Modelling in bond graph to obtain reduced models of systems with singular perturbations is applied. This singularly perturbed system is characterized by having three timescales, i.e., slow, medium, and fast dynamics. From a bond graph whose storage elements have an integral causality assignment (BGI), the mathematical model of the complete system can be determined. By assigning a derivative causality assignment to the storage elements for the fast dynamics and maintaining an integral causality assignment for the slow and medium dynamics on the bond graph, reduced models for the slow and medium dynamics are obtained. When a derivative causality to the storage elements for the fast and medium dynamics is assigned and an integral causality assignment to the slow dynamics is applied, the most reduced model is determined. Finally, the proposed methodology to the Ward Leonard system is applied.

1. Introduction

The singular perturbation theory provides a powerful tool for modelling and control design in dealing with the presence of parasitic parameters, for example, small time constants due to inductances, capacitances, and moment of inertia which increase the system order. The systems containing parasitic parameters are characterized by having slow and fast dynamics, and they are considered as timescale systems. In addition, the parasitic parameters can be neglected, and reduced order systems can be obtained [1, 2].

The mathematical model of systems represents a fundamental task in control theory. Frequently, these systems determine high-order state equations. However, many systems can contain different timescales due to the parasitic parameters. Thus, these systems can have different dynamics, and applying the appropriate procedures, these dynamics can be separated. The corresponding subsystems of these dynamics permit to have reduced models whose analysis and control design are direct and simpler [1, 2].

There are many papers and books dedicated to system analysis with singular perturbations. Also, many applications and control design for singularly perturbed systems have been proposed [1, 2]. For example, systems with high-gain feedback using singular perturbation methods have been studied in [3]. Nonlinear optimal control applying singular perturbations techniques is proposed in [4]. The nonlinear and robust composite control for a DC motor is presented in [5].

The control for a class of nonlinear singularly perturbed systems by exact design with manifold is introduced in [6]. The robust stability applied to singularly perturbed systems is described in [7]. The robustness of output feedback
controllers for systems with singular perturbations is proposed in [8]. The observer problem by the need to obtain the states for a system with singular perturbations is solved in [9]. The composite control of singular perturbations systems using sliding modes is proposed in [10]. The feedback control design for systems with three timescales is introduced in [11]. A procedure to obtain reduced models of a system with multitudescales and applied to an electrical power system is presented in [12].

The nonlinear system analysis with multiple timescales is proposed in [13]. The problem for controlling linear systems with multiparameter singular perturbations is solved in [14]. The calculation of the average on cycles of three timescales systems is described in [15]. The asymptotic stability of a singularly perturbed nonlinear with three timescales is studied in [16]. The multiparameter asymptotic stability analysis for the three timescale singular perturbation problem of an autonomous helicopter is proposed in [17]. The analysis and control design of systems with two and three timescales and the application to fuel cells are presented in [18].

Bond graph theory offers a modelling and control platform for systems with different energy domains where important results in analysis and synthesis of systems have been found [19–21].

Some interesting papers using bond graphs for singular perturbations methods can be cited. The modelling and simplification of two timescale systems through bond graphs is described in [22]. A reciprocal system to obtain the fast dynamics is introduced in [23]. The quasi-steady-state model of a system in the physical domain is proposed in [24]. The analysis of a class of nonlinear systems with singular perturbations in bond graph is presented in [25]. Approximate bond graph models for two timescale systems are proposed in [26]. The analysis of a singular perturbed system with a feedback and observer in the physical domain is presented in [27].

A steel frame structure based on a bond graph model of distributed system using lumping technique is proposed in [28]. Also, this methodology can be used for stability and sensitivity analyses.

The slow and fast dynamics of a system represented by a bond graph determining the causal loop gains can be obtained [23, 29] where the bond graph methodology is a useful tool for system analysis.

In this paper, the modelling in bond graph of systems with three timescales is introduced. This bond graph is formed by three groups of storage elements in integral causality assignment (BGI). These storage elements represent the slow, medium, and fast dynamics of the system.

During the dynamic performance of a system with three timescales when the fast dynamics have converged, reduced systems can be obtained in a bond graph approach whose storage elements for the fast dynamics have to accept a derivative causality assignment and the storage elements for the medium and slow dynamics an integral causality is assigned. However, the subsystem for medium dynamics still has singular perturbation parameters. Hence, the most reduced system can be gotten by assigning a derivative causality to the storage elements of the fast and medium dynamics and an integral causality to the slow dynamics.

Therefore, the contribution of this paper is to obtain the quasi-steady-state models of three timescale systems modelled by bond graphs.

This paper is organized as follows: Section 2 gives the three timescale systems in the algebraic approach. The modelling in bond graph of systems with three timescales is described in Section 3. The reduced models obtaining the quasi-steady-state models in a bond graph approach is presented in Section 4. A case study of a Ward Leonard system applying the proposed methodology is developed in Section 5. Finally, in Section 6, the conclusions are given.

2. Three Timescale Systems

Consider a system with singular perturbations which is decomposed into slow, medium, and fast dynamics described by

\[
\begin{bmatrix}
    \dot{x}_1(t)
    \\
    \dot{\varepsilon}_1 x_2(t)
    \\
    \dot{\varepsilon}_2 x_3(t)
\end{bmatrix} =
\begin{bmatrix}
    A_{11} & A_{12} & A_{13} \\
    A_{21} & A_{22} & A_{23} \\
    A_{31} & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t)
\end{bmatrix} +
\begin{bmatrix}
    B_1 \\
    B_2 \\
    B_3
\end{bmatrix} u(t),
\]

(1)

with the output

\[
y(t) =
\begin{bmatrix}
    C_1 & C_2 & C_3
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t)
\end{bmatrix} + Du(t),
\]

(2)

where \(x_1(t)\) is an \(n \times 1\) vector, \(x_2(t)\) is an \(m \times 1\) vector, \(x_3(t)\) is an \(l \times 1\) vector, \(u(t)\) is an \(p \times 1\) vector input and \(y(t)\) is an \(q \times 1\) vector output.

It is assumed that system (1) has three different groups of eigenvalues: \(n\) eigenvalues are close to the origin and \(m\) and \(l\) eigenvalues are far from the origin, respectively. The eigenspectrum \(e(A)\) of system (1) in the increasing order of absolute values is

\[
e(A) = \{p_{s1}, \cdots, p_{sm}, p_{m1}, \cdots, p_{mm}, p_{f1}, \cdots, p_{fl}\},
\]

\[
e(A_j) = \{p_{si}, \cdots, p_{sm}\},
\]

\[
e(A_m) = \{p_{m1}, \cdots, p_{mm}\},
\]

\[
e(A_f) = \{p_{f1}, \cdots, p_{fl}\}.
\]

(3)

System (1) is said to possess a three timescale property, if the largest absolute eigenvalue of the slow eigenspectrum \(e(A_s)\) is much smaller than the smallest absolute eigenvalue of the fast one eigenspectrum \(e(A_f)\) and if the largest absolute eigenvalue of the fast one eigenspectrum \(e(A_m)\) is much smaller than the smallest absolute eigenvalue of the fast two eigenspectrum \(e(A_f)\); that is [2],

\[
\varepsilon_1 \equiv \frac{\|p_{m}\|}{\|p_{m1}\|} \ll 1,
\]

(4)

\[
\varepsilon_2 \equiv \frac{\|p_{mn}\|}{\|p_{f1}\|} \ll 1.
\]

