Research Article

Asymptotic Analysis and Error Estimate for Rosenau-Burgers Equation

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In this work, the asymptotic stability result for Rosenau-Burgersequation is established, under appropriate assumptionson steady state eigenvalue problemand the forcing function. In addition, we propose and analyzealinearized numerical method for solving this nonlinear Rosenau-Burgers equation. We prove that the numerical scheme is unconditionally stable, and the error estimate showsthat thenumerical method is in the order of $O(\Delta t^2 + N^{-m})$, where $\Delta t$, $N$, and $m$ are, respectively, step of time, polynomial degree, and regularity of $u$. Numerical examples are illustrated to verify the theoretical results.

1. Introduction

In the research of dynamic dense discrete system, it was shown that the KdV equation can not completely describe the interaction between waves and waves, and in order to overcome the shortcomings of this KdV equation, Rosenau [1, 2] proposed the following Rosenau equation:

$$u_t + u_{xxxx} + u_x + uu_x = 0. \quad (1)$$

For further consideration of dissipation in dynamic system, the term $-\alpha u_{xx}$ is included in the above equatoin. The resulting new equation is called the Rosenau-Burgers equation:

$$u_t + u_{xxxx} - \alpha u_{xx} + u_x + uu_x = 0. \quad (2)$$


Besides the theoretical analysis, some recent contributions focus on using numerical methods to approximate the solutions of Rosenau-Burgers equation. Chung [10] & Sank [11] introduced finite element Galerkin method for solving a Rosenau equation. They obtained the existence and uniqueness of solutions and the error estimates of the solutions are also discussed. Chung [12], Omrani [13], and Feng [14] presented a conservative finite difference scheme for solving Rosenau equation. Convergence, stability, and error estimate of full discrete scheme are also proved. Manickam et al. [15] applied orthogonal cubic spline collocation method to approximate Rosenau equation, and the error estimates are obtained in both $L^2$ norm and $L^\infty$ norm.

In this paper, we consider the following Rosenau-Burgers equation:

$$u_t + u_{xxxx} - \alpha u_{xx} + u_x + uu_x = f(x,t), \quad (3)$$

$$u(x,0) = u_0(x), \quad (4)$$

$$u(x,t) = u(x + 2L,t), \quad (5)$$

where $\alpha > 0$.

In this article, the convergence of the Rosenau-Burgers equation to its steady-state problem is discussed, under the assumption that the corresponding linearized steady-state eigenvalue problem has a positive minimum eigenvalue and
appropriate condition on the forcing function. Moreover, we present numerical method to solve Rosenau-Burgers equation, and the proposed scheme is performed by combining Crank-Nicolson approach in time and Fourier-spectral in space. Our rigorous analysis result shows that the scheme is unconditionally stable, and the numerical method leads to second order in time and spectral accuracy in space.

The rest of the article is organized in the following way. Section 2 will study asymptotic behaviour of Rosenau-Burgers equation. In Section 3, we will discuss stability and error estimate for the full discrete scheme. In Section 4, we present some numerical experiments to illustrate the validity of the numerical method. The conclusions of this paper are given in Section 5.

2. Asymptotic Behaviour of Rosenau-Burgers Equation

In this section, we will investigate the asymptotic behavior of the solution as \( t \to \infty \). Let \( \lim_{t \to \infty} u(x, t) = u^\infty(x) \), where \( u^\infty \) is the steady state solution of (3)-(5), satisfying

\[
-\alpha u_{xx} + u_x + u^\infty u_x = f^\infty, \quad t \in (0, \infty), \tag{6}
\]

where \( u_{xx}^\infty = \lim_{t \to \infty} u_{xx}(x, t), \quad u_{xx}^\infty = \lim_{t \to \infty} u_{xx}(x, t), \quad f^\infty(x) = \lim_{t \to \infty} f(x, t). \tag{7}
\]

Assume the eigenvalue problem

\[
-\alpha \phi_{xx} + \phi_x + u_{xx}^\infty \phi = \lambda \phi, \quad x \in (0, \infty), \tag{8}
\]

\[
\phi(x) = \phi(x + 2L), \tag{9}
\]

has a minimum eigenvalue \( \lambda_0 > 0 \).

