega, \mathbb{R}^n)) \) the vector space of all bounded continuous functions from \( Z \) to \( L^p(\Omega, \mathbb{R}^n) \), where \( L^p(\Omega, \mathbb{R}^n) \) denotes the collection of all \( p \)-th integrable \( \mathbb{R}^n \)-valued random variables. Then, \( BC(Z, L^p(\Omega, \mathbb{R}^n)) \) is a Banach space with the norm \( X_p = \sup_{k \in Z} |X|_p, |X|_p = \max_{k \in \mathbb{N}} (E|X|_p)^{1/p}, \forall X = \{x_1, x_2, \ldots, x_n\} \in BC(Z, L^p(\Omega, \mathbb{R}^n)), \) where \( p > 1 \) and \( E(\cdot) \) stands for the expectation operator with respect to the given probability measure \( P \). [a, b]_Z = [a, b] \cap Z, \forall a, b \in \mathbb{R}.

2. Model Formulation and Preliminaries

2.1. Discrete Analogue of System (3). Consider the following stochastic functional differential equations:

\[
dx(t) = -a(t)x(t)dt + F(t, x(t), x(t - \tau(t)))dt + G(t, x(t), x(t - \eta(t)))d\omega(t),
\]

which yields the following stochastic functional differential equations with piecewise constant arguments:

\[
\begin{aligned}
x_k(t+1) &= e^{-a_k(t)}x_k(t) + \sum_{j=1}^{m} b_{kj}(k)f_j(y_j(k)) + \sum_{j=1}^{m} c_{ij}(t)f_j(y_j(k-\tau_{ij}(k))) + \sum_{j=1}^{m} a_{ij}f_j(y_j(k-\tau_{ij}(k))) \\
&+ \frac{1}{\lambda} T_{ij}y_j(k) + \frac{1}{\nu} \beta_{ij}f_j(y_j(k-\tau_{ij}(k))) + \frac{1}{\mu} S_{ij}y_j(k) + I_k(t) + \sum_{j=1}^{m} d_{ij}(k)f_j(y_j(k-\eta_{ij}(k)))d\omega_j(k),
\end{aligned}
\]

\[
\begin{aligned}
y_j(t+1) &= e^{-\gamma_j(t)}y_j(t) + \frac{1}{\alpha_j(t)} \sum_{i=1}^{n} \tilde{b}_{ji}(k)\tilde{f}_i(x_i(k)) + \sum_{i=1}^{n} \tilde{c}_{ji}(k)\tilde{f}_i(x_i(k-\tau_{ji}(k))) + \sum_{i=1}^{n} \tilde{a}_{ji}(k)\tilde{f}_i(x_i(k-\eta_{ji}(k))) \\
&+ \frac{1}{\rho} \tilde{T}_{ji}y_i(k) + \frac{1}{\rho} \tilde{\beta}_{ji}f_i(x_i(k-\tau_{ji}(k))) + \sum_{i=1}^{n} \tilde{S}_{ji}y_i(k) + \tilde{I}_j(k) + \sum_{i=1}^{n} \tilde{d}_{ji}(k)\tilde{f}_i(x_i(k-\tilde{\eta}_{ji}(k)))d\tilde{\omega}_i(k),
\end{aligned}
\]

where \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \).

\[
dx(t) = -a(t)x(t)dt + F(t, x(t), x(t - \tau(t)))dt + G(t, x(t), x(t - \eta(t)))d\omega(t),
\]

where \( \lfloor t \rfloor \) denotes the integer part of \( t \). Here, the discrete analogue of the stochastic parts of system (4) is obtained by the Euler scheme, i.e.,

\[
dx(t) = -a(k)x(t)dt + F(k, x(t), x(k - \tau(k)))dt + G(k, x(t), x(k - \eta(k)))d\omega(t).
\]

Integrating the above equation from \( k \) to \( t \) and letting \( t \to k + 1 \), we achieve the discrete analogue of system (4) as follows:

\[
x(k + 1) = e^{-a(k)}x(k) + \frac{1-e^{-a(k)}}{a(k)} [F(k, x(k), x(k - \tau(k))) + G(k, x(k), x(k - \eta(k)))d\omega(k)],
\]

where \( k \in \mathbb{Z} \). System (7) is the discrete analogue of system (4) by the semidiscrete method. By a similar discussion as that in system (7), we obtain the semidiscrete analogue of system (3) as follows:

2.2. Volterra Additive Equation for the Solution of System (8).

**Lemma 1.** Assume that \( X = \{x_i\} \) is a solution of system (8), and then \( X \) can be expressed as
Proof. Let

\[ F_i(k, x) = \sum_{j=1}^{m} b_{ij}(k) f_j(y_j(k)) \]
\[ + \sum_{j=1}^{m} c_{ij}(k) f_j(y_j(k - \tau_{ij}(k))) \]
\[ + \sum_{j=1}^{m} d_{ij}(k) \sigma_j(y_j(k - \eta_{ij}(k))) \Delta \omega_j(k), \]

\[ k \in \mathbb{Z}, i = 1, 2, \ldots, n. \]  

(10)

By \( \Delta [u(k) v(k)] = [\Delta u(k)] v(k) + u(k + 1) [\Delta v(k)] \) and the first equation of system (8), it gets

\[ \Delta \left[ \prod_{s=0}^{k-1} e^{\alpha_i(s)} x_i(k) \right] = \prod_{s=0}^{k-1} \frac{e^{\alpha_i(s)} [1 - e^{-\alpha_i(s)}]}{\alpha_i(k)} F_i(k, x), \]

where \( i = 1, 2, \ldots, n \) and \( k \in \mathbb{Z} \). So,

\[ \sum_{s=0}^{k-1} \Delta \left[ \prod_{s=0}^{k-1} e^{\alpha_i(s)} x_i(v) \right] = \prod_{s=0}^{k-1} \frac{e^{\alpha_i(s)} [1 - e^{-\alpha_i(s)}]}{\alpha_i(v)} F_i(v, x) \]

is equivalent to

(12)

\[ x_i(k) = \prod_{s=0}^{k-1} e^{-\alpha_i(s)} x_i(k_0) + \sum_{s=0}^{k-1} \prod_{s=0}^{k-1} \frac{e^{\alpha_i(s)} [1 - e^{-\alpha_i(s)}]}{\alpha_i(v)} \left[ \sum_{j=1}^{m} b_{ij}(v) f_j(y_j(v)) + \sum_{j=1}^{m} c_{ij}(v) f_j(y_j(v - \tau_{ij}(v))) + \sum_{j=1}^{m} d_{ij}(v) \sigma_j(y_j(v - \eta_{ij}(v))) \Delta \omega_j(v) \right], \]

(9)

\[ y_j(k) = \prod_{s=0}^{k-1} e^{-\eta_i(s)} y_j(k_0) + \sum_{s=0}^{k-1} \prod_{s=0}^{k-1} \frac{e^{-\eta_i(s)} [1 - e^{-\eta_i(s)}]}{\eta_i(v)} \left[ \sum_{i=1}^{n} \bar{\beta}_{ij}(v) f_i(x_i(v)) + \sum_{i=1}^{n} \bar{\beta}_{ij}(v) f_i(x_i(v - \bar{\tau}_{ij}(v))) + \sum_{i=1}^{n} \bar{\beta}_{ij}(v) f_i(x_i(v - \bar{\tau}_{ij}(v))) \Delta \bar{\omega}_i(v) \right], \]

(13)

where \( k_0 \in \mathbb{Z}, k \in (k_0, +\infty) \), \( i = 1, 2, \ldots, n \), and \( j = 1, 2, \ldots, m \).