(5)
By neglecting the fast dynamics of the complete system \((e_2 = 0)\), the solution of the third line of (1) determines
\[
\dot{x}_3(t) = -A_{31}^{-1}A_{31} \dot{x}_1(t) - A_{33}^{-1}A_{32} \dot{x}_2(t) - A_{33}^{-1}B_3 u(t), \tag{6}
\]
and from the first and second lines of (1) and (6), the reduced models are expressed by
\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} \\
\bar{A}_{21} & \bar{A}_{22}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2
\end{bmatrix} u(t), \tag{7}
\]
where
\[
\begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} \\
\bar{A}_{21} & \bar{A}_{22}
\end{bmatrix} =
\begin{bmatrix}
A_{11} - A_{13} A_{31}^{-1} A_{31} & A_{12} - A_{13} A_{33}^{-1} A_{32} \\
A_{21} - A_{23} A_{31}^{-1} A_{31} & A_{22} - A_{23} A_{33}^{-1} A_{32}
\end{bmatrix}.
\tag{8}
\]

The output of this reduced system is
\[
\bar{y}(t) = \begin{bmatrix}
\bar{C}_1 & \bar{C}_2
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} + \bar{D} u(t), \tag{9}
\]
where
\[
\begin{bmatrix}
\bar{C}_1 & \bar{C}_2
\end{bmatrix} =
\begin{bmatrix}
C_1 - C_3 A_{33}^{-1} A_{31} & C_2 - C_3 A_{33}^{-1} A_{32}
\end{bmatrix},
\bar{D} = D - C_3 A_{33}^{-1} B.
\tag{10}
\]

From the second line of (7) and doing \(e_1 = 0\), the medium dynamics can also be removed by
\[
\bar{x}_2(t) = - (\bar{A}_{22})^{-1} \bar{A}_{21} \dot{x}_1(t) - (\bar{A}_{22})^{-1} \bar{B}_2 u(t), \tag{11}
\]
and substituting (11) into the first line of (7), the quasi-steady-state model is
\[
\dot{x}_1(t) = \bar{A}_{11} \bar{x}_1(t) + \bar{B}_1 u(t), \tag{12}
\]
where
\[
\bar{A}_{11} = \bar{A}_{11} - \bar{A}_{12} (\bar{A}_{22})^{-1} \bar{A}_{21},
\bar{B}_1 = \bar{B}_1 - \bar{A}_{12} (\bar{A}_{22})^{-1} \bar{B}_2. \tag{13}
\]

The output for the most reduced model is given by
\[
y(t) = \bar{C}_1 \bar{x}_1(t) + \bar{D} u(t), \tag{14}
\]
where
\[
\bar{C}_1 = C_1 - C_2 (\bar{A}_{22})^{-1} \bar{A}_{21},
\bar{D} = D - C_2 (\bar{A}_{22})^{-1} \bar{B}_2. \tag{15}
\]

Bond graph models of singularly perturbed systems are described in the next section.

3. Modelling in Bond Graph of Systems with Singular Perturbations

A bond graph is a graphical representation of a system where the power interactions are described by lines called bonds. These bonds indicate the interactions between power variable pairs called effort \(e(t)\) and flow \(f(t)\), which is shown in Figure 1.

In order to obtain the sets of equations of a system modelled by bond graphs, the constitutive relations of the elements are required. These relationships can be dynamic or algebraic depending on the element and by the cause-effect assignment. In the bond graph, a bond with the causal stroke determines the causality assignment and the assignments of the half arrow and the causal stroke are independent as is shown in Figure 2.

The following physical elements can be used to build a dynamic system:

(i) The Active 1 Ports or Sources Denoted by \((MS, MSf)\). These sources have only one causality, and this is shown in Figure 3.

(ii) The Passive 1 Ports. These elements are as follows:

(a) Resistance taking whatever causality shown in Figure 4
(b) Capacitance and inertance elements in an integral causality assignment where the input variable is integrated to produce the output variable which is shown in Figure 5
(c) Capacitance and inertance in a derivative causality assignment where the input and output have a derivative operation, and this is shown in Figure 6

(iii) The Ideal 2 Port Elements Denoted by \((TF, GY)\) Representing Transformers and Gyrators. Figure 7 shows these elements.

(iv) The 3 Port Junctions Denoted by \((1, 0)\) which Are Junctions That Determine the Different Connections between the Elements. These junctions are shown in Figure 8.

By using physical elements and junction structures, one can analyze systems containing complex multiport components applying bond graphs. Hence, a bond graph model with a preferred integral causality assignment (BGI) of a system with three timescales is shown in Figure 9.

The block diagram of Figure 9 contains the following:

(i) Source field denoted by \((MS, MSf)\) that determines the plant input \(u(t) \in \mathbb{R}^P\).
(ii) Junction structure denoted by \((0, 1, TF, GY)\) with 0 and 1 junctions, transformers TF, and gyrators GY.
(iii) Energy storage field denoted by \((C, I)\) that defines energy variables \(q(t)\) and \(p(t)\) associated with \(C\) and \(I\) elements divided by the following:

(a) The states for the slow dynamics \(x_1(t) \in \mathbb{R}^n\) and \(x_1^d(t) \in \mathbb{R}^n\) associated with the storage elements in integral and derivative causality assignment, respectively
(b) The co-energy vectors for the slow dynamics \(z_1(t) \in \mathbb{R}^n\) and \(z_1^d(t) \in \mathbb{R}^n\) of the storage elements in integral and derivative causality assignment, respectively
The states for the medium dynamics $x_2(t) \in \mathbb{R}^{m_0}$ and $x_2^d(t) \in \mathbb{R}^{m_d}$ associated with storage elements in integral and derivative causality assignment, respectively.

The co-energy vectors for the medium dynamics $z_2(t) \in \mathbb{R}^{a_0}$ and $z_2^d(t) \in \mathbb{R}^{a_d}$ of the storage elements in integral and derivative causality assignment, respectively.

The states for the fast dynamics $x_3(t) \in \mathbb{R}^{l}$ and $x_3^d(t) \in \mathbb{R}^{l_d}$ associated with the storage elements in integral and derivative causality assignment, respectively.

The co-energy vectors for the fast dynamics $z_3(t) \in \mathbb{R}^{l}$ and $z_3^d(t) \in \mathbb{R}^{l_d}$ of the storage elements in integral and derivative causality assignment, respectively.

Energy dissipation field denoted by $(R)$ that defines $D_{in}(t) \in \mathbb{R}^{l}$ and $D_{out}(t) \in \mathbb{R}^{l}$ as a mixture of power variables $e(t)$ and $f(t)$ indicating the energy exchanges between the dissipation field and the junction structure.

$$D_{out}(t) = LD_{in}(t).$$

Then, the matrices of the state variable representation are defined by
\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix} = 
\begin{bmatrix}
E_1 & 0 & 0 \\
0 & E_2F_2 & 0 \\
0 & 0 & E_3F_3
\end{bmatrix}^{-1} 
\cdot 
\begin{bmatrix}
(S_{11}^{11} + S_{12}^{11}M_{S_{21}^{11}})F_1 \\
(S_{11}^{12} + S_{12}^{12}M_{S_{21}^{12}})F_2 \\
(S_{11}^{13} + S_{12}^{13}M_{S_{21}^{13}})F_3
\end{bmatrix} 
\cdot 
\begin{bmatrix}
S_{12}^{11} + S_{12}^{12}M_{S_{22}^{11}} \\
S_{12}^{12} + S_{12}^{13}M_{S_{22}^{12}} \\
S_{12}^{13} + S_{12}^{13}M_{S_{22}^{13}}
\end{bmatrix} 
\]

and the matrices of the output equation are described by

\[
\begin{bmatrix}
C_1 & C_2 & C_3
\end{bmatrix} = 
\begin{bmatrix}
(S_{31}^{11} + S_{32}^{11}M_{S_{21}^{11}})F_1 \\
(S_{31}^{12} + S_{32}^{12}M_{S_{21}^{12}})F_2 \\
(S_{31}^{13} + S_{32}^{13}M_{S_{21}^{13}})F_3
\end{bmatrix},