Note that, for any \( \phi \in H^2 \cap H_1^p, \) it holds that

\[
\alpha \| \phi_x \|^2 + \int_0^{2L} u_{xx}^\infty \phi_x^2 \, dx = \lambda \| \phi \|^2 \geq \lambda_0 \| \phi \|^2. \tag{10}
\]

Then, we have

\[
(\alpha \phi_{xx} + \phi_x + u_{xx}^\infty \phi_x + u_{xx}^\infty \phi, \phi) = \alpha \| \phi_x \|^2 + \frac{1}{2} \int_0^{2L} u_{xx}^\infty \phi_x^2 \, dx,
\]

\[
\geq \frac{\lambda_0}{2} \| \phi_x \|^2 + \frac{\alpha}{2} \| \phi_x \|^2 \geq \frac{\alpha}{2} \| \phi_x \|^2. \tag{11}
\]

Poincaré inequality [16]: For any \( \phi \in H_1^p \), there holds

\[
\| \phi_x \|^2 \leq (1/\sqrt{\lambda_1}) \| \phi_{xx} \|^2, \quad k = 0, 1, \quad \text{where} \lambda_1 = \pi^2/L^2 \text{ is the minimum eigenvalue of the eigenvalue problem:} -\phi_{xx} = \lambda \phi, \quad \phi(0) = u(2L).
\]

Let \( z = u - u^\infty \), from (3)-(5) and (6)-(7), we have

\[
z_t - z_{xxxx} + \alpha z_{xx} + z_x + u^\infty z_x + z u_{xx}^\infty = F, \tag{12}
\]

\[
z(x, 0) = u_0 - u^\infty = z_0, \tag{13}
\]

\[
z(x, t) = z(x + 2L, t), \tag{14}
\]

where \( F = f - f^\infty \).

The weak formulation of (12)-(14) is to find \( z(t) \in H^2_{per}(R), \) such that

\[
(\partial_t z, v) + (z_{xx}, v_{xx}) + \alpha (z_x, v_x) + (z, v) + (z_x, v) + (u^\infty z_x + zu_{xx}^\infty) = (F, v), \quad \forall v \in H^2_{per}, \tag{15}
\]

\[
(A2) \ u_0 \in H^2_{per}, \text{ and } \int_0^T \| F(s) \|^2 \, ds \leq M, M > 0.
\]

Theorem 1. Under assumptions (A1) and (A2), there holds

\[
\| z(t) \|^2_{H^2_{per}} \leq \frac{2}{\alpha} \left[ \frac{L^2}{\pi^2} + \frac{L^4}{\pi^4} \right] \int_0^t \| F(s) \|^2 \, ds + \frac{\pi^2 + L^2}{\pi^2} \| z_0 \|^2_{H^2_{per}}, \tag{16}
\]

moreover, for given \( \varepsilon > 0 \), there exists \( T > 0 \), such that

\[
\| z(t) \|^2 + \frac{\pi^2}{\varepsilon^2} \leq \varepsilon, \quad \forall t > T. \tag{17}
\]

Proof. Set \( v = z \) in (15), and we obtain

\[
\frac{d}{dt} \left( \| z \|^2 + \| z_{xx} \|^2 \right) + \alpha \| z_x \|^2 \leq 2 \sqrt{\lambda_1} \| F \| \| z_x \| \leq \frac{2}{\alpha \lambda_1} \| F \|^2 + \frac{\alpha}{2} \| z_x \|^2. \tag{19}
\]

Now, integrating with respect to time from 0 to t, we obtain

\[
\| z(t) \|^2 + \| z_{xx} \|^2 + \frac{\alpha}{2} \int_0^t \| z_x (s) \|^2 \, ds \leq \frac{2}{\alpha \lambda_1} \int_0^t \| F(s) \|^2 \, ds + \| z_0 \|^2 + \| z_{0xx} \|^2. \tag{20}
\]