2.3. Some Lemmas.

Lemma 2 (Minkowski inequality) [48]. Assume that \( p \geq 1 \), \( E[\xi]^p < \infty, E[\eta]^p < \infty \), and then

\[ (E[\xi + \eta]^p)^{1/p} \leq (E[\xi]^p)^{1/p} + (E[\eta]^p)^{1/p}. \]

(14)

Lemma 3 (Hölder inequality) [48]. Assume that \( p > 1 \), and then

\[ \sum_{k} |a_k b_k| \leq \left[ \sum_{k} |a_k| \right]^{1-(1/p)} \left[ \sum_{k} |b_k| \right]^{1/p}. \]

If \( p = 1 \), then \( \sum_k |a_k b_k| \leq (\sum_k |a_k|) (\sup_k |b_k|) \).

Lemma 4 [35]. Suppose that \( g \in L^2([a, b], \mathbb{R}) \), and then

\[ E \left[ \sup_{t \in [a, b]} \left| \int_{a}^{b} g(s) d\omega(s) \right|^p \right] \leq C_p E \left[ \int_{a}^{b} |g(t)|^2 dt \right]^{p/2}, \]

(16)
where
\[
C_p = \begin{cases} 
\left( \frac{12}{p} \right)^{p/2}, & 0 < p < 2, \\
4, & p = 2, \\
\left[ \frac{p^{p-1}}{(p-1)!} \right]^{p/2}, & p > 2.
\end{cases}
\]

Lemma 5. Assume that \( \{x(k) : k \in \mathbb{Z} \} \) is real-valued stochastic process and \( \omega(k) \) is the standard Brownian motion, and then
\[
E[x(k)\Delta \omega(k)]^p \leq C_p E|x(k)|^p, \quad k \in \mathbb{Z},
\]
where \( C_p \) is defined as that in Lemma 4, in which \( p > 0 \).

Proof. By Lemma 4, it follows that
\[
E[x(k)\Delta \omega(k)]^p = E \left[ \int_k^{k+1} x(s)dw(s) \right]^p \leq C_p E \left[ \int_k^{k+1} x^2(s)ds \right]^{p/2}.
\]
This completes the proof.

Lemma 6 [49]. Suppose \( X = \{x_i\} \) and \( Y = \{y_j\} \) are two states of system (8), and then we have
\[
\begin{align*}
\left| \sum_{j=1}^n \alpha_{ij} f_j(x_i) - \sum_{j=1}^n \alpha_{ij} f_j(y_j) \right| & \leq \sum_{j=1}^n |\alpha_{ij}| \left| f_j(x_i) - f_j(y_j) \right|, \\
\left| \sum_{j=1}^m \beta_{ij} f_j(x_i) - \sum_{j=1}^m \beta_{ij} f_j(y_j) \right| & \leq \sum_{j=1}^m |\beta_{ij}| \left| f_j(x_i) - f_j(y_j) \right|, \\
& \quad i = 1, 2, \ldots, n,
\end{align*}
\]

Lemma 7 [50]. Assume that \( \Lambda \) is a closed convex nonempty subset of a Banach space \( \mathcal{X} \). Suppose further that \( \mathcal{B} \) and \( \mathcal{C} \) map \( \Lambda \) into \( \mathcal{X} \) such that
\begin{enumerate}
\item \( x, y \in \Lambda \) implies that \( \mathcal{B}x + \mathcal{C}y \in \Lambda \)
\item \( \mathcal{B} \) is continuous and \( \mathcal{B}\Lambda \) is contained in a compact set
\item \( \mathcal{C} \) is a contraction mapping
\end{enumerate}
Then, there exists a \( z \in \Lambda \) with \( z = \mathcal{B}z + \mathcal{C}z \).

Set \( a^u = \sup_{k \in \mathbb{Z}} |f^u(k)| \) and \( f^l = \inf_{k \in \mathbb{Z}} |f^l(k)| \) for bounded function \( f \) defined on \( \mathbb{Z} \). Throughout this paper, suppose that the following conditions are satisfied:

\begin{enumerate}
\item [(H1)] \( a_i > 0 \) and \( \overline{a}_j > 0 \), \( i = 1, 2, \ldots, n \), and \( j = 1, 2, \ldots, m \).
\item [(H2)] There exists several constants \( L^f_j, L^n_j, \overline{L}^f_j, \) and \( \overline{L}^n_j \) such that
\[
\begin{align*}
|f_j(u) - f_j(v)| & \leq L^f_j |u - v|, \\
|\sigma_j(u) - \sigma_j(v)| & \leq L^n_j |u - v|, \\
|\overline{f}_j(u) - \overline{f}_j(v)| & \leq \overline{L}^f_j |u - v|, \\
|\overline{\sigma}_j(u) - \overline{\sigma}_j(v)| & \leq \overline{L}^n_j |u - v|,
\end{align*}
\]
for all \( u, v \in \mathbb{R} \), \( i = 1, 2, \ldots, n \), and \( j = 1, 2, \ldots, m \).
\end{enumerate}