D = S_{33}^{11} + S_{32}^{11}M_{S_{23}^{11}},
\]

where

\[
E_1 = I - S_{11}^{11}(F_1^d)^{-1}S_{41}^{11}F_1,
\]

\[
E_2 = I - S_{11}^{12}(F_2^d)^{-1}S_{41}^{12}F_2,
\]

\[
E_3 = I - S_{11}^{13}(F_3^d)^{-1}S_{41}^{13}F_3,
\]

\[
M = L(I - S_{22}L)^{-1}.
\]
4. Reduced Models in the Physical Domain

An important advantage of the bond graph methodology is the capacity of deriving different models. Hence, the reduced models for the different dynamics of three timescale systems can be obtained. A direct and interesting result to get the quasi-steady-state model in a bond graph approach is proposed in [24, 26, 27]. However, these published results have been extended for three timescale systems in this paper.

A bond graph called singularly perturbed bond graph (SPBG_H) is proposed where the derivative causality to the storage elements of the fast dynamics is assigned and the storage elements of the slow and medium dynamics maintain an integral causality assignment whose scheme is shown in Figure 10.

In order to obtain the SPBG_H, the key vectors $z_3$ and $x_3$ of Figure 10 have been changed with respect to Figure 9. By the causality assignment of the storage elements, the dissipation elements may have different causality from the corresponding BG1 and the new key vectors are defined by

$$D^h_{out}(t) = L_h D^h_{in}(t),$$

where $L_h$ is the constitutive relationship of the dissipation elements in the bond graph SPBG_H.

The following lemma permits us to determine reduced systems with singular perturbations in the physical domain:

**Lemma 1.** Consider three timescale systems modelled by bond graphs where storage elements for the slow and medium dynamics have an integral causality assignment and the storage elements for the fast dynamics have a derivative causality which has been assigned whose junction structure is described by

$$\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{z}_2(t) \\
D_{in}(t) \\
z_1(t) \\
\dot{x}_3(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
H_{11}^1 & H_{12}^1 & H_{13}^1 & H_{14}^1 & H_{15}^1 & H_{16}^1 \\
H_{21}^1 & H_{22}^1 & H_{23}^1 & H_{24}^1 & H_{25}^1 & H_{26}^1 \\
H_{31}^1 & H_{32}^1 & H_{33}^1 & H_{34}^1 & H_{35}^1 & H_{36}^1 \\
H_{41}^1 & H_{42}^1 & H_{43}^1 & H_{44}^1 & H_{45}^1 & H_{46}^1 \\
H_{51}^1 & H_{52}^1 & H_{53}^1 & H_{54}^1 & H_{55}^1 & H_{56}^1 \\
H_{61}^1 & H_{62}^1 & H_{63}^1 & H_{64}^1 & H_{65}^1 & H_{66}^1 \\
0 & H_{12}^2 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
z_2(t) \\
d_{in}(t) \\
z_1(t) \\
\dot{x}_3(t) \\
\dot{x}_2(t)
\end{bmatrix} + \begin{bmatrix}
A_{11}^h & A_{12}^h & A_{13}^h & A_{14}^h & A_{15}^h & A_{16}^h \\
A_{21}^h & A_{22}^h & A_{23}^h & A_{24}^h & A_{25}^h & A_{26}^h \\
A_{31}^h & A_{32}^h & A_{33}^h & A_{34}^h & A_{35}^h & A_{36}^h \\
A_{41}^h & A_{42}^h & A_{43}^h & A_{44}^h & A_{45}^h & A_{46}^h \\
A_{51}^h & A_{52}^h & A_{53}^h & A_{54}^h & A_{55}^h & A_{56}^h \\
A_{61}^h & A_{62}^h & A_{63}^h & A_{64}^h & A_{65}^h & A_{66}^h \\
B_1^h & B_2^h
\end{bmatrix} \begin{bmatrix} u(t) \\
\end{bmatrix},$$

and then a state-space representation of linearly independent state variables of a reduced system is defined by

$$\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
A_{11}^h & A_{12}^h & A_{13}^h & A_{14}^h & A_{15}^h & A_{16}^h \\
A_{21}^h & A_{22}^h & A_{23}^h & A_{24}^h & A_{25}^h & A_{26}^h \\
A_{31}^h & A_{32}^h & A_{33}^h & A_{34}^h & A_{35}^h & A_{36}^h \\
A_{41}^h & A_{42}^h & A_{43}^h & A_{44}^h & A_{45}^h & A_{46}^h \\
A_{51}^h & A_{52}^h & A_{53}^h & A_{54}^h & A_{55}^h & A_{56}^h \\
A_{61}^h & A_{62}^h & A_{63}^h & A_{64}^h & A_{65}^h & A_{66}^h \\
B_1^h & B_2^h
\end{bmatrix} \begin{bmatrix} u(t) \\
\end{bmatrix},$$

where $L_h$ is the constitutive relationship of the dissipation elements in the bond graph SPBG_H.
Slow

\[ \mathbf{A}_{h11} \mathbf{A}_{h12} \mathbf{A}_{h21} \mathbf{A}_{h22} = \begin{bmatrix} E_{h1} & 0 \\ 0 & E_{h2} \end{bmatrix}^{-1} \left[ (H_{11}^{h1} + H_{12}^{h1} M_h H_{11}^{h2}) F_1 \right. \\
\left. (H_{12}^{h1} + H_{12}^{h1} M_h H_{12}^{h2}) F_1 \right], \]

(34)

\[
\begin{bmatrix}
B_{h1}^* \\
B_{h2}^*
\end{bmatrix} = \begin{bmatrix} E_{h1} & 0 \\ 0 & E_{h2} \end{bmatrix}^{-1} \begin{bmatrix} H_{11}^{h1} + H_{12}^{h1} M_h H_{23} \\ H_{12}^{h1} + H_{12}^{h1} M_h H_{23} \end{bmatrix}, \quad (35)
\]

with

\[
E_{h1} = 1 - H_{11}^{h1} (r_d^{h1})^{-1} H_{41}^{h1} F_1, \quad (36)
\]

\[
E_{h2} = 1 - H_{12}^{h1} (r_d^{h1})^{-1} H_{42}^{h1} F_2, \quad (37)
\]

\[
M_h = L_h (I - H_{22} L_h)^{-1}, \quad (38)
\]

and the real roots of the algebraic equation are given by

\[
\bar{x}_i(t) = \left[ \begin{array}{c}
A_{h11}^* \\
A_{h21}^*
\end{array} \right] \bar{x}_i(t) + B_{h1}^* u(t), \quad (39)
\]

where

\[
\begin{bmatrix}
A_{h11}^* \\
A_{h21}^*
\end{bmatrix} = F_3^{-1} \begin{bmatrix} (H_{11}^{h1} + H_{12}^{h1} M_h H_{11}^{h2}) F_1 \right. \\
\left. (H_{12}^{h1} + H_{12}^{h1} M_h H_{12}^{h2}) F_2 \right], \quad (40)
\]

\[
B_{h1}^* = F_3^{-1} (H_{11}^{h1} + H_{12}^{h1} M_h H_{23}), \quad (41)
\]

with the output

\[
\bar{y}_i(t) = \left[ \begin{array}{c}
C_{h11}^* \\
C_{h21}^*
\end{array} \right] \bar{x}_i(t) + D_{h1}^* u(t), \quad (42)
\]

where

\[
\begin{bmatrix}
C_{h11}^* \\
C_{h21}^*
\end{bmatrix} = \begin{bmatrix} (H_{11}^{h1} + H_{12}^{h1} M_h H_{12}^{h2}) F_1 \right. \\
\left. (H_{12}^{h1} + H_{12}^{h1} M_h H_{12}^{h2}) F_2 \right], \quad (43)
\]

\[
D_{h1}^* = H_{33} + H_{32} M_h H_{23}. \quad (44)
\]

The proof is presented in Appendix A.