Applying the Poincaré inequalities, we obtain

\[
\| z(t) \|^2 + \| z_{xx} \|^2 = \| z(t) \|^2 + \frac{L^2}{\pi^2 + L^2} \| z_{xx} \|^2 + \frac{\pi^2}{\pi^2 + L^2} \| z_x \|^2 \geq \frac{\pi^2}{\pi^2 + L^2} \| z_{xx} \|^2 \geq \frac{\pi^2}{\pi^2 + L^2} \| z(t) \|^2_{H^2_{per}}. \tag{21}
\]

Substituting (21) into (20), we have

\[
\| z(t) \|^2_{H^2_{per}} \leq \frac{2}{\alpha} \left[ \frac{L^2}{\pi^2} + \frac{L^4}{\pi^4} \right] \int_0^t \| F(s) \|^2 \, ds + \frac{\pi^2 + L^2}{\pi^2} \| z_0 \|^2_{H^2_{per}}. \tag{22}
\]
On the other hand, using Poincaré inequality again, we have
\[
\frac{\alpha}{2} \int_0^t \| z_x(s) \|^2 \, ds = \frac{\alpha}{2} \int_0^t \frac{\pi^4}{\pi^4 + L_2^4} \| z_x(s) \|^2 + \frac{L_4}{\pi^4 + L_2^4} \| z_x(s) \|^2 \, ds
\]
\[
\geq \frac{\alpha}{2} \int_0^t \left( \| z(s) \|^2 + \frac{\pi^2}{L_2^2} \| z_x(s) \|^2 \right) \, ds
\]
where \( \gamma = \alpha \pi^2 L_2^2 / (2 \pi^4 + L_4^2) \). Substituting (23) into (20) yields
\[
\| z(t) \|^2 + \frac{\pi^2}{L_2^2} \| z_x(t) \|^2 + \gamma \int_0^t \left( \| z(s) \|^2 + \frac{\pi^2}{L_2^2} \| z_x(s) \|^2 \right) \, ds
\]
\[
\leq \frac{2}{\alpha \lambda_1} \left( \int_0^t F(s) \, ds \right) + \| z_0 \|^2_{H^1_\text{ref}}.
\]
That is,
\[
\lim_{t \to \infty} \left( \| z(t) \|^2 + \frac{\pi^2}{L_2^2} \| z_x(t) \|^2 \right) = 0.
\]
Hence, we finish the proof of (17).

\[
\square
\]

3. Time-Discrete and Error Analysis

In this section, we present the semidiscrete scheme for the solution of (3)-(5). First, we introduce a Crank-Nicholson method to discrete time. Let \( K \) be a positive integer, \( \Delta t = T/K \) is the time step, and \( t_n = n \Delta t, \; n = 0, 1, \ldots, K-1 \) are the mesh points.

Consider the following time-discrete scheme:
\[
\frac{u^{n+1} - u^n}{\Delta t} + \partial_t u^{n+1/2} - \frac{1}{\Delta t} \left( \partial_x^2 u^{n+1} - \partial_x^2 u^n \right)
\]
\[
+ \frac{1}{6} \left( 2 \partial_t u^{n+1/2} \left( 3u^n - u^{n-1} \right) \right)
\]
\[
+ u^{n+1/2} \partial_x \left( 3u^n - u^{n-1} \right) - \alpha \partial_x^2 u^{n+1/2} = 0,
\]
\[
n \geq 1.
\]

We have the following stability result.

Theorem 2. The semidiscrete scheme (26) is unconditionally stable, such that
\[
E(u^{n+1}) \leq E(u^n), \; n = 0, 1, \ldots, K-1,
\]
where
\[
E(u^n) = \| u^n \|^2 + \| \partial_x^2 u^n \|^2.
\]

Proof. Taking the inner product of (26) with \( 2 \Delta t u^{n+1/2} \), we get
\[
\| u^{n+1} \|^2_0 - \| u^n \|^2 + \| \partial_x^2 u^{n+1} \|^2 - \| \partial_x^2 u^n \|^2
\]
\[
+ 2 \Delta t \| \partial_t u^{n+1/2} \|^2 = 0.
\]
That is,
\[
\| u^{n+1} \|^2 + \| \partial_x^2 u^{n+1} \|^2 \leq \| u^n \|^2 + \| \partial_x^2 u^n \|^2.
\]
This concludes the proof.