3. 2p-th Mean Almost Periodic Sequence Solution

Define
\[
\begin{align*}
a^u_i &= \max_{1 \leq i \leq n} a^u_i, \\
a^l_i &= \min_{1 \leq i \leq n} a^l_i, \\
a^u_j &= \max_{1 \leq j \leq m} \overline{a}_j, \\
a^l_j &= \max_{1 \leq j \leq m} \overline{a}_j, \\
D^+ &= \max_{1 \leq j \leq m} \sum_{i=1}^m \left( b^+_{ij} + \sigma^+_{ij} + |\alpha_{ij}| + |\beta_{ij}| \right) L^f_j, \\
D^- &= \max_{1 \leq j \leq m} \sum_{i=1}^m \left( b^-_{ij} + \sigma^-_{ij} + |\alpha_{ij}| + |\beta_{ij}| \right) L^n_j, \\
K^+ &= \max_{1 \leq i \leq n} \sum_{j=1}^m d^+_j L^f_j, \\
K^- &= \max_{1 \leq i \leq n} \sum_{j=1}^m d^-_j L^n_j, \\
r_{2p} &= \max \left\{ \frac{1 - e^{-a^u}}{a^l (1 - e^{-a^-})} \left( D^+ + K^+ C_{2p} \right), \frac{1 - e^{-a^-}}{a^l (1 - e^{-a^-})} \left( D^- + K^- C_{2p} \right) \right\}, \\
\beta_{2p} &= \frac{\alpha_{2p}}{1 - r_{2p}}.
\end{align*}
\]
\[ \alpha_{2p} := \max \{ e_{2p}, \tilde{e}_{2p} \}, \]
\[ e_{2p} := \frac{(1 - e^{-a})}{a' - (1 - e^{-a})} \max_{1 \leq i < n} \left[ \sum_{j=1}^{m} \left( b_{ij} + c_{ij} + \left| \alpha_{ij} \right| + \left| \beta_{ij} \right| \right) f_{j}(0) \right] \]
\[ + \sum_{j=1}^{m} \left( |T_{ij}| + |S_{ij}| \right) \left[ \mu_{j} + I_{j} + \sum_{i=1}^{n} \left( \tilde{d}_{ij}^{T} \sigma_{ij}(0) C_{ij} \right) \right], \]
\[ \tilde{e}_{2p} := \frac{(1 - e^{-a'})}{a' - (1 - e^{-a'})} \max_{1 \leq i < n} \left[ \sum_{i=1}^{n} \left( \tilde{b}_{ij}^{T} \tilde{c}_{ij} + |\tilde{\alpha}_{ij}| + |\tilde{\beta}_{ij}| \right) \tilde{f}_{i}(0) \right] \]
\[ + \sum_{i=1}^{n} \left( |\tilde{T}_{ij}| + |\tilde{S}_{ij}| \right) \left[ \tilde{\mu}_{i} + \tilde{I}_{i} + \sum_{i=1}^{n} \left( \tilde{d}_{ij} \tilde{\sigma}_{ij}(0) C_{ij} \right) \right]. \]
\[ (23) \]

Definition 3.1 [34]. A stochastic process \( X \in BC(\mathbb{Z}; L^{p}(\Omega; \mathbb{R}^{m+n})) \) is said to be \( p \)-th mean almost periodic sequence if for each \( \varepsilon > 0 \), there exists an integer \( l(\varepsilon) > 0 \) such that each interval of length \( l(\varepsilon) \) contains an integer \( \alpha \) for which

\[ |X(k + \omega) - X(k)|_{p} = \max_{1 \leq i \leq n} \left( E|x_{i}(k + \omega) - x_{i}(k)|^{p}\right)^{1/p} < \varepsilon, \]
\[ \forall k \in \mathbb{Z}. \]  
\[ (24) \]

A stochastic process \( X \), which is \( 2 \)-nd mean almost periodic sequence, is called square-mean almost periodic sequence. Like for classical almost periodic functions, the number \( \omega \) is called an \( \varepsilon \)-translation of \( X \).

Theorem 1. Assume that all of the coefficients in system (8) are almost periodic sequences, and then \((H_{1})-(H_{2})\) hold and the following condition is satisfied:

\[ (H_{3}) \quad r_{2p} < 1, \text{ where } 2p > 1 \]

Then, there exists a \( 2p \)-th mean almost periodic sequence solution \( X \) of system (8) with \( \|X\|_{2p} \leq \beta_{2p} \).

Proof. Let \( \Lambda \subseteq BC(\mathbb{Z}; L^{p}(\Omega; \mathbb{R}^{m+n})) \) be the collection of all \( 2p \)-th mean almost periodic sequences \( X \) satisfying the inequality \( \|X\|_{2p} \leq \beta_{2p} \).

Define \( \Phi X(k) = \mathcal{B}X(k) + \mathcal{C}X(k) \), where

\[ \Phi X(k) = \{(\Phi X)(k); (\Phi X)_{n+1}(k)\} = \{(\Phi X)_{1}(k), \ldots, (\Phi X)_{n}(k), (\Phi X)_{n+1}(k), \ldots, (\Phi X)_{n+m}(k)\}^{T}, \]
\[ (25) \]

\[ (\Phi X)_{i}(k) = \mathcal{B}X(k) + (\mathcal{C}X)(k), \]
\[ (26) \]

\[ (\Phi X)_{n+j}(k) = \mathcal{B}X_{n+j}(k) + (\mathcal{C}X)_{n+j}(k), \]
\[ (27) \]
where \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \), and \( k \in \mathbb{Z} \).

Let \( X^0 = \{x^0_i; y^0_j\} \) be defined as

\[
\begin{align*}
\left\{x^0_i\right\} & = \max_{1 \leq i \leq n, \ k \in \mathbb{Z}} \left\{ \frac{1}{a_i(v)} \left| e^{-a_i(v)} \prod_{i=\mu}^{\nu} \left| \frac{j_i(v)}{a_i(v)} \right| - e^{-a_i(v)} \prod_{i=\mu}^{\nu} \left| \frac{j_i(v)}{a_i(v)} \right| \right| \right\}^{1/2p} \\
\left\{y^0_j\right\} & = \max_{1 \leq j \leq m, \ k \in \mathbb{Z}} \left\{ \frac{1}{\mu_j} \left| \frac{e^{-\mu_j(v)} \prod_{i=\mu}^{\nu} \left| \frac{j_i(v)}{\mu_j(v)} \right| - e^{-\mu_j(v)} \prod_{i=\mu}^{\nu} \left| \frac{j_i(v)}{\mu_j(v)} \right| \right| \right\}^{1/2p}
\end{align*}
\]

By Minkowski inequality in Lemma 2 and Hölder inequality in Lemma 3, we have

\[
\begin{align*}
x^0_i(k) &= \sum_{v=\mu}^{\nu} \prod_{i=\mu}^{\nu} e^{-a_i(v)} \left[ \sum_{j=0}^{m} b_{ij}(v)f_j(0) + \sum_{j=0}^{m} c_{ij}(v)f_j(0) + \sum_{j=0}^{m} \alpha_{ij}f_j(0) + \sum_{j=0}^{m} T_{ij}\mu_j + \sum_{j=0}^{m} \beta_{ij}f_j(0) \right] \\
&+ \sum_{j=0}^{m} S_{ij}\mu_j + \sum_{j=0}^{m} d_{ij}(v)\sigma_j(0)\Delta w_j(v) + I_i(v),
\end{align*}
\]

\[
y^0_j(k) = \sum_{v=\mu}^{\nu} e^{-a_i(v)} \left[ \sum_{j=0}^{m} \tilde{b}_{ij}(v)\tilde{f}_j(0) + \sum_{j=0}^{m} \tilde{c}_{ij}(v)\tilde{f}_j(0) + \sum_{j=0}^{m} \tilde{\alpha}_{ij}\tilde{f}_j(0) + \sum_{j=0}^{m} \tilde{T}_{ij}\mu_j + \sum_{j=0}^{m} \tilde{\beta}_{ij}\tilde{f}_j(0) \right] \\
&+ \sum_{i=0}^{n} \tilde{S}_{ji}\mu_i + \sum_{i=0}^{n} \tilde{d}_{ji}(v)\tilde{\sigma}_i(0)\Delta \tilde{w}_i(v) + \tilde{f}_j(v),
\]
From Lemma 6, it gets from the above inequality that