A scheme to obtain the quasi-steady-state model of a three timescale system by removing medium and fast dynamics called SPBG is shown in Figure 11.

The storage elements for the medium and fast dynamics have a derivative causality assignment and the storage elements for the slow dynamics an integral causality is assigned in Figure 11. The new key vectors denoted by \( D_{out}'(t) \) and \( D_{in}'(t) \) for the dissipation elements of the SPBG are described by

\[
D_{out}'(t) = L \cdot D_{in}'(t), \quad (45)
\]

where \( L \) is the constitutive relationship.

The mathematical description of a system with singular perturbations related to Figure 11 in the following lemma is proposed.

**Lemma 2.** Consider a three timescale system modelled by bond graphs with storage elements for the slow dynamics having an integral causality assignment and a derivative causality to the storage elements for the medium and fast dynamics is assigned whose junction structure is defined by

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t) \\
\dot{x}_4(t)
\end{bmatrix} = \begin{bmatrix}
R_{11}^{11} & R_{12}^{11} & R_{13}^{11} & R_{14}^{11} \\
R_{21}^{11} & R_{22}^{11} & R_{23}^{11} & R_{24}^{11} \\
R_{31}^{11} & R_{32}^{11} & R_{33}^{11} & R_{34}^{11} \\
R_{41}^{11} & R_{42}^{11} & R_{43}^{11} & R_{44}^{11}
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t)
\end{bmatrix} + \begin{bmatrix}
\sum z_1(t) \\
\sum z_2(t) \\
\sum z_3(t) \\
\sum z_4(t)
\end{bmatrix} \begin{bmatrix}
\cdot x_1(t) \\
\cdot x_2(t) \\
\cdot x_3(t) \\
\cdot x_4(t)
\end{bmatrix},
\]

(46)

and then the quasi-steady-state model of the system is given by

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t) \\
\dot{x}_4(t)
\end{bmatrix} = \begin{bmatrix}
A_{11}^{11} & A_{12}^{11} & A_{13}^{11} & A_{14}^{11} \\
A_{21}^{11} & A_{22}^{11} & A_{23}^{11} & A_{24}^{11} \\
A_{31}^{11} & A_{32}^{11} & A_{33}^{11} & A_{34}^{11} \\
A_{41}^{11} & A_{42}^{11} & A_{43}^{11} & A_{44}^{11}
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t)
\end{bmatrix} + \begin{bmatrix}
\sum z_1(t) \\
\sum z_2(t) \\
\sum z_3(t) \\
\sum z_4(t)
\end{bmatrix} \begin{bmatrix}
\cdot x_1(t) \\
\cdot x_2(t) \\
\cdot x_3(t) \\
\cdot x_4(t)
\end{bmatrix},
\]

(47)
where

\[ A'_1 = E_r^{-1} \left( R_{11}^{(1)} + R_{12}^{(1)} M_r R_{21}^{(1)} \right) F_1, \]  
(48)

\[ B'_1 = E_r^{-1} \left( R_{12}^{(1)} + R_{12}^{(1)} M_r R_{22}^{(1)} \right), \]  
(49)

being

\[ E_r = I - R_{14}^{(1)} \left( F_1 \right)^{-1} R_{41} F_1, \]  
(50)

\[ M_r = L_r \left( I - R_{22} L_r \right)^{-1}, \]  
(51)

with the real roots of the algebraic equation for the medium dynamics described by

\[ \bar{x}_2(t) = A'_2 \bar{x}_2(t) + B'_2 u(t), \]  
(52)

where

\[ A'_2 = F_2^{-1} \left( R_{11}^{(2)} + R_{12}^{(2)} M_r R_{21}^{(2)} \right) F_1, \]  
(53)

\[ B'_2 = F_2^{-1} \left( R_{12}^{(2)} + R_{12}^{(2)} M_r R_{22}^{(2)} \right), \]  
(54)

and the output for the reduced model is

\[ \gamma(t) = C'_2 \bar{x}_2(t) + D' u(t), \]  
(55)

where

\[ C'_2 = \left( R_{31}^{(2)} + R_{32} M_r R_{22}^{(2)} \right) F_1, \]  
(56)

\[ D' = R_{33} + R_{32} M_r R_{23}. \]  
(57)

The proof is presented in Appendix B.

5. Case Study

The DC motor has widely been used in variable drive systems. Originally, the speed of DC motors was controlled by adjusting the current of the shunt field. This method of control was replaced by the Ward Leonard system [30]. This system is a motor-generator set to supply an adjustable voltage to the variable speed DC drive motor which is shown in Figure 12.

The Ward Leonard system is formed by an armature winding for the generator with resistance \( R_g \) and inductance \( L_g \), and the electromechanical conversion between velocity input \( w \) and the generated voltage is denoted by \( n_g \); the armature winding for the motor has the resistance \( R_m \) and inductance \( L_m \), the connection between the two armature windings of the generator and motor is done by a capacitor \( C_c \) and a resistance \( R_c \), and the motor drives a mechanical load from the electromechanical conversion \( n_m \). The mechanical load is the coupling of two inertia \( I_m \) and \( I_c \) by using a mechanical transformer \( a \) and a spring \( K_c \) with the damping \( b_c \).

The BGI of the Ward Leonard system is shown in Figure 13.

The key vectors and the constitutive relationships of the BGI for the slow dynamics are

\[ x_1 = \begin{bmatrix} p_{14} \\ q_{19} \end{bmatrix}; \]
\[ x_1^* = \begin{bmatrix} e_{14} \\ e_{19} \end{bmatrix}; \]
\[ z_1 = \begin{bmatrix} f_{14} \\ e_{19} \end{bmatrix}; \]
\[ F_1 = \text{diag} \left\{ \frac{1}{J_m}, \frac{1}{K_c} \right\}, \]
\[ x_1^d = p_{14}; \]
\[ x_1^d = e_{14}; \]
\[ z_1^d = f_{14}; \]
\[ F_1^d = \frac{1}{J_c}, \]

for the medium dynamics,

\[ x_2 = \begin{bmatrix} p_4 \\ p_{10} \end{bmatrix}; \]
\[ x_2^* = \begin{bmatrix} e_4 \\ e_{10} \end{bmatrix}; \]
\[ z_2 = \begin{bmatrix} f_4 \\ f_{10} \end{bmatrix}; \]
\[ F_2 = \text{diag} \left\{ \frac{1}{L_g}, \frac{1}{L_m} \right\}, \]

for the fast dynamics,

\[ x_3 = q_7; \]
\[ x_3^* = f_7; \]
\[ z_3 = e_7; \]
\[ F_3 = \frac{1}{C_c}, \]

