Research Article

MHD Slip Flow of CNT-Ethylene Glycol Nanofluid due to a Stretchable Rotating Disk with Cattaneo–Christov Heat Flux Model

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1. Introduction

Nanofluids have gained remarkable attention of researchers due to their inspiring heat transfer in various industrial and engineering applications. Common working fluids such as water, engine oils, and ethylene glycol have restricted thermal performances which limit their usage in modern-day cooling applications. Nanofluids consist of nanoscale particles such as copper, alumina, carbides, nitrides, metal oxides, graphite, and carbon nanotubes which enhance the thermal conductivity of base fluids (Mahanthesh et al. [1] and Ahmad et al. [2]). These nanofluids have widespread applications in modern systems of heating and cooling, solar cells, generation of new fuels, hybrid-powered engines, cancer therapy, drug delivery, and medicine (Hsiao [3] and Aziz et al. [4]). Due to these various applications of nanofluids, many studies associated with the flow of nanofluids have been conducted. For instance, Prasannakumara et al. [5] examined the boundary layer flow and heat transfer of fluid particles suspension with nanoparticles over a nonlinear stretching sheet. Khan et al. [6] analyzed three-dimensional steady MHD flow of Powell–Eyring nanofluid with convective and nanoparticles mass flux conditions. Also, Khan et al. [7] investigated unsteady three-dimensional Sisko magneto-nanofluid flow with heat absorption and temperature-dependent thermal conductivity. Recently, three-dimensional bioconvection nanofluid flow from a biaxial stretching sheet was presented by Amirsom et al. [8]. Besides, convinced works in this direction are explained by Shashikumar et al. [9], Khan et al. [10], Uddin et al. [11], and Khan et al. [12].

In the last two decades, heat transfer of carbon nanofluids has received considerable attention as a result of their extensive applications in the fields of nanotechnology and medicine. Nanoparticles of carbon nanotubes (CNTs) were...
first discovered in 1991 by Lijima [13]. Carbon nanotubes are allotropes of carbon with a tube-shaped nanostructure such as frames of carbon atoms with diameter ranges from 1 to 100 nm. They have a remarkable conductivity which supports them to form a network of conductive tubes. Because of their configuration, some CNTs are good conductor of heat and electricity whereas others act as semiconductors. CNTs have 15 times thermal conductivity and 1000 times capability of copper and 200 times power and 5 times resistance of steel (Prajapati et al. [14] and Khalid et al. [15]). Carbon nanotubes have significance applications in nanotechnology, hardware, optics, energy storage, biomedical, ceramic, thermal defense, and various fields of material sciences and engineering (Hayat et al. [16]). Carbon nanotubes used in nanofluids are generally divided into single-wall carbon nanotubes (SWCNTs) and multiwall carbon nanotubes (MWCNTs) contingent on their number of concentric layers of rolled graphene sheets. SWCNTs contain catalyst for their synthesis, and they are prepared from covering layer of graphene sheets whereas MWCNTs can be formed without catalyst and comprise of numerous rolled layers of graphite with complex structure (Khan et al. [17]).

Because of their unique morphology, electronic structural, mechanical properties, and innovative physicochemical features, CNTs are the most resourceful material of this century. Moreover, the presence of carbon chains in CNTs does not convey any hazard to the atmosphere (Alsagri et al. [18]). Having all these implication of CNTs in mind, Kumaresan et al. [19] experimentally studied the heat transfer characteristics of nanofluids containing CNTs, and they disclosed nanofluids at very low nanoparticle volume fraction enhance high heat transfer rate. Heat transfer performance of CNT nanofluids flow through a horizontal tube was examined by Ding et al. [20]. They obtained essential progress of the convective heat transfer which mainly depends on Reynolds number and solid volume fraction of CNTs. The synthetic engine oil and ethylene glycol thermal conductivity improvement in the presence of MWCNTs was designated by Liu et al. [21]. They revealed that CNT-ethylene glycol nanofluid have better thermal conductivity as compared with ethylene glycol base fluid. Also, Mahanthesh et al. [22] investigated the Marangoni transport of dissipating SWCNT and MWCNT with water nanofluids under the influence of magnetic force and radiation. They reported that the thermal distribution of SWCNT nanoliquid is better than MWCNT nanoliquid. Some contributions on CNT-based nanofluids can be reviewed (Aman et al. [23], Khan et al. [23], Asadi et al. [24, 25], and Nasir et al. [26]).