\[
\begin{align*}
\{x_i^0\}_{2p} & \leq \max_{1 \leq j \leq n} \sup_{k \in \mathbb{Z}} \left[ \frac{(1 - e^{-a'})}{a'(1 - e^{-a'})} \left( \sum_{j=1}^{m} \left( b_{ij}^e + e_{ij}^e \right) f_j(0) + \sum_{j=1}^{m} \left( |\alpha_{ij}| + |\beta_{ij}| \right) |f_j(0)| \right) + \sum_{j=1}^{m} \left( |T_{ij}| + |S_{ij}| \right) |\mu_j| + l_i^p \right] \right. \\
& \quad \left. + \sum_{j=1}^{m} d_{ij}' \sigma_j(0) \left[ \sum_{v=\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_i(s)} \frac{1}{a_i(v)} \right]^{\frac{1}{2} - \frac{1}{2p}} \right] \right) \right]^{1/2p} \right] \right]^{1/2p} \\
& \leq \frac{(1 - e^{-a'})}{a'(1 - e^{-a'})} \max_{1 \leq j \leq n} \left[ \sum_{i=1}^{m} \left( b_{ij}^e + e_{ij}^e + |\alpha_{ij}| + |\beta_{ij}| \right) |f_j(0)| + \sum_{i=1}^{m} \left( |T_{ij}| + |S_{ij}| \right) |\mu_j| + l_i^p + \sum_{j=1}^{m} d_{ij}' \sigma_j(0) \right]^{1/2p} \\
& \leq e_{2p} .
\end{align*}
\]

Similarly, we have

\[
\|y_i^0\|_{2p} \leq \frac{(1 - e^{-a'})}{a'(1 - e^{-a'})} \max_{1 \leq j \leq m} \left[ \sum_{i=1}^{n} \left( \bar{b}_{ji}^e + \bar{e}_{ji}^e + |\bar{\alpha}_{ji}| + |\bar{\beta}_{ji}| \right) |\bar{f}_i(0)| + \sum_{i=1}^{n} \left( |\bar{T}_{ji}| + |\bar{S}_{ji}| \right) |\bar{\mu}_j| + l_j^p + \sum_{i=1}^{n} \bar{d}_{ji}^* \bar{\sigma}_i(0) \right]^{1/2p} = \bar{e}_{2p} .
\]

By the above inequalities, it concludes

\[
\|X^0\|_{2p} \leq \max\{e_{2p}, \bar{e}_{2p}\} = \alpha_{2p} .
\]

It follows (25)–(27) that

\[
\|\Phi X_i(k) - x_i^0(k)\|_{2p} \leq \max_{1 \leq j \leq n} \sup_{k \in \mathbb{Z}} \left[ \sum_{i=1}^{m} b_{ij}^e L_i^{2p} \right]^{1/2p} \\
+ \max_{1 \leq j \leq m} \sup_{k \in \mathbb{Z}} D_i^* \left[ \sum_{i=1}^{n} e^{-a_i(s)} \frac{1}{a_i(v)} \right]^{1/2p} \\
+ \max_{1 \leq j \leq m} \sup_{k \in \mathbb{Z}} K^* \left[ \sum_{i=1}^{n} e^{-a_i(s)} \frac{1}{a_i(v)} \right]^{1/2p} ,
\]

where $L_i$, $D_i$, and $K_i$ are defined in the text.
which yields from Lemmas 2 and 3 that

\[
\left\| \left( \Phi X \right)_i (k) - x_i^0 (k) \right\|_{2p}^n \leq \max_{i \in \mathbb{N}} \sup_{k \in \mathbb{Z}} \sum_{j=1}^m b_{ij} L_j \left\{ \sum_{v=\infty}^{k-1} \prod_{s=0}^{v+1} \frac{e^{-\alpha_i (v)} \left( 1 - e^{-\alpha_i (v)} \right)}{a_i (v)} \right\}^{p-1} \times \sum_{v=\infty}^{k-1} \prod_{s=0}^{v+1} \frac{e^{-\alpha_i (v)} \left( 1 - e^{-\alpha_i (v)} \right)}{a_i (v)} E \left[ y_j (v) \right]^{2p} \right) \]

\[
+ \max_{i \in \mathbb{N}, k \in \mathbb{Z}} \sup_{j=1}^m D_i^{\ast} + \max_{i \in \mathbb{N}, k \in \mathbb{Z}} K_i^{\ast} \left\{ \sum_{v=\infty}^{k-1} \prod_{s=0}^{v+1} \frac{e^{-\alpha_i (v)} \left( 1 - e^{-\alpha_i (v)} \right)}{a_i (v)} E \left[ y_j (v - \tau_i (v)) \right]^{2p} \right) \]

\[
\leq \sum_{v=\infty}^{k-1} \prod_{s=0}^{v+1} \frac{e^{-\alpha_i (v)} \left( 1 - e^{-\alpha_i (v)} \right)}{a_i (v)} E \left[ y_j (v - \eta_j (v)) \Delta w_j (v) \right]^{2p} \right) \]

(34)

where \( D_i^{\ast} = D_i^* - \sum_{j=1}^m b_{ij} L_j \), \( i, j = 1, 2, \ldots, m \). Applying Lemma 5 to the above inequality, it derives

\[
\left\| \left( \Phi X \right)_i (k) - x_i^0 (k) \right\|_{2p}^n \leq \frac{1}{d} \left( 1 - e^{-\alpha_i} \right) \left\{ D_i^* + K_i^{\ast} C_{2p} \right) \]

\[
\cdot \left\| X \right\|_{2p} \leq r_{2p} \left\| X \right\|_{2p} \leq \frac{r_{2p} \alpha_{2p}}{1 - r_{2p}} \]

(35)

Similarly,

\[
\left\| \left( \Phi X \right)_{m+j} (k) - y_j^0 (k) \right\|_{2p}^m \leq \frac{1}{d} \left( 1 - e^{-\alpha_i} \right) \left\{ D_i^* + K_i^{\ast} C_{2p} \right) \left\| X \right\|_{2p} \]

\[
\leq r_{2p} \left\| X \right\|_{2p} \leq \frac{r_{2p} \alpha_{2p}}{1 - r_{2p}} \]

(36)

Together with the above inequalities, we obtain

\[
\left\| \Phi X \right\|_{2p} \leq \frac{r_{2p} \alpha_{2p}}{1 - r_{2p}} \]

(37)

Hence, for \( X = \{ x_i \}, y_j \in \Lambda \), it leads from (32) and (37) to

\[
\left\| \Phi X \right\|_{2p} \leq \left\| X \right\|_{2p} + \left\| \Phi X - X \right\|_{2p} \leq \alpha_{2p} \frac{r_{2p} \alpha_{2p}}{1 - r_{2p}} = \frac{\alpha_{2p}}{1 - r_{2p}} \]

(38)

From (38), \( \mathcal{B} \Lambda \) is uniformly bounded. Together with the continuity of \( \mathcal{B} \), for any bounded sequence \( \{ \phi_n \} \) in \( \Lambda \), we know that there exists a subsequence \( \{ \phi_{n_k} \} \) in \( \Lambda \) such that \( \{ \mathcal{B} (\phi_{n_k}) \} \) is convergent in \( \mathcal{B} (\Lambda) \). Therefore, \( \mathcal{B} \) is compact on \( \Lambda \). Then, condition (2) of Lemma 7 is satisfied.