for the dissipation elements,

\[ D_{in} = \begin{bmatrix} f_{13} & e_{20} & f_3 & f_9 & e_6 \end{bmatrix}^T; \]
\[ D_{out} = \begin{bmatrix} e_{13} & f_{20} & e_5 & e_9 & f_6 \end{bmatrix}^T; \]
\[ L = \text{diag} \left\{ \frac{1}{b_m}, \frac{1}{b_k}, \frac{1}{R_g}, \frac{1}{R_m}, \frac{1}{R_c} \right\}, \]

and the input \( u = f_1 \).
The system order is \(n = 2, n_f = 1, m = 2,\) and \(l = 1\); thus, \(n + m + l = 5\). The junction structure of the BGI is described by

\[
\begin{bmatrix}
    e_{14} \\
    f_{19} \\
    e_4 \\
    e_{10} \\
    f_7 \\
    f_{13} \\
    e_{20} \\
    f_3 \\
    f_9 \\
    e_6 \\
    f_{18}
\end{bmatrix} = \begin{bmatrix}
    0 & -a & 0 & n_m & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -a \\
    a & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & n_g & 0 \\
    -n_m & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
    0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
    f_{14} \\
    e_{19} \\
    f_4 \\
    e_{10} \\
    f_7 \\
    f_{13} \\
    e_{20} \\
    f_3 \\
    f_9 \\
    e_6 \\
    f_{18}
\end{bmatrix}.
\]  

(63)

From (27), (58), (59), and (63),

\[
E_1 = \begin{bmatrix}
    1 + a & 0 & \int_j \\
    \int_m & 0 & 0 \\
    0 & 1 & 0
\end{bmatrix}.
\]  

(64)
In order to remove the fast dynamics of the system, SPBG\textsubscript{H} is shown in Figure 14.

From the BGI in Figure 13, a derivative causality assignment to the storage element for the fast dynamics \( C : C_e \) is applied which is shown in Figure 14. Hence, the new key vectors and the constitutive relationship for the dissipation elements are defined by

\[
\begin{bmatrix}
D^h_m & = & \begin{bmatrix} f_{13} & e_{20} & f_3 & f_9 & f_6 \end{bmatrix}^T; \\
D^h_{\text{out}} & = & \begin{bmatrix} e_{13} & f_{20} & e_3 & e_9 & e_6 \end{bmatrix}^T; \\
L^h & = & \text{diag}\left\{ b_m, \frac{1}{b_k}, R_g, R_m, R_e \right\}.
\end{bmatrix}
\]

From (39)–(41) with (58)–(61), (66), and (67), the matrices to get the algebraic roots for \( x_3 \) are

\[
\begin{bmatrix}
e_{14}\\f_{19}\\e_4\\e_{10}\\e_7\\f_{13}\\e_{20}\\f_3\\f_9\\f_6\\f_{18}\end{bmatrix} = \begin{bmatrix} 0 & -a & n_m & 0 & -1 & 0 & 0 & 0 & 0 & -a & 0 \end{bmatrix}^T \begin{bmatrix} f_{14}\\e_{19}\\f_4\\f_{10}\\e_7\\f_{13}\\e_3\\e_9\\e_6\\f_1\\e_{18} \end{bmatrix}
\]

The corresponding junction structure of the bond graph of Figure 14 is given by

\[
A^h_{31} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad A^h_{32} = \frac{C_e R_e - C_e R_e}{L_g L_m}.
\]
From the second lines of (34) and (35) with (58), (60), (66), and (67), the matrices of the reduced system for the medium dynamics are

\[
A_{21}^h = \begin{bmatrix}
0 & 0 \\
-\frac{n_m L_m}{J_m} & 0
\end{bmatrix},
\]

(71)

\[
A_{22}^h = \begin{bmatrix}
-(R_g + R_c) & \frac{L_g}{L_m} R_c \\
\frac{L_m R_c}{L_g} & -(R_m + R_c)
\end{bmatrix},
\]

(72)

\[
B_2^h = \begin{bmatrix}
L_g n_g \\
0
\end{bmatrix}.
\]

(73)

In this case, \( E_1^h = E_1 \), and from the first lines of (34) and (35) with (58), (60), (66), and (67), the matrices of the reduced system for the slow dynamics are

\[
E_{1h} A_{11}^h = \begin{bmatrix}
-b_m & -\frac{a}{J_m} \\
\frac{a}{J_m} & -1 & \frac{K_c}{b_k K_c}
\end{bmatrix},
\]

\[
E_{1h} A_{12}^h = \begin{bmatrix}
0 & n_m \\
0 & 0
\end{bmatrix},
\]

(74)

\[
E_{1h} B_1^h = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

(75)

By substituting (68) to (76) into (33) and (39), the quasi-steady-state model is defined by

\[
\begin{pmatrix}
\left(1 + a \frac{J_c}{J_m}\right) P_{14}
\end{pmatrix}
= \begin{bmatrix}
\frac{-b_m}{J_m} & -\frac{a}{K_c} & 0 & \frac{n_m}{L_m} \\
\frac{a}{J_m} & -1 & \frac{1}{b_k K_c} & 0 & 0 \\
0 & 0 & -(R_g + R_c) & \frac{L_g}{L_m} R_c \\
\frac{-n_m L_m}{J_m} & 0 & \frac{L_m R_c}{L_g} & -(R_m + R_c)
\end{bmatrix}
\begin{bmatrix}
\tilde{P}_{14} \\
\tilde{P}_{19} \\
\tilde{P}_4 \\
\tilde{P}_{10}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} u.
\]

(77)

By assigning a derivative causality to the storage elements for the medium dynamics, \( I : L_g \) and \( I : L_m \), the SPBG of Figure 15.

With the SPBG of Figure 15, the new key vectors and the constitutive relationship for the dissipation elements are

\[
D_m^f = \begin{bmatrix}
f_{13} & e_{20} & e_3 & e_9 & f_6
\end{bmatrix}^T,
\]

\[
D_m^h = \begin{bmatrix}
e_{13} & f_{20} & f_3 & f_9 & e_6
\end{bmatrix}^T,
\]

(78)

\[
L' = \text{diag}\left(b_m, \frac{1}{b_k}, \frac{1}{R_g}, \frac{1}{R_m}, R_c\right),
\]
and the junction structure is described by

\[
\begin{bmatrix}
    e_{14} \\
    f_{19} \\
    f_{4} \\
    f_{10} \\
    e_{7} \\
    f_{13} \\
    e_{20} \\
    f_{3} \\
    f_{9} \\
    f_{6} \\
    f_{18}
\end{bmatrix} = \begin{bmatrix}
    0 & -a & 0 & 0 & 0 & -1 & 0 & 0 & n_m & 0 & 0 & -a \\
    a & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & n_g & 0 \\
    0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
    a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(79)

In this case, \( R_{22} \neq 0 \), and from (51), (78), and (79),

\[
M_r = \begin{bmatrix}
    \Delta b_m & 0 & 0 & 0 & 0 \\
    0 & \frac{\Delta}{b_k} & 0 & 0 & 0 \\
    0 & 0 & R_c + R_m & R_c & R_c R_m \\
    0 & 0 & R_c & R_c + R_g & -R_c R_g \\
    0 & 0 & -R_c R_m & R_c R_g & R_c R_g R_m
\end{bmatrix}
\]

(80)

where \( \Delta = R_c R_g + R_c R_m + R_g R_m \).