We will consider Fourier-Galerkin spectral method for the discretization equations (27). We will present some error estimate for full-discretization schemes. First, let us define
\[
S_N = \left\{ u \mid u(x) = \sum_{|k| \leq N/2} \tilde{u}_k \phi_k, \phi_k = e^{-ikx} \right\}.
\]

Denote \( \pi_N : L^2 \rightarrow S_N \) to be the \( L^2 \)-projection operator which satisfies
\[
(\pi_N \phi - \phi, \psi) = 0, \quad \forall \psi \in S_N.
\]

We also define the \( H^2 \)-projection operator \( \pi_N^H : H^2 \rightarrow S_N \) by
\[
(\partial_x^2 \pi_N \phi, \partial_x^2 \psi) = 0, \quad \forall \psi \in S_N.
\]

We have the following estimate [17]:
\[
\| \phi - \pi_N \phi \| \leq cN^{-m} \| \phi \|_m, \quad \forall \phi \in H^m, \; m \geq 0.
\]

Consider the full-discretization Fourier-Galerkin spectral method to (26) as follows: find \( u^{n+1}_N \in S_N \), such that
\[
\frac{1}{\Delta t} \left( u^{n+1}_N - u^n_N, \psi_N \right) + \left( \partial_t u^{n+1/2}_N, \psi_N \right) - \frac{1}{\Delta t} \left( \partial_x^2 u^{n+1}_N - \partial_x^2 u^n_N \right)
\]
\[
- \partial_x^2 u^n_N, \psi_N \right) + \frac{1}{6} \left( 2 \partial_t u^{n+1/2}_N \left( 3u^n_N - u^{n-1}_N \right) \right) \psi_N
\]
\[
+ u^{n+1/2}_N \partial_x \left( 3u^n_N - u^{n-1}_N \right) - \alpha \partial_x^2 u^{n+1/2}_N, \psi_N \right) = 0,
\]
\[
n \geq 1, \; \psi \in S_N.
\]

Theorem 3. Let \( u^{n+1}_N \mid_{n=1}^{M-1} \) be the solutions of (26), and we derive that
\[
\| u^{n+1}_N \|^2 + \| \partial_x^2 u^{n+1}_N \|^2 \leq \| u^0_N \|^2 + \| \partial_x^2 u^0_N \|^2.
\]

We denote the truncation error \( r^{n+1/2}_1(x) = r^{n+1/2}_1(x) + r^{n+1/2}_2(x) \), here,
\[
r^{n+1/2}_1(x) = \frac{u(x, t_{n+1}) - u(x, t_n)}{\Delta t} - \partial_t u(x, t_{n+1/2}),
\]
\[
r^{n+1/2}_2(x) = \frac{\partial_x^2 u(x, t_{n+1}) - \partial_x^2 u(x, t_n)}{\Delta t}
\]
\[
- \partial_x^2 \partial_t u(x, t_{n+1/2}).
\]
From Taylor expansion, we have
\[
\left\| \frac{u_{n+1/2} - u_n}{\Delta t} \right\|_0^2 \leq c \Delta t^2.
\]
(38)

We also define the following error functions:
\[
e^d_n = \pi x_n (t_n) - u_n^e, \\
e^n = u(t_n) - \pi N u(t_n), \\
e^n = e^n + e^d.
\]
(39)

We show the error estimate of the full-discretization problem (35) in the following theorem.