The next step is proving condition (1) of Lemma 7. Now, we consist in proving the 2p-th mean almost periodicity of \( \mathcal{B} X (\cdot) \) and \( \mathcal{B} X (\cdot) \). Since \( X (\cdot) \) is a 2p-th mean almost periodic sequence and all the coefficients in system (8) are almost periodic sequences, for all \( \epsilon > 0 \) there exists \( l_\epsilon > 0 \) such that every interval of length \( l_\epsilon > 0 \) contains a \( \omega \) with the property that

\[
\left| E \left[ x_i (k + \omega) - x_i (k) \right]^{2p} \right|^{1/2p} < \epsilon,
\]

\[
\left| \alpha_i (k + \omega) - \alpha_i (k) \right| < \epsilon,
\]

\[
\left| \beta_i (k + \omega) - \beta_i (k) \right| < \epsilon,
\]

\[
\left| \gamma_i (k + \omega) - \gamma_i (k) \right| < \epsilon,
\]

\[
\left| \partial_i L_j (k + \omega) - \partial_i L_j (k) \right| < \epsilon,
\]

\[
\left| \partial_i M_j (k + \omega) - \partial_i M_j (k) \right| < \epsilon,
\]

\[
\left| \partial_i N_j (k + \omega) - \partial_i N_j (k) \right| < \epsilon,
\]

\[
\left| \partial_i (k + \omega) - \partial_i (k) \right| < \epsilon,
\]

\[
\left| \eta_j (k + \omega) - \eta_j (k) \right| < \epsilon,
\]

\[
\left| \eta_j (k + \omega) - \eta_j (k) \right| < \epsilon,
\]

\[
\left| \eta_j (k + \omega) - \eta_j (k) \right| < \epsilon,
\]

\[
\left| I_j (k + \omega) - I_j (k) \right| < \epsilon,
\]

\[
\left| J_j (k + \omega) - J_j (k) \right| < \epsilon,
\]

(39)

where \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \), and \( k \in \mathbb{Z} \). By (26), (27), and (H2), we could easily find a positive constant \( M \) such that
\[
\begin{align*}
\left\{ E \left( \frac{d}{dx} X_i (k + \omega) - \left( \frac{d}{dx} X_i \right) (k) \right)^{2p} \right\}^{1/2p} &\leq M \sup_{1 \leq n \leq N, k \in Z} \left\{ E \left| x_i (k + \omega) - x_i (k) \right|^{2p} \right\}^{1/2p} < M, \\
\left\{ E \left( \frac{d}{dx} X_{nj} (k + \omega) - \left( \frac{d}{dx} X_{nj} \right) (k) \right)^{2p} \right\}^{1/2p} &\leq M \sup_{1 \leq j \leq m, k \in Z} \left\{ E \left| y_j (k + \omega) - y_j (k) \right|^{2p} \right\}^{1/2p} < M, \\
\left\{ E \left( \frac{d}{dx} X_i (k + \omega) - \left( \frac{d}{dx} X_i \right) (k) \right)^{2p} \right\}^{1/2p} &\leq M \sup_{1 \leq n \leq N, k \in Z} \left\{ E \left| x_i (k + \omega) - x_i (k) \right|^{2p} \right\}^{1/2p} < M, \\
\left\{ E \left( \frac{d}{dx} X_{nj} (k + \omega) - \left( \frac{d}{dx} X_{nj} \right) (k) \right)^{2p} \right\}^{1/2p} &\leq M \sup_{1 \leq j \leq m, k \in Z} \left\{ E \left| y_j (k + \omega) - y_j (k) \right|^{2p} \right\}^{1/2p} < M,
\end{align*}
\]
\[\text{(40)}\]
\[\text{(41)}\]

where \(i = 1, 2, \ldots, n\), \(j = 1, 2, \ldots, m\), and \(k \in Z\). From (40) and (41), \(\mathcal{O}_X (\cdot)\) and \(\mathcal{O}_Y (\cdot)\) are 2\(p\)-th mean almost periodic processes. Further, by (38), it is easy to obtain that
\[
\mathcal{O}_X + \mathcal{O}_Y \in \Lambda, \forall X, Y \in \Lambda. \text{ Then, condition (1) of Lemma 7 holds.}
\]

Finally, \(\forall X = \{x_i; y_j\}, X^* = \{x_i^*; y_j^*\} \in \Lambda\), from (27), it yields
\[
\begin{align*}
\| (\mathcal{C}X_i (k) - (\mathcal{C}X^*_i) (k)) \|_{2p} &\leq \frac{1 - e^{-\alpha t}}{\omega} \max_{1 \leq k \leq N} \sup_{1 \leq n \leq N, k \in Z} \left\{ E \left[ \sum_{i=1}^{k-1} \prod_{j=1}^{m} d_{ij} (v) \left( \frac{\sigma_j (x_i (v) - \eta_{ij} (v))}{\alpha (1 - e^{-\omega t})} \right) \Delta w_j (v) \right]^{2p} \right\} \\
&\leq \frac{1 - e^{-\alpha t}}{\omega} \max_{1 \leq k \leq N, 1 \leq j \leq m} \sup_{k \in Z} \left\{ \left[ \sum_{i=1}^{k-1} \prod_{j=1}^{m} e^{-\omega t} \right]^{2p-1} \times \sum_{i=1}^{k-1} \prod_{j=1}^{m} e^{-\omega t} \right\} \cdot E \left[ x_i (v - \eta_{ij} (v)) - y_j (v - \eta_{ij} (v)) \right] \Delta w_j (v) \right\}^{2p} \\
&\leq \frac{K^* C_{2p}^{1/2p} (1 - e^{-\alpha t})}{\omega (1 - e^{-\omega t})} \| X - Y \|_{2p} \\
&\leq r_{2p} \| X - Y \|_{2p},
\end{align*}
\]
\[\text{(42)}\]

similarly,
\[
\begin{align*}
\| (\mathcal{C}X_{nj} (k) - (\mathcal{C}X^*_{nj}) (k)) \|_{2p}^m &\leq \frac{K^* C_{2p}^{1/2p} (1 - e^{-\alpha t})}{\omega (1 - e^{-\omega t})} \| X - Y \|_{2p}^m \\
&\leq r_{2p} \| X - Y \|_{2p}.
\end{align*}
\]
\[\text{(43)}\]

From the above inequalities, it leads
\[
\| \mathcal{O}_X - \mathcal{O}_Y \|_{2p} \leq r_{2p} \| X - Y \|_{2p}.
\]
\[\text{(44)}\]

In view of \((H_j)\), \(\mathcal{C}\) is a contraction mapping. Hence, condition (3) of Lemma 7 is satisfied. Therefore, all the conditions in Lemma 7 hold. By Lemma 7, system (8) has a 2\(p\)-th mean almost periodic sequence solution. This completes the proof. \(\square\)

According to Theorem 1, we can easily obtain the following theorem.