Rotating disk-induced flow of fluids have significance applications in numerous engineering and industrial sectors such as rotating machinery, electric power generating system, computer storage devices, air cleaning machines, crystal growth processes, gas turbine rotors, food processing, medical equipment, and others. Due to these, immense papers have been published regarding the flow of fluid due to rotating disk. For example, Hayat et al. [27] and Mustafa [28] analyzed MHD flow of nanofluid by a rotating disk with partial slip effects. Nanofluid flow near a stretchable rotating disk with axial magnetic field and convective conditions was analyzed by Mushtaq and Mustafa [29]. Also, Imtiaz et al. [30] studied the radiative flow of CNTs between stretchable rotating disks with convective conditions. Recently, the impacts of exponential space-dependent heat source on MHD slip flow of SWCNT and MWCNT nanoliquids past a stretchable rotating disk was investigated by Mahanthesh et al. [31]. Their result established that the thermal field for SWCNT-nanoliquid is higher than MWCNT-nanoliquid. Additionally, some efforts in this direction are published by Hayat et al. [32, 33], Khan et al. [34], and Mahanthesh et al. [35].

The dynamics of heat transfer is very useful due to its abundant applications in industrial, engineering, and biomedical applications. Fourier was the first scholar who established the most successful classical heat flux model in continuum mechanics (Akbar et al. [36]). The main drawback of this model is the temperature field, and the whole system is instantly affected by initial disturbance. To avoid this unrealistic feature, first, Cattaneo [37] improved Fourier’s law by adding thermal relaxation time which tolerates the heat flux. Then, Christov [38] more modified the Cattaneo heat flux model from Maxwell–Cattaneo’s model. Also, the uniqueness of Cattaneo–Christov model for incompressible flow of fluids was tested by Tibblo and Zampoli [39]. Presently, much interest has been shown in the study of Cattaneo–Christov heat flux mode (Kundu et al. [40], Makinde et al. [41], Gangadhar et al. [42], and Hayat et al. [43]).

Motivated by the above cited literatures, the purpose of this article is to study the flow of CNTs with ethylene glycol nanofluids due to stretchable rotating disk with Cattaneo–Christov heat flux model. Governing equations of the flow are highly nonlinear-coupled differential equations and solved numerically by employing spectral quasi-linearization method (SQLM). The spectral collocation method converges fast and provides more accurate approximations with less grid points (Mota [44], Ibrahim and Tulu [45], and Uddin et al. [46]). To the best of the authors’ knowledge, no analysis has been published so far in this direction. Particularly, examining the effect of the Cattaneo–Christov heat flux model on SWCNTs and MWCNTs with ethylene glycol (C2H8O2) nanofluids flow and heat transfer using SQLM would be the contribution of this paper to the existing body of knowledge. Moreover, the effects of embedded parameters such as nanoparticle volume fraction, axially applied magnetic field, stretching factor, velocity, and thermal slip factors on velocity and temperature fields as well as skin friction coefficient and local heat transfer rate are examined, and the results are presented and discussed using graphs and tables.

2. Mathematical Description of Problem

We consider steady and incompressible flow of nanofluid due to stretchable rotating disk. We assume a nonrotating cylindrical coordinate frame (r, θ, z), and the velocity...
components \((v_r, v_\theta, v_z)\) represent in the directions of increasing \((r, \theta, z)\). The disk is rotated along \(z\)-axis with angular velocity \(\Omega\), and also, it is stretched in the radial direction with stretching rate \(s\). The nanofluid flow is exposed to the magnetic field of uniform strength \(B_0\) acting parallel to the \(z\)-direction (Figure 1). Induced magnetic field caused by the motion of electrically conducting nanofluid is not considered because it is very small compared with \(B_0\). The surface temperature of the disk and the ambient temperature are, respectively, denoted by \(T_w\) and \(T_\infty\). Moreover, effects of velocity and thermal slip boundary conditions are considered, and the Cattaneo–Christov heat flux model is used to analyze the heat transfer.

Assuming \((\partial p/\partial r) = (\partial p/\partial z) = 0\), the governing equations for mass, momentum, and thermal energy transfer of Casson nanofluid flow past a stretchable rotating disk with axially applied magnetic field and slip boundary conditions are given as follows (Aziz et al. [4] and Hayat et al. [33]):

\[
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0, \tag{1}
\]

\[
\rho_{nf} \left( \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right) = \mu_{nf} \left[ 1 + \frac{1}{\gamma} \right] \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} \right) + \frac{\partial^2 v_z}{\partial z^2} - \frac{\sigma_{nf} B_0^2 v_r}{r^2}, \tag{2}
\]

\[
\rho_{nf} \left( \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + \frac{\partial v_\theta}{\partial z} \right) = \mu_{nf} \left[ 1 + \frac{1}{\gamma} \right] \left( \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} \right) + \frac{\partial^2 v_z}{\partial z^2} - \frac{\sigma_{nf} B_0^2 v_\theta}{r^2}, \tag{3}
\]