**Theorem 4.** Suppose \(u(x, t)\) is the exact solution of (3)-(5) and \(\{u_n^{n+1}\}_{n=1}^{M-1}\) are the solutions of (38); then, we have
\[
E(u(\cdot, t_n) - u_n^e) \leq c (\Delta t^2 + N^{2-m}).
\]
(40)

**Proof.** Subtracting (35) from a reformulation of (3) at \(t_{n+1/2}\), we obtain
\[
E(\pi x_n (t_n) - u_n^e) + \left( \begin{array}{c}
\pi x_n (t_n) - u_n^e \\
\partial_x u(t_{n+1/2}) \\
u_n^{n+1/2} \partial_x \left( 3 u_n^e - u_n^{n-1} \right)
\end{array} \right) + \frac{1}{\Delta t} \left( \begin{array}{c}
\partial_x u(t_{n+1/2}) \\
u_n^{n+1/2} \partial_x \left( 3 u_n^e - u_n^{n-1} \right)
\end{array} \right) = \frac{1}{\Delta t} \left( \begin{array}{c}
\pi x_n (t_n) - u_n^e \\
u_n^{n+1/2} \partial_x \left( 3 u_n^e - u_n^{n-1} \right)
\end{array} \right) + \frac{1}{\Delta t} \left( \begin{array}{c}
\pi x_n (t_n) - u_n^e \\
u_n^{n+1/2} \partial_x \left( 3 u_n^e - u_n^{n-1} \right)
\end{array} \right).
\]
(41)

Using Cauchy-Schwarz and Young's inequality, we have
\[
Q_1 \leq \Delta t \left\| \frac{\pi x_n (t_n) - u_n^e}{\gamma N} \right\|^2 + \Delta t \left\| \frac{\pi x_n (t_n) - u_n^e}{\gamma N} \right\|^2 + 2 \Delta t \left\| \pi x_n (t_n) - u_n^e \right\|^2,
\]
(43)

\[
Q_2 \leq \Delta t \left\| \pi x_n (t_n) - u_n^e \right\|^2 + \Delta t \left\| \pi x_n (t_n) - u_n^e \right\|^2,
\]
(44)

where
\[
D_1 = \frac{1}{3} u(t_{n+1/2}) \partial_x u(t_{n+1/2}) - \frac{1}{3} (3 u_n^e - u_n^{n-1}) \partial_x u \left( \begin{array}{c}
u_n^{n+1/2} \partial_x \left( 3 u_n^e - u_n^{n-1} \right)
\end{array} \right),
\]
(45)

\[
D_2 = \frac{1}{3} u(t_{n+1/2}) \partial_x u(t_{n+1/2}) - \frac{1}{3} (3 u_n^e - u_n^{n-1}) \partial_x u \left( \begin{array}{c}
u_n^{n+1/2} \partial_x \left( 3 u_n^e - u_n^{n-1} \right)
\end{array} \right),
\]
(46)
Applying Taylor expansion and Young’s inequality,

\[
\frac{1}{3} \| D_2 \|^2 \leq \Delta t^3 \| \partial_x u \|^2_{\infty} \int_{t_n}^{t_{n+1}} \| \partial_t^3 u \|^2 dt + \Delta t^3 \| \partial_x u \|^2_{\infty} \int_{t_n}^{t_{n+1}} \| \partial_t \partial_x^2 u \|^2 dt \\
\leq c \left( \Delta t^4 + \| \partial_x e_N \|^2 + \| \partial_t \partial_x e_N \|^2 \right).
\]

Then, we obtain

\[
E(\tilde{e}^{n+1}) - E(\tilde{e}^n) \leq \Delta t \| \partial_t e_N \|_{\infty}\| \partial_t e_N \|_{\infty}^2 + 2 \Delta t \| \partial_t \partial_x e_N \|_{\infty}^2 \\
+ \int_{t_n}^{t_{n+1}} \| (\pi_N - I) \partial_x u \|^2 dt + \int_{t_n}^{t_{n+1}} \| (\pi_N - I) \partial_t \partial_x u \|^2 dt + \int_{t_n}^{t_{n+1}} \| (\pi_N - I) \partial_t \partial_x u \|^2 dt + \Delta t \| \partial_x e_N \|_{\infty}^2 \\
+ c \Delta t \left( \| \partial_x (3e_N - e_{n-1}) \|^2 + \| 3e_N - e_{n-1} \|^2 \right) \]