By substituting (58), (60), (79), and (80) into (53),

\[
A^r_{21} = \frac{1}{\Delta} \begin{bmatrix}
    -\frac{n_m L_g}{J_m} R_c \\
    0 \\
    \frac{-n_m L_g}{J_m} R_c \\
    0
\end{bmatrix}
\]

(81)

from (54), (60), (79), and (80),

\[
B^2_r = \frac{1}{\Delta} \begin{bmatrix}
    n_y L_g (R_c + R_m) \\
    n_y L_m R_c
\end{bmatrix}
\]

(82)

and from (52), (81), and (82), the real roots of the medium dynamics are

\[
\begin{bmatrix}
    \bar{p}_4 \\
    \bar{p}_{19}
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
    \frac{-n_m L_g}{J_m} R_c & 0 \\
    \frac{-n_m L_m}{J_m} (R_c + R_g) & 0
\end{bmatrix}
\]

(83)

In this case, \( E_r = E_j \) and substituting (58), (79), and (80) into (48), the state matrix for the reduced slow system is given by
Figure 16: State variables of the complete system: (a) slow dynamics; (b) medium dynamics; (c) fast dynamics.

Figure 17: Continued.
Figure 17: State variables of the reduced system: (a) slow dynamics $p_{14}$; (b) slow dynamics $q_{19}$; (c) medium dynamics $p_4$; (d) medium dynamics $p_{10}$.

Figure 18: State variables for the slow dynamics: (a) $p_{14}$ and $\tilde{p}_{14}$; (b) $q_{19}$ and $\tilde{q}_{19}$.

Figure 19: Sensitivity of $p_{10}$ respect to $C_e$. 
and from (88) and (89), we can conclude that the Ward Leonard system is a three timescale system.

Now, the simulation of this case study is obtained using 20-sim software. Figure 16 shows the state variables of the complete system.

The performance of the reduced system given in (77) is shown in Figure 17. Note that, the exact and reduced models are very close.

Figure 18 shows the most reduced system defined in (86). When the slow dynamics of a system are important for analysis or synthesis, the quasi-steady-state model instead of the original system can be used.

The use of sensitivity analysis can give information about the coupling of the different timescales. In general, systems with different timescales are by definition weakly coupled. However, in singular perturbation theory, this coupling is taken into account. At each step of reduction, the coupling of the neglected timescale is taken into account in the remaining equations. Sensitivity according to [31, 32] gives the augmented bond graph from which the sensitivity (92) is calculated:

\[ \mathcal{L} \left( \frac{\partial y}{\partial \Delta} \right) = \left[ C_n (sI - A_n)^{-1} C_m + D_n \right] (\mathcal{ZW})(s) \]

where \( C_m \) is selected as \( P_{10} \) which is a medium dynamics, and it is coupled to the fast and slow dynamics.

The sensitivity of \( p_{10} \) respect to \( C_c \) is given by

\[ \frac{\partial p_{10}}{\partial C_c} = \frac{n_p S_1}{C_c K_c L_g b_k (J_c a^2 + J_m)} \frac{1}{\Delta^2} \frac{\partial \Delta}{\partial C_c} \]

where \( \Delta = \text{det} (sI - A) \) and

\[ S_1 = \left( b_m + J_m s + a^2 b_k + J_c a^2 s + J_m K_c b_k s^2 + K_c b_k b_m s + J_c K_c a^2 b_k s^2 \right) \]  \hspace{1cm} (92)

The sensitivity of \( p_{10} \) respect to \( J_m \) is

\[ \frac{\partial p_{10}}{\partial C_c} = \frac{n_p S_1}{C_c K_c L_g b_k (J_c a^2 + J_m)} \frac{1}{\Delta^2} \frac{\partial \Delta}{\partial J_m} \]

In the case of singular perturbation, the sensitivity due to the changes in \( C_c \) and \( J_m \) can be better represented by the difference between the original state variable and the sensitivities in (91) and (93). Figures 19 and 20 present the sensitivity respect to \( C_c \) and \( J_m \).

6. Conclusion

A three timescale system modelled by bond graphs has been presented. The state-space representation of the full system including slow, medium, and fast dynamics based on a bond
graph model with all storage elements in an integral causality assignment is determined. In order to obtain reduced models in the physical domain, two lemmas are proposed. Lemma 1 establishes a derivative causality assignment to the storage elements for the fast dynamics and an integral causality assignment to the storage elements for the slow and medium dynamics into the bond graph called SPBGs which is defined by using Lemma 2, and then the junction structure and the quasi-steady-state model through this lemma are determined.

Undoubtedly, the algebraic approach given in Section 2 to find reduced models of singularly perturbed systems can be used. However, the mathematical model of this system must be gotten. Also, the quasi-steady-state model removing the fast dynamics given by using (7) requires that the submatrix $A_{33}$ has to be nonsingular and an algebraic process. If SPBGs can be obtained, then $A_{33}$ is nonsingular and Lemma 1 gives a direct way to find the previous results. The quasi-steady-state model for the slow dynamics by using the algebraic approach is given by $A_1(t) = [A_{11} - A_{13}A_{33}^{-1}(A_{31} - (A_{12} - A_{13}A_{33}^{-1}A_{32})(A_{22} - A_{23}A_{33}^{-1}A_{31})^{-1}(A_{21} - A_{23}A_{33}^{-1}A_{31})]x(t) + [B_1 - A_{13}A_{33}^{-1}B_3 - (A_{12} - A_{13}A_{33}^{-1}A_{32})(A_{22} - A_{23}A_{33}^{-1}A_{31})^{-1}(B_2 - A_{23}A_{33}^{-1}A_{31})]u(t)$, and it is clear that the bond graph defined by SPBGs gives the same result in a direct and easier way.

Thus, reduced models based on the manipulation of the causality of the bond graphs have been proposed. A class of nonlinear systems and linear time varying (LTV) systems with three timescales represent the future works of this paper.

**Appendix**

**A. Proof of Lemma 1**

From the seventh and eighth lines of (32) with (16) to (21) and applying the derivative with respect to the time,

$$x_1^d(t) = \left(I - H_{14}^{11}(F_1^{11})^{-1}H_{14}^{11}F_1^{11}\right)x_1^d(t),$$

$$x_2^d(t) = \left[I - H_{14}^{11}(F_1^{11})^{-1}H_{14}^{11}F_1^{11}\right]x_2^d(t),$$

and from the fifth line of (32) with (31)

$$D_{in}^h(t) = (I - H_{22}L_0^{-1}) (H_{21}^{11}z_1(t) + H_{21}^{12}z_2(t) + H_{21}^{13}z_3(t) + H_{23}u(t)).$$