\[
\rho_{nf} \left( \frac{\partial v_z}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right) = \mu_{nf} \left[ 1 + \frac{1}{\gamma} \right] \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right), \tag{4}
\]

\[
\left( \rho c_p \right)_{nf} \left( \frac{\partial T}{\partial r} + \frac{v_r}{r} \frac{\partial T}{\partial z} \right) = -\nabla \cdot \mathbf{q}, \tag{5}
\]

with boundary conditions
\[
\begin{align*}
v_r & = rs + L_0 \left( 1 + \frac{1}{\gamma} \right) \frac{\partial v_r}{\partial z}, \\
v_\theta & = r \Omega + L_0 \left( 1 + \frac{1}{\gamma} \right) \frac{\partial v_\theta}{\partial z}, \\
v_z & = 0, \\
T & = T_w + N_0 \frac{\partial T}{\partial z} \text{ at } z = 0, \\
v_r & \rightarrow 0, \quad v_\theta \rightarrow 0, \quad T \rightarrow T_\infty \text{ as } z \rightarrow \infty.
\end{align*} \tag{6}
\]

where \(L_0\) and \(N_0\) are the velocity slip and thermal slip factors, respectively.

In this study, even if nanofluids using a two-component model are considered, we assumed that there is no agglomeration of CNT nanoparticles within the nanofluid, and the base fluid and CNTs are assumed to be in thermal equilibrium and no slip occurs between them. Therefore, the heat flux vector \(\mathbf{q}\) satisfying the Cattaneo–Christov diffusion model can be adapted for nanofluid and given as (Akbar et al. [36], and Gangadhar et al. [42])

\[
\lambda_1 \left[ \frac{\partial q}{\partial t} + \nabla \cdot \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{v} \right] + \mathbf{q} = -k \nabla T, \tag{7}
\]

where \(\lambda_1\) is the relaxation time of heat flux, \(k\) is the thermal conductivity, and \(\mathbf{v}\) is the velocity vector. Equation (7) gives Fourier’s law when \(\lambda_1 = 0\).

For steady flow, equation (7) is simplified into

\[
\lambda_1 \left[ \nabla \cdot ((\nabla \cdot \mathbf{q}) \mathbf{v}) \right] + \nabla \cdot \mathbf{q} = -k \nabla^2 T, \tag{8}
\]

Substituting equations (8) into (5) and simplifying, we get the following energy equation:

\[
\frac{\partial T}{\partial r} + \frac{v_r}{r} \frac{\partial T}{\partial z} = \alpha_{nf} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) - \lambda_1 \left[ \frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2} + 2 v_r v_z \frac{\partial^2 T}{\partial r \partial z} \right. \\
& \left. + \left( \frac{v_r}{r} \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) \frac{\partial T}{\partial r} + \left( v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) \frac{\partial T}{\partial z} \right], \tag{9}
\]

The non-Newtonian Casson parameter \(\gamma\) is defined as \(\gamma = \mu_{nf} \sqrt{2 \pi c / T_{nf}}\), whereas the rheological equation of state for the Cauchy stress tensor of Casson fluid flow can be expressed as (Khalid et al. [15] and Makinde et al. [41])
where \( k_{ij} = (1/2)((\partial v_i/\partial x_j) + (\partial v_j/\partial x_i)) \) is the rate of strain tensor, \( \pi_k k_{ij} \) is the \( \text{i}^{th} \) and \( j^{th} \) product of the components of the deformation rate tensor with itself, \( \pi_k \) is the critical value of \( \pi \), \( \mu_k \) is the Casson fluid dynamic viscosity coefficient, \( T_y \) is the yield stress of Casson fluid, and \( v_i \) and \( v_j \) are the velocity components.

The thermophysical properties of the nanoparticles (SWCNTs and MWCNTs) and base fluid (ethylene glycol) are given in Table 1 (Khalid et al. [15] and Yunus and Ghajar [47]).