\textbf{Theorem 2.} Assume that all conditions in Theorem 1 hold and all coefficients in system (8) are periodic sequences, and then system (8) admits at least one 2\(p\)-th mean periodic sequence solution.

\textbf{Remark 1.} In view of the definition of \(r_{2p}\) in Theorem 1, \(a_i\) and \(\bar{a}_i\) in system (8) are bigger, then the probability of \(r_{2p} < 1\) is bigger, where \(i = 1, 2, \ldots, n\) and \(j = 1, 2, \ldots, m\). The parameters \(T_{ij}, S_{ij}, H_{ij}, \bar{T}_{ij}, \bar{S}_{ij}, \bar{H}_{ij}, I_{ij}, J_{ij}\) have no effect on the value of \(r_{2p}\), but they have effects on the boundedness of almost periodic sequence solutions of system (8), where \(i = 1, 2, \ldots, n\) and \(j = 1, 2, \ldots, m\). In order to derive the existence of almost periodic sequence solutions of system (8), we would better choose bigger \(a_i\) and \(\bar{a}_i\), and the other parameters should be smaller except \(T_{ij}, S_{ij}, H_{ij}, \bar{T}_{ij}, \bar{S}_{ij}, \bar{H}_{ij}, I_{ij}, J_{ij}\), where \(i = 1, 2, \ldots, n\) and \(j = 1, 2, \ldots, m\).

\textbf{4. 2\(p\)-th Moment Global Exponential Stability}

Suppose that \(X = \{x_i; y_j\}\) with initial value \(\varphi = \{\varphi_i; \psi_j\}\) and \(X^* = \{x_i^*; y_j^*\}\) with initial value \(\varphi^* = \{\varphi_i^*; \psi_j^*\}\) are arbitrary two solutions of system (8). For convenience, let
\begin{equation}
\gamma_{2p} = \max_{1 \leq i \leq n, 1 \leq j \leq m} \sup_{a \in [-\rho_a, 0]} \left\{ \left( E|\varphi_i(s) - \varphi_i^*(s)|^{2p} \right)^{1/2p}, \left( E|\psi_j(s) - \psi_j^*(s)|^{2p} \right)^{1/2p} \right\},
\end{equation}

\begin{equation}
\mu_0 = \max_{(i,j)} \left\{ a_{ij}^m, \eta_{ij}^m, \tau_{ij}^m, \sigma_{ij}^m \right\}.
\end{equation}

Definition 4.1 [35]. System (8) is said to be $2p$-th moment global exponential stability if there are positive constants $k_0$, $M$, and $\lambda$ such that

\begin{equation}
|X(k) - X^*(k)|_{2p} = \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \left( E|x_i(k) - x_i^*(k)|^{2p} \right)^{1/2p}, \left( E|y_j(k) - y_j^*(k)|^{2p} \right)^{1/2p} \right\} < M\gamma_{2p} e^{-\lambda k},
\end{equation}

where $\forall k > k_0$, $k \in \mathbb{Z}$. The 2-nd moment global exponential stability is called square-mean global exponential stability.

\begin{equation}
|x_i(k) - x_i^*(k)| \leq \sum_{s=0}^{k-1} e^{-a_i(s)} |\varphi_i(0) - \varphi_i^*(0)| + \frac{1 - e^{-a_i}}{a_i} \sum_{s=0}^{k-1} \prod_{i=1}^{k-1} e^{-a_i(s)} \sum_{j=1}^{n} \left( b_{ij}^L \psi_j(v) - y_j^*(v) \right) + \left( \epsilon_{ij} + |\alpha_{ij}| + |\beta_{ij}| \right) L_i \left[ y_j(v - \zeta_{ij}(v)) - y_j^*(v - \zeta_{ij}(v)) \right] + d_{ij}^L \left| y_j(v - \eta_{ij}(v)) - y_j^*(v - \eta_{ij}(v)) \right| \Delta w_j(v),
\end{equation}

\begin{equation}
|y_j(k) - y_j^*(k)| \leq \sum_{s=0}^{k-1} e^{-\alpha_j(s)} |\varphi_j(0) - \varphi_j^*(0)| + \frac{1 - e^{-\alpha_j}}{\alpha_j} \sum_{s=0}^{k-1} \prod_{i=1}^{k-1} e^{-\alpha_j(s)} \sum_{i=1}^{n} \left( \widetilde{b}_{ji}^L \psi_i(v) - x_i^*(v) \right) + \left( \omega_{ij} + |\bar{\alpha}_{ij}| + |\bar{\beta}_{ij}| \right) L_i \left[ x_j(v - \bar{\tau}_{ji}(v)) - x_j^*(v - \bar{\tau}_{ji}(v)) \right] + d_{ji}^L \left| x_j(v - \bar{\eta}_{ji}(v)) - x_j^*(v - \bar{\eta}_{ji}(v)) \right| \Delta w_i(v),
\end{equation}

**Theorem 3.** Assume that $(H_1)$–$(H_3)$ hold, and then system (8) is $2p$-th moment globally exponentially stable.

**Proof.** From Lemmas 1 and 6, it follows that

\begin{equation}
|x_i(k) - x_i^*(k)| \leq \sum_{s=0}^{k-1} e^{-a_i(s)} |\varphi_i(0) - \varphi_i^*(0)| + \frac{1 - e^{-a_i}}{a_i} \sum_{s=0}^{k-1} \prod_{i=1}^{k-1} e^{-a_i(s)} \sum_{j=1}^{n} \left( b_{ij}^L \psi_j(v) - y_j^*(v) \right) + \left( \epsilon_{ij} + |\alpha_{ij}| + |\beta_{ij}| \right) L_i \left[ y_j(v - \zeta_{ij}(v)) - y_j^*(v - \zeta_{ij}(v)) \right] + d_{ij}^L \left| y_j(v - \eta_{ij}(v)) - y_j^*(v - \eta_{ij}(v)) \right| \Delta w_j(v),
\end{equation}

\begin{equation}
|y_j(k) - y_j^*(k)| \leq \sum_{s=0}^{k-1} e^{-\alpha_j(s)} |\varphi_j(0) - \varphi_j^*(0)| + \frac{1 - e^{-\alpha_j}}{\alpha_j} \sum_{s=0}^{k-1} \prod_{i=1}^{k-1} e^{-\alpha_j(s)} \sum_{i=1}^{n} \left( \widetilde{b}_{ji}^L \psi_i(v) - x_i^*(v) \right) + \left( \omega_{ij} + |\bar{\alpha}_{ij}| + |\bar{\beta}_{ij}| \right) L_i \left[ x_j(v - \bar{\tau}_{ji}(v)) - x_j^*(v - \bar{\tau}_{ji}(v)) \right] + d_{ji}^L \left| x_j(v - \bar{\eta}_{ji}(v)) - x_j^*(v - \bar{\eta}_{ji}(v)) \right| \Delta w_i(v),
\end{equation}
where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k \in [-\mu_0, +\infty)$. For convenience, let
\begin{align*}
a_0 &= 1 - e^{-\alpha u} \\
\bar{a}_0 &= 1 - e^{-\alpha u} \tag{49}
\end{align*}