Adding up for \( n = 1, \ldots, k \) and using the following inequality,

\[
\| \partial_x u \|^2 \leq c \left( \| u \|^2 + \| \partial^2_x u \|^2 \right).
\]

we have

\[
E(\tilde{e}^{n+1}) \leq E(\tilde{e}^n) + c \left( \Delta t^4 + N^4 - 2m \right) + c \Delta t \sum_{n=1}^{k} E(\tilde{e}^n).
\]

4. Numerical Results

This section presents several numerical examples to confirm the accuracy and applicability of schemes (35) for solving Rosenau-Burgers equation. First, we let

\[
u_n^N = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k^n \exp \left( -\frac{inkx}{L} \right)
\]

And we obtain the following linear system:

\[
\frac{1}{\Delta t} \left( \tilde{u}_k^{n+1} - \tilde{u}_k^n \right) \left( 1 + \left( \frac{nk}{L} \right)^4 \right) + \left( \frac{ink}{L} + \alpha \left( \frac{nk}{L} \right)^2 \right) \tilde{u}_k^{n+1/2} + \frac{1}{6} \left( 3u_N^n - u_{N-1}^n \right) \partial_x u_N^{n+1/2} \\
+ \frac{1}{6} \left( u_N^n - u_{N-1}^n \right) u_N^{n+1/2} \bigg|_{k=0},
\]

where \( \tilde{u}_k \) or \( |u|_k \) represents the kth mode Fourier coefficient of the function u.

4.1. Verification of the Temporal Convergence Order. We use the following quantity to compute the convergence rate in time direction [18]:

\[
p = \log_2 \left( \frac{\| u_N^{n=2\Delta t} - u_N^\alpha \|^2}{\| u_N^{n=2\Delta t} - u_N^{n=\Delta t} \|^2} \right).
\]

The full discrete problem (52) is solved in \( \Omega = (0, 2\pi) \) with \( T = 1 \) and \( N = 60 \). Tables 1–3 display the temporal convergence orders for different values of \( \alpha \) and initial condition \( u_0 \). As shown in Tables 1–3, our numerical scheme (52) is of the 2nd order accuracy in time, which is confirmed with the theoretical result in Theorem 4.

4.2. Asymptotic Properties of Solutions. In order to verify the asymptotic behavior of \( z(x, t) \), the effect of \( \alpha \) and initial condition \( u_0 \) will be investigated. Let \( \alpha = 1 \) and \( f = \exp(-t) \cos x - \exp(-t) \sin x + \exp(-2t) \cos x \sin x \); then, the exact solution is \( u(x, t) = \exp(-t) \sin x \), and \( e^\infty = \cos \). The numerical result in Figure 1 shows the convergence of the solution \( u(x, t) \) to its steady-state solution \( u^\infty \). When \( \alpha = 1 \), and \( f = \left( \exp(-t) + 1 \right) \cos x - 2 \exp(-t) \cos x - \sin x \left( \exp(-t) + 1 \right) - \cos x \sin x \left( \exp(-t) + 1 \right) \) in (3), the exact solution is \( u = \left( \exp(-t) + 1 \right) \cos x \), the steady-state solution is \( u^\infty = \cos x \), and \( f^\infty = \cos x - \sin x - \sin x \cos x \). It is easy to verify that both \( u^\infty \) and \( F \) satisfy the assumptions (A1) and (A2). Figure 2 indicates that the numerical solution
Table 1: Temporal convergence orders for $u_0 = \cos x$.

<table>
<thead>
<tr>
<th>$\alpha \ \Delta t$</th>
<th>$\Delta t=1.00E-1$</th>
<th>$\Delta t=5.00E-2$</th>
<th>$\Delta t=1.00E-2$</th>
<th>$\Delta t=5.00E-3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0$</td>
<td>1.9933</td>
<td>2.0010</td>
<td>2.0009</td>
<td>2.0005</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>1.9428</td>
<td>1.9712</td>
<td>1.9942</td>
<td>1.9972</td>
</tr>
<tr>
<td>$\alpha = 3$</td>
<td>1.8833</td>
<td>1.9262</td>
<td>1.9825</td>
<td>1.9910</td>
</tr>
<tr>
<td>$\alpha = 5$</td>
<td>1.7717</td>
<td>1.8503</td>
<td>1.9629</td>
<td>1.9810</td>
</tr>
</tbody>
</table>

Table 2: Temporal convergence orders for $u_0 = \sin x + \cos x$.