From the first to fourth lines of (32) with (31),

$$\begin{bmatrix} x_1^d \\ x_2^d \\ z_3 \\ z_3^d \end{bmatrix} = \begin{bmatrix} H_{11}^{11} & H_{11}^{12} & H_{11}^{13} & H_{11}^{14} \\ H_{11}^{21} & H_{11}^{22} & H_{11}^{23} & H_{11}^{24} \\ H_{11}^{31} & H_{11}^{32} & H_{11}^{33} & H_{11}^{34} \\ H_{11}^{41} & H_{11}^{42} & H_{11}^{43} & H_{11}^{44} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_3^d \end{bmatrix} + \begin{bmatrix} H_{11}^{12} \\ H_{11}^{22} \\ H_{11}^{32} \\ H_{11}^{42} \end{bmatrix}u + \begin{bmatrix} H_{11}^{13} \\ H_{11}^{23} \\ H_{11}^{33} \\ H_{11}^{43} \end{bmatrix}H_{hiD_{in}^h,}$

$$\begin{bmatrix} z_1^d \\ z_2^d \\ z_3^d \\ z_3^d \end{bmatrix} = \begin{bmatrix} H_{11}^{11} & H_{11}^{12} & H_{11}^{13} & H_{11}^{14} \\ H_{11}^{21} & H_{11}^{22} & H_{11}^{23} & H_{11}^{24} \\ H_{11}^{31} & H_{11}^{32} & H_{11}^{33} & H_{11}^{34} \\ H_{11}^{41} & H_{11}^{42} & H_{11}^{43} & H_{11}^{44} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_3^d \end{bmatrix} + \begin{bmatrix} H_{11}^{12} \\ H_{11}^{22} \\ H_{11}^{32} \\ H_{11}^{42} \end{bmatrix}u + \begin{bmatrix} H_{11}^{13} \\ H_{11}^{23} \\ H_{11}^{33} \\ H_{11}^{43} \end{bmatrix}H_{hiD_{in}^h,}$

(A.5)

Considering the linearly independent state variables, (A.5) with (36) and (37) can be reduced to
and by substituting (A.12), (A.13), and (A.14) into (6), the solution for the fast dynamics (39) is proved.

Now, substituting (A.10) into the second line of (A.6),

\[ \varepsilon_1 x_2 = A_{21}^h - A_{23}^h (A_{33}^h)^{-1} A_{31}^h x_1 + A_{22}^h - A_{23}^h (A_{33}^h)^{-1} A_{32}^h x_2 + A_{23}^h (A_{33}^h)^{-1} x_3 + B_{2}^h - A_{23}^h (A_{33}^h)^{-1} B_{3}^h u, \]

(A.15)

and comparing the second line of (1) with (A.15), the relationships between BGI and SPBG\(\tilde{h}\) for the medium dynamics are

\[ A_{23}^h = A_{23} A_{33}^{-1} \varepsilon_2, \]

(A.16)

\[ A_{21}^h = A_{21} - A_{23} A_{33}^{-1} A_{31}, \]

(A.17)

\[ A_{22}^h = A_{22} - A_{23} A_{33}^{-1} A_{32}, \]

(A.18)

\[ B_{2}^h = B_{2} - A_{23} A_{33}^{-1} B_{3}, \]

(A.19)

From (A.10) and the first line of (A.6),

\[ x_1 = \left[ A_{11}^h - A_{13}^h (A_{33}^h)^{-1} A_{31}^h \right] x_1 + \left[ A_{12}^h - A_{13}^h (A_{33}^h)^{-1} A_{32}^h \right] x_2 + \left[ A_{13}^h (A_{33}^h)^{-1} \right] x_3 + \left[ B_{1}^h - A_{13}^h (A_{33}^h)^{-1} B_{3}^h \right] u, \]

(A.20)

Comparing the first line of (1) with (A.20), the relationships between BGI and SPBG\(\tilde{h}\) for the slow dynamics are

\[ A_{13}^h = A_{13} A_{33}^{-1} \varepsilon_2, \]

(A.21)

\[ A_{11}^h = A_{11} - A_{13} A_{33}^{-1} A_{31}, \]

(A.22)

\[ A_{12}^h = A_{12} - A_{13} A_{33}^{-1} A_{32}, \]

(A.23)

\[ B_{1}^h = B_{1} - A_{13} A_{33}^{-1} B_{3}, \]

(A.24)

from (A.17) to (A.19) with (A.22) to (A.24), the quasi-steady-state model given by (33) is proved.

By substituting (A.10) into (A.9),

\[ y = \left[ C_1^h - C_3^h (A_{33}^h)^{-1} A_{31}^h \right] x_1 + \left[ C_2^h - C_3^h (A_{33}^h)^{-1} A_{32}^h \right] x_2 + C_3 (A_{33}^h)^{-1} x_3 + \left[ D^h - C_3 (A_{33}^h)^{-1} B_{3}^h \right] u, \]

(A.25)

Comparing (2) with (A.25), the relationships between BGI and SPBG\(\tilde{h}\) for the output are
\[ C_3^h = C_3 A_{33}^{-1} \varepsilon_2, \quad (A.26) \]
\[ C_1^h = C_1 - C_3 A_{33}^{-1} A_{31}, \quad (A.27) \]
\[ C_2^h = C_2 - C_3 A_{33}^{-1} A_{32}, \quad (A.28) \]
\[ D_3^h = D - C_3 A_{33}^{-1} B_3, \quad (A.29) \]

and from (9), (A.27), (A.28), and (A.29), (42) is proved.

**B. Proof of Lemma 2**

From the derivative with respect to time of the last line of (46) with (16) and (17),

\[ \dot{x}_1^d = \left( F_t \right)^{-1} R_{41}^i F_1 \dot{x}_1, \quad (B.1) \]

from the sixth line of (46) with (45),

\[ D_m' = (I - R_{22} L_r)^{-1} \]

\[ \left( R_{11}^i z_1 + R_{31}^i \dot{x}_2 + R_{32}^i \dot{x}_3 + R_{33}^i x_d^d + R_{34}^i \dot{x}_4^d + R_{35}^i \dot{x}_3 \right), \quad (B.2) \]

\[
\begin{bmatrix}
I - R_{14} F_t^{-1} R_{41} F_1 & \dot{x}_1^d \\
z_2 & 0 \\
z_3 & 0 \\
z_4^d & 0 \\
z_3 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
R_{11}^i & R_{12}^i & R_{13}^i & R_{14}^i & R_{15}^i \\
R_{21}^i & R_{22}^i & R_{23}^i & R_{24}^i & R_{25}^i \\
R_{31}^i & R_{32}^i & R_{33}^i & R_{34}^i & R_{35}^i \\
R_{41}^i & R_{42}^i & R_{43}^i & R_{44}^i & R_{45}^i \\
R_{51}^i & R_{52}^i & R_{53}^i & R_{54}^i & R_{55}^i \\
\end{bmatrix}
\begin{bmatrix}
z_2 \\
z_3 \\
z_4^d \\
z_3 \\
\end{bmatrix}
+ \begin{bmatrix}
R_{11}^i \cdot \\
R_{12}^i \cdot \\
R_{13}^i \cdot \\
R_{14}^i \cdot \\
R_{15}^i \cdot \\
\end{bmatrix}
\begin{bmatrix}
R_{11}^i \cdot \\
R_{12}^i \cdot \\
R_{13}^i \cdot \\
R_{14}^i \cdot \\
R_{15}^i \cdot \\
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1^d \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4^d \\
\dot{x}_3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_{11}^i M_{12} R_{21}^i + R_{12}^i M_{12} R_{21}^i + R_{13}^i M_{12} R_{21}^i + R_{14}^i M_{12} R_{21}^i + R_{15}^i M_{12} R_{21}^i \\
R_{21}^i M_{12} R_{21}^i + R_{22}^i M_{12} R_{21}^i + R_{23}^i M_{12} R_{21}^i + R_{24}^i M_{12} R_{21}^i + R_{25}^i M_{12} R_{21}^i \\
R_{31}^i M_{12} R_{21}^i + R_{32}^i M_{12} R_{21}^i + R_{33}^i M_{12} R_{21}^i + R_{34}^i M_{12} R_{21}^i + R_{35}^i M_{12} R_{21}^i \\
R_{41}^i M_{12} R_{21}^i + R_{42}^i M_{12} R_{21}^i + R_{43}^i M_{12} R_{21}^i + R_{44}^i M_{12} R_{21}^i + R_{45}^i M_{12} R_{21}^i \\
R_{51}^i M_{12} R_{21}^i + R_{52}^i M_{12} R_{21}^i + R_{53}^i M_{12} R_{21}^i + R_{54}^i M_{12} R_{21}^i + R_{55}^i M_{12} R_{21}^i \\
\end{bmatrix}
\begin{bmatrix}
\dot{z}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4^d \\
\dot{x}_3 \\
\end{bmatrix}
+ \begin{bmatrix}
R_{11}^i \cdot \\
R_{12}^i \cdot \\
R_{13}^i \cdot \\
R_{14}^i \cdot \\
R_{15}^i \cdot \\
\end{bmatrix}
\begin{bmatrix}
R_{11}^i \cdot \\
R_{12}^i \cdot \\
R_{13}^i \cdot \\
R_{14}^i \cdot \\
R_{15}^i \cdot \\
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{u} \\
\dot{u} \\
\dot{u} \\
\dot{u} \\
\end{bmatrix}
\]