Similar to the argument as that in (37), it gets from (47) that

\begin{equation}
|Z(k)|_{2p} \leq e^{-d_k} \gamma_{2p} + \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 k \bar{p}_{ij} L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

\begin{equation}
+ \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

\begin{equation}
+ \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

\begin{equation}
\leq e^{-d_k} \gamma_{2p} + \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

\begin{equation}
\quad + \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

\begin{equation}
\quad + \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

Similarly, it follows from (48) that

\begin{equation}
|W(k)|_{2p} \leq e^{-d_k} \gamma_{2p} + \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

\begin{equation}
\quad + \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

\begin{equation}
\quad + \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

\begin{equation}
\quad + \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

\begin{equation}
\quad + \max_{1 \leq j \leq m} \sum_{i=1}^m a_0 \left( c_{ij} + \alpha_{ij} + \beta_{ij} \right) L_j \left\{ \sum_{s=0}^{k-1} e^{-d(s-k-1)} \left[ \Delta w_j(s) \right]_{2p} \right\}^{1/2p} \nonumber
\end{equation}

Be aware of $(H_2)$ in Theorem 1, there exists a constant $\lambda > 0$ small enough such that
\[
\max_{1 \leq i \leq n} \sum_{j=1}^{n} e^{1} a_{ij} \left[ b_{ij} L_{j}^f + e^{\mu_{j}} \left( c_{ij} + |\alpha_{ij}| + |\beta_{ij}| \right) L_{j}^f + e^{\mu_{j}} C_{2p}^{1/2p} d_{ij}^{p} L_{j}^f \right] \leq \rho \leq 1, \tag{52}
\]

\[
\max_{1 \leq i \leq m} \sum_{j=1}^{m} e^{1} a_{ij} \left[ b_{ij} L_{j}^f + e^{\mu_{j}} \left( c_{ij} + |\alpha_{ij}| + |\beta_{ij}| \right) L_{j}^f + e^{\mu_{j}} C_{2p}^{1/2p} d_{ij}^{p} L_{j}^f \right] \leq \rho \leq 1.
\]

Define \( V(k) = \max \left\{ |Z(k)|_{2p}, |W(k)|_{2p} \right\} \), \( k \in \mathbb{Z} \).

Next, we claim that there exists a constant \( M_0 > 1 \) such that

\[
V(k) \leq M_0 y_{2p} e^{-\lambda_k}, \quad \forall k \in [-\mu_0, +\infty). \tag{53}
\]

If (53) is invalid, then there must exist \( k_0 \in (0, +\infty) \) such that one of the following two cases holds:

\[
|Z(k_0)|_{2p} \leq e^{-d k_0} y_{2p} + \max \left\{ \sum_{j=1}^{m} a_{ij} b_{ij} L_{j}^f M_0 y_{2p} \left[ \sum_{s=0}^{k_0-1} e^{-d (k-s-1)} \right] \sum_{s=0}^{k_0-1} e^{-d (k-s-1) e^{-2p k_0}} \right\}^{1/2p}
\]

\[
+ \max \left\{ \sum_{j=1}^{m} a_{ij} M_0 y_{2p} \left[ c_{ij} + |\alpha_{ij}| + |\beta_{ij}| \right] L_{j}^f + C_{2p}^{1/2p} d_{ij}^{p} L_{j}^f \right\}^{1/2p}
\]

\[
\leq \sum_{s=0}^{k_0-1} e^{-d (k-s-1)} \left( M_0 y_{2p} \right) \left[ b_{ij} L_{j}^f + e^{\mu_{j}} \left( c_{ij} + |\alpha_{ij}| + |\beta_{ij}| \right) L_{j}^f + e^{\mu_{j}} C_{2p}^{1/2p} d_{ij}^{p} L_{j}^f \right] \left[ 1 - e^{-d (k-s-1) e^{-2p k_0}} \right]^{1/2p}
\]

\[
\leq \sum_{s=0}^{k_0-1} e^{-d (k-s-1)} \left( M_0 y_{2p} \right)^{1/2p} \left[ 1 - e^{-d (k-s-1) e^{-2p k_0}} \right]^{1/2p}
\]

\[
\leq M_0 y_{2p} e^{-\lambda_k}
\]

\[
+ \max \left\{ \sum_{j=1}^{m} a_{ij} b_{ij} L_{j}^f + e^{\mu_{j}} \left( c_{ij} + |\alpha_{ij}| + |\beta_{ij}| \right) L_{j}^f + e^{\mu_{j}} C_{2p}^{1/2p} d_{ij}^{p} L_{j}^f \right\} \left[ 1 - e^{-d (k-s-1) e^{-2p k_0}} \right]^{1/2p}
\]

\[
\leq M_0 y_{2p} e^{-\lambda_k} \left\{ e^{-d (k-s-1) k_0} + \rho \left[ 1 - e^{-d (k-s-1) k_0} \right] \right\}
\]

\[
\leq M_0 y_{2p} e^{-\lambda_k}.
\]
In the fourth inequality from the bottom of (54), we use the fact \(1 - e^{-\alpha k} \leq 1 - e^{-\alpha k_0}\), and then system (8) admits a 2\(p\)-th moment globally exponentially stable. This completes the proof. 

According to Theorems 1 and 2, we can easily obtain the following theorem.

**Theorem 4.** Assume that all conditions in Theorem 1 hold, and then system (8) admits a 2\(p\)-th mean almost periodic sequence solution, which is 2\(p\)-th moment globally exponentially stable. Further, if all coefficients in system (8) are periodic sequences, then system (8) admits at least one 2\(p\)-th mean periodic sequence solution, which is globally exponentially stable.

**Proof.** The result can be easily obtained by Theorem 3, so we omit it. This completes the proof. 

**Remark 2.** Assume that \(X(k) = (x_1(k), x_2(k), \ldots, x_n(k))\) is a solution of (1), and then the length of \(X(k)\) is usually measured by \(\|X\|_\infty = \sup_{k \in Z, \xi \in X} \max |x_i(k)|\). However, if \(X(k)\) is a solution of stochastic system, its length should not be measured by \(X_{\infty}\), because \(X(k)\) is a random variable. In this paper, we use norm \(\|X\|_{2p} = \max_{k \in Z} \sup_{\xi \in X} |x_i(k)|\). Owing to the expectation \(E\) and order \(p\) in \(\|X\|_{2p}\), the computing processes of this paper are more complicated than those in literatures [41–47]. It is worth mentioning that Minkowski inequality in Lemma 2 and Hölder inequality in Lemma 3 are crucial to the computing processes. The facts above are obvious from the computations of (32), (42), (50), and (54) in the proofs of Theorems 1 and 3. Further, the stochastic terms \(d_{ij} \sigma_j \Delta w_j\) and \(\tilde{d}_{ij} \tilde{\sigma}_j \tilde{\Delta} \tilde{w}_j\) in system (8) also increase the complexity of computing. This point is also clear from the computations of (42) and (50).