<table>
<thead>
<tr>
<th>$\alpha \ \Delta t$</th>
<th>$\Delta t=1.00E-1$</th>
<th>$\Delta t=5.00E-2$</th>
<th>$\Delta t=1.00E-2$</th>
<th>$\Delta t=5.00E-3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0$</td>
<td>1.9870</td>
<td>1.9992</td>
<td>2.0007</td>
<td>2.0005</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>1.9329</td>
<td>1.9667</td>
<td>1.9934</td>
<td>1.9968</td>
</tr>
<tr>
<td>$\alpha = 3$</td>
<td>1.8715</td>
<td>1.9187</td>
<td>1.9806</td>
<td>1.9901</td>
</tr>
<tr>
<td>$\alpha = 5$</td>
<td>1.7528</td>
<td>1.8371</td>
<td>1.9593</td>
<td>1.9791</td>
</tr>
</tbody>
</table>

Table 3: Temporal convergence orders for $u_0 = \sin 3x \cos 2x$.

<table>
<thead>
<tr>
<th>$\alpha \ \Delta t$</th>
<th>$\Delta t=1.00E-1$</th>
<th>$\Delta t=5.00E-2$</th>
<th>$\Delta t=1.00E-2$</th>
<th>$\Delta t=5.00E-3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0$</td>
<td>1.9959</td>
<td>2.0008</td>
<td>2.0006</td>
<td>2.0004</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>1.9542</td>
<td>1.9767</td>
<td>1.9953</td>
<td>1.9978</td>
</tr>
<tr>
<td>$\alpha = 3$</td>
<td>1.8928</td>
<td>1.9325</td>
<td>1.9840</td>
<td>1.9918</td>
</tr>
<tr>
<td>$\alpha = 5$</td>
<td>1.7906</td>
<td>1.8635</td>
<td>1.9664</td>
<td>1.9828</td>
</tr>
</tbody>
</table>

of $u(x,t)$ converges to $u^\infty$ as $t \rightarrow \infty$. In Figure 3, we plot the numerical solution at $\alpha = 0$, we can observe that $u(x,t)$ oscillates around 0. In Figures 4-5, we observed that the numerical solutions decay almost to zero when $\alpha$ becomes larger. This implies that the impact of $\alpha$ is significant in the long time behavior. The figures are for $z(x,t)$ indicated in the result discussions.

The reference solutions were obtained by using the numerical scheme (35) with $N = 128$ and $\delta t = 5 \times 10^{-5}$. We present $L^2$ and $H^1$ errors for $u$. The results are summarized in Figure 6, in which we conclude that the full-discrete scheme (35) reaches the second-order convergence rate in time direction. In order to test the spatial spectral accuracy, error functions for $N$ are given in Figure 7, which show that our numerical method has good convergence behavior. We observe it reaches spectral convergence in space, if the error in spatial direction is negligible.

5. Conclusion

This article studies the asymptotic stability of Rosenau-Burgers equation. Under suitable assumptions on an eigenvalue problem and the forcing function, the convergence order of the Rosenau-Burgers equation to its steady-state problem is derived. Then, we propose a linearized numerical method for solving this nonlinear Rosenau-Burgers equation. The numerical method is combined with a finite difference scheme in time and Fourier-spectral method in space. We have derived a discrete stability inequality and error estimate for the numerical scheme which leads to 2nd-order accuracy in time and spectral accuracy in space. Numerical examples are illustrated to verify the theoretical results.
Figure 3: The graph of approximation solutions at $\alpha = 0$, $\Delta t = 0.01$, $N = 60$, $u_0 = \sin 3x \cos 2x$.

Figure 4: The graph of approximation solutions at $\alpha = 1$, $\Delta t = 0.01$, $N = 60$, $u_0 = \sin 3x \cos 2x$.

Figure 5: The graph of approximation solution at $\alpha = 5$, $\Delta t = 0.01$, $N = 60$, $u_0 = \sin 3x \cos 2x$.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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