Base fluid and the nanoparticles must be in thermal equilibrium and no slip should take place between them. The nanofluid effective density \( \rho_{nf} \), specific heat capacity \( (\rho c_p)_{nf} \), dynamic viscosity \( \mu_{nf} \) and electrical conductivity \( \sigma_{nf} \) are given as (Alsagri et al. [18] and Khalid et al. [15])

\[
\rho_{nf} = \phi_v \rho_{CNTs} + (1 - \phi_v) \rho_f, \\
(\rho c_p)_{nf} = \phi_v (\rho c_p)_{CNTs} + (1 - \phi_v) (\rho c_p)_{f}, \\
\mu_{nf} = \frac{1}{(1 - \phi_v)^{2}}, \\
\sigma_{nf} = 1 + \frac{3((\sigma_{CNTs}/\sigma_f) - 1) \phi_v}{\left((\sigma_{CNTs}/\sigma_f) + 2\right) - ((\sigma_{CNTs}/\sigma_f) - 1) \phi_v}.
\]

(11)

where \( \phi_v \) is the nanoparticle volume fraction; \( \rho_{CNTs} \), \( (\rho c_p)_{CNTs} \), and \( \sigma_{CNTs} \) are, respectively, density, specific heat capacity, and electrical conductivity of CNTs, whereas \( \rho_f \), \( (\rho c_p)_{f} \), \( \mu_f \) and \( \sigma_f \) are, respectively, density, specific heat capacity, dynamic viscosity, and electrical conductivity of the base fluid (ethylene glycol). The nanoparticle volume fraction \( \phi_1 \), \( \phi_2 \), and \( \phi_3 \) are defined as

\[
\phi_1 = (1 - \phi_v)^{2.5} \left(1 - \phi_v + \phi_v \frac{\rho_{CNTs}}{\rho_f}\right), \\
\phi_2 = 1 - \phi_v + \phi_v \frac{(\rho c_p)_{CNTs}}{(\rho c_p)_{f}}, \\
\phi_3 = \left(1 + \frac{3((\sigma_{CNTs}/\sigma_f) - 1) \phi_v}{\left((\sigma_{CNTs}/\sigma_f) + 2\right) - ((\sigma_{CNTs}/\sigma_f) - 1) \phi_v} \right) \frac{\rho_f}{(1 - \phi_v) \rho_f + \phi_v \rho_{CNTs}}.
\]

(13)

For effective thermal conductivity of CNT nanofluid \( k_{nf} \), Xue’s model which is effective for spherical and elliptical shape is utilized as follows (Alsagri et al. [18]):

\[
k_{nf} = \frac{1 - \phi_v + 2 \phi_v \left(k_{CNTs}/(k_{CNTs} - k_f)\right) \ln((k_{CNTs} - k_f)/2k_f)}{1 - \phi_v + 2 \phi_v \left(k_{CNTs}/(k_{CNTs} - k_f)\right) \ln((k_{CNTs} - k_f)/2k_f)}
\]

(10)

### 3. Numerical Method of Solutions

We introduce the nondimensional variables based on [4] and [36]:

\[
v_r = r \Omega f'(\xi), \\
v_\theta = r \Omega h(\xi), \\
v_z = -(2 \Omega \nu_f)^{1/2} f(\xi), \\
\theta(\xi) = \frac{T - T_\infty}{T_w - T_\infty}, \\
\xi = \frac{2 \Omega f(\xi)^{1/2}}{v_f}.
\]

(14)

Hence, the continuity equation (1) is satisfied, and the transformed equations for momentums and energy are found as follows:

\[
2\left(1 + \frac{1}{\gamma}\right) f''' + \phi_1 \left[2 f f'' - f'^2 + h^2 - \phi_3 M f'\right] = 0,
\]

(15)

\[
2\left(1 + \frac{1}{\gamma}\right) h''' + \phi_1 \left[2 f h'' - 2 f' h - \phi_3 M h\right] = 0,
\]

(16)

\[
\theta'' + \phi_2 \frac{k_f}{k_{nf}} \left[f \theta'' - \alpha \left(f'^2 \theta'' + f f' \theta''\right)\right] = 0,
\]

(17)

with transformed boundary conditions:

\[
f = 0, \\
f'(0) = A + \left(1 + \frac{1}{\gamma}\right) \delta f''(0), \\
h(0) = 1 + \left(1 + \frac{1}{\gamma}\right) \delta f'(0), \\
\theta(0) = 1 + \beta \theta'(0), \\
f'(\infty) = h(\infty) = \theta(\infty) \longrightarrow 0,
\]

where \( M = \sigma_f B_0^2 / \rho_f \Omega \) is the magnetic parameter, \( Pr = \nu_f / \alpha_f \) is a Prandtl number, \( \alpha = 2 \lambda_f \Omega \) is the thermal relaxation parameter, \( A = s / \Omega \) is the scaled stretching parameter, \( \delta = L_0 \sqrt{2 \Omega / v_f} \) is the velocity slip parameter, and \( \beta = N_0 \sqrt{2 \Omega / v_f} \) is the thermal slip parameter.

The radial and tangential directions shear stress at the surface of the disc for Casson