**5. Examples and Computer Simulations**

**Example 5.1.** Consider the following continuous-time BAM neural networks with random perturbation:

\[
\begin{cases}
    dx(t) = [-x(t) + 0.01 \sin(\sqrt{5} t) \sin(y(t))] dt + 0.05 \cos(\sqrt{7} t) \cos(y(t - 1)) dt + 0.01 \Delta w(t), \\
    dy(t) = [-0.2 y(t) + 0.015 \sin(\sqrt{21} t) \cos(x(t - 1))] dt + 0.03 \cos(\sqrt{23} t) \Delta w(t), \\
\end{cases}
\]

**5.1. Semidiscrete Model.** Based on model (55), we obtain the following semidiscrete model by using the semidiscretization technique:

\[
\begin{align*}
    x(k + 1) &= e^{-1} x(k) + (1 - e^{-1}) [0.01 \sin(\sqrt{5} k) \sin(y(k)) \\
                   &+ 0.05 \cos(\sqrt{7} k) \cos(y(k - 1)) + 0.01 \Delta w(k)], \\
    y(k + 1) &= e^{-0.2} y(k) + \frac{1 - e^{-0.2}}{0.2} \\
             &+ 0.03 \cos(\sqrt{23} k) \Delta w(k),
\end{align*}
\]

where \(k \in \mathbb{Z}\).

**Remark 3.** In literature [44], Huang et al. studied model (1) and obtained some sufficient conditions for the existence of a unique almost periodic sequence solution which is globally attractive. In [45], they considered system (2) and studied the dynamics of \(2^N\) almost periodic sequence solutions. But neither of them considered the random factors. Such as, the obtained results in [44, 45] cannot be applied to the study for stochastic model (56). Therefore, the work in this paper complements the corresponding results in [44, 45].

**5.2. Discrete Model Formulated by the Euler Scheme.** Based on model (55), we obtain the following discrete-time model by using the Euler method:

\[
\begin{align*}
    x(k + 1) &= 0.01 \sin(\sqrt{5} k) \sin(y(k)) \\
             &+ 0.05 \cos(\sqrt{7} k) \cos(y(k - 1)) + 0.01 \Delta w(k), \\
    y(k + 1) &= 0.8 y(k) + 0.15 \sin(\sqrt{21} k) \cos(x(k - 1)) \\
             &+ 0.03 \cos(\sqrt{23} k) \Delta w(k),
\end{align*}
\]

where \(k \in \mathbb{Z}\).

In Figures 1 and 2, we give two plots of numerical solutions which are produced by continuous-time model (55), semidiscrete model (56), and Euler-discretization model (57), respectively. Compared with Euler-discretization model (57), semidiscrete model (56) gives a more accurate characterization for continuous-time model (55).

**Remark 4.** In literature [51–54], the authors discussed the dynamics of periodic solutions of discrete-time cellular neural networks formulated by the Euler scheme. From the above discussion, semidiscrete stochastic system (8) gives a more accurate and realistic formulation for studying the dynamics of discrete-time cellular neural networks. In a way, the work of this paper complements and improves some corresponding results in [51–54].
Corresponding to system (8), we have 
\[ a' = a^u = 1, \]
\[ \bar{a}' = \bar{a}^u = 1, \]
\[ L_f = L_f' = L_f'' = 1, \]
\[ \bar{b}_{ij} = 0.01, \quad \bar{c}_{ij} = 0.05, \]
\[ \alpha_{11} = \alpha_{12} = 0.1, \quad \beta_{11} = \beta_{12} = 0.02, \quad \alpha_{21} = \alpha_{22} = 0.04, \quad \beta_{21} = \beta_{22} = 0.2, \]
\[ d_{ij} = 0.02, \quad i, j = 1, 2. \]
Taking \( p = 2 \), by simple calculation,
\[ C_4^{1/4} = 4, \]
\[ D^* \approx 0.72, \]
\[ K^* \approx 0.04, \]
\[ r_4 \approx 0.88 < 1. \]

According to Theorems 1 and 3, system (56) admits a 4-th mean almost periodic sequence solution, which is 4-th moment globally exponentially stable (see Figures 3–5).

Figure 3 depicts a numerical solution \((x, y)\) of semidiscrete stochastic model (56). Observe that the trajectories of \((x, y)\) demonstrate almost periodic oscillations. Figures 4 and 5 display three numerical solutions of semidiscrete stochastic model (56) at different initial values \((1.5, 2.5)\), \((2.5, 1.5)\), and \((0.2, 0.3)\), respectively. They are shown that semidiscrete stochastic model (56) is 4-th moment globally exponentially stable.

Example 5.2. Consider the corresponding determinant model of system (56) as follows:
Figure 4: 4-th moment global exponential stability of state variable $x$ of model (56).

Figure 5: 4-th moment global exponential stability of state variable $y$ of model (56).

Figure 6: Comparison of state variable $x$ between stochastic model (56) and determinant model (59).
\[ x(k+1) = e^{-1}x(k) + (1 - e^{-1})(0.01 \sin(\sqrt{5}k) \sin(y(k))) + 0.05 \cos(\sqrt{7}k) \cos(y(k-1)) + 0.01, \]
\[ y(k+1) = e^{-0.2}y(k) + \frac{1 - e^{-0.2}}{0.2}[0.15 \sin(\sqrt{21}k) \cos(x(k-1)) + 0.03 \cos(\sqrt{23}k)], \]

where \( k \in \mathbb{Z} \). In Figures 6 and 7, we give the results of contrast between stochastic model (56) and determinant model (59). Figures 6 and 7 indicate that the effects of stochastic perturbation on state variables \( x \) and \( y \) are significant. And the stochastic influence on state variable \( y \) is more obvious than that on state variable \( x \). In Figure 8, we give a result of globally exponentially stable contrast between stochastic model (56) and determinant model (59). Figure 8 reveals that the convergence speed of stochastic model (56) is faster than determinant model (59).
Remark 5. From example 5.2, stochastic disturbance brings a positive effect on the global exponential stability of the models.

6. Conclusions and Future Works

In this paper, we formulate a discrete analogue of BAM neural networks with stochastic perturbations and fuzzy operations by using semidiscretization technique. The existence of $2p$-th mean almost periodic sequence solutions and $2p$-th moment global exponential stability for the above models are investigated with the help of Krasnoselskii’s fixed point theorem and stochastic theory. The main results obtained in this paper are completely new, and the methods used in this paper provide a possible technique to study $2p$-th mean almost periodic sequence solution and $2p$-th moment global exponential stability of semidiscrete models with stochastic perturbations and fuzzy operations.

With a careful observation of Theorems 1 and 3, it is not difficult to discover that

1. The methods used in this paper can be applied to study other types of neural networks, such as impulsive neural networks, high-order neural networks on time scales, etc.
2. Other types of fuzzy neural networks could be investigated, such as Takagi-Sugeno fuzzy neural networks
3. Other dynamic behaviours of system (8) should be further discussed
4. The case of $2p \in (0, 1]$ could be further explored

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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