From (50) and (B.4), the reduced state-space representation can be written by

\[
\begin{bmatrix}
E \cdot \dot{x}_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix}
A_{11}' & A_{12}' & A_{13}' \\
A_{21}' & A_{22}' & A_{23}' \\
A_{31}' & A_{32}' & A_{33}' \\
\end{bmatrix}
\begin{bmatrix}
F_1 & 0 & 0 \\
0 & F_2 & 0 \\
0 & F_3 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
+ \begin{bmatrix}
B_{11}' \\
B_{12}' \\
B_{13}' \\
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{u} \\
\dot{u} \\
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
A_{11}' & A_{12}' & A_{13}' \\
A_{21}' & A_{22}' & A_{23}' \\
A_{31}' & A_{32}' & A_{33}' \\
\end{bmatrix}
= \begin{bmatrix}
R_{11}^i + R_{11}^i R_{12} M_{12} R_{21}^i + R_{12}^i M_{12} R_{21}^i + R_{13}^i M_{12} R_{21}^i + R_{14}^i M_{12} R_{21}^i + R_{15}^i M_{12} R_{21}^i \\
R_{21}^i + R_{21}^i R_{22} M_{12} R_{21}^i + R_{22}^i M_{12} R_{21}^i + R_{23}^i M_{12} R_{21}^i + R_{24}^i M_{12} R_{21}^i + R_{25}^i M_{12} R_{21}^i \\
R_{31}^i + R_{31}^i R_{32} M_{12} R_{21}^i + R_{32}^i M_{12} R_{21}^i + R_{33}^i M_{12} R_{21}^i + R_{34}^i M_{12} R_{21}^i + R_{35}^i M_{12} R_{21}^i \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
A_{11}' & A_{12}' & A_{13}' \\
A_{21}' & A_{22}' & A_{23}' \\
A_{31}' & A_{32}' & A_{33}' \\
\end{bmatrix}
= \begin{bmatrix}
R_{11}^i + R_{11}^i R_{12} M_{12} R_{21}^i + R_{12}^i M_{12} R_{21}^i + R_{13}^i M_{12} R_{21}^i + R_{14}^i M_{12} R_{21}^i + R_{15}^i M_{12} R_{21}^i \\
R_{21}^i + R_{21}^i R_{22} M_{12} R_{21}^i + R_{22}^i M_{12} R_{21}^i + R_{23}^i M_{12} R_{21}^i + R_{24}^i M_{12} R_{21}^i + R_{25}^i M_{12} R_{21}^i \\
R_{31}^i + R_{31}^i R_{32} M_{12} R_{21}^i + R_{32}^i M_{12} R_{21}^i + R_{33}^i M_{12} R_{21}^i + R_{34}^i M_{12} R_{21}^i + R_{35}^i M_{12} R_{21}^i \\
\end{bmatrix}
\]
where $\text{Cr}$ is a relationship between BGI and SPBG proved. and by using (B.10) to (B.12), the algebraic equation (52) is proved. Equation (B.16) is rewritten by and from (B.5) to (B.7), the expressions (48), (49), (53) and (54) are proved.

The reduced system of (B.5) can be written by

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
A'_{11} & A'_{12} \\
A'_{21} & A'_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
B'_1 \\
B'_2
\end{bmatrix}u,
$$

(B.8)

where $A'_{11} = E_r^{-1}A'_{11}F_1$, $A'_{12} = E_r^{-1}A'_{12}F_1$, $A'_{21} = F_1^{-1}A'_{21}F_1$, and $B'_1 = E_r^{-1}B'_1$. From the second line of (B.8),

$$
\dot{x}_2 = -(A'_{22})^{-1}A'_{21}x_1 + (A'_{22})^{-1}x_2 - (A'_{22})^{-1}B'_2u. 
$$

(B.9)

Comparing (7) with (B.9) and doing $\dot{x}_2 = 0$, the relationships between BGI and SPBG are

$$
A'_{22} = (\overline{A'_{22}})^{-1}e_1, 
$$

(B.10)

$$
A'_{21} = -\overline{(A_{22})^{-1}}\overline{A_{21}}, 
$$

(B.11)

and by using (B.10) to (B.12), the algebraic equation (52) is proved.

By substituting (B.9) into the first line of (B.8), the relationships between BGI and SPBG are

$$
A'_{12} = \overline{A_{12}}(\overline{A_{22}})^{-1}e_1, 
$$

(B.13)

$$
A'_{11} = \overline{A_{11}} - \overline{A_{12}}(\overline{A_{22}})^{-1}\overline{A_{21}}, 
$$

(B.14)

and from (B.13) to (B.15) with (12), (47) is proved. From the seventh line of (46) with (B.2) and (51),

$$
y = (R'_{11} + R_{32}M_{r}R'_{21})x_1 + (R'_{21} + R_{32}M_{r}R'_{21})x_2 \\
+ (R^{13}_{31} + R_{32}M_{r}R_{21}^{13})x_3 + (R_{33} + R_{32}M_{r}R_{23})u,
$$

(B.16)

and by using (B.16), the expressions (56) and (57) are proved. Equation (B.16) is rewritten by

$$
y = C'_1x_1 + C'_2x_2 + C'_3x_3 + D'u, 
$$

(B.17)

where $C'_1 = (R'_{11} + R_{32}M_{r}R'_{21})F_1$, $C'_2 = R'_{12} + R_{32}M_{r}R'_{21}$, $C'_3 = R^{13}_{31} + R_{32}M_{r}R_{21}^{13}$, and $D' = R_{33} + R_{32}M_{r}R_{23}$. By substituting (B.9) into (B.17) with $x_4 = 0$,

$$
y = C'_1(A'_{22})^{-1}A'_{21}x_1 + C'_1(A'_{22})^{-1}x_2 \\
- C'_2(A'_{22})^{-1}B'_2u + D'u, 
$$

(B.18)

and comparing (B.18) with $y = C_1x_1 + C_2x_2 + Du$, the relationships between BGI and SPBG are

$$
C'_1 = C_1 - C_2(A_{22})^{-1}A_{21},
$$

(B.19)

$$
C'_1 = C_1 - C_2(A_{22})^{-1}A_{21},
$$

(B.20)

and from (14), the output of this reduced system given by (55) is proved.

**Data Availability**

The typical and traditional data used to support the findings of this study are included within the article. In particular, we used the data for the Ward-Leonard system from the typical DC machines with mechanical loads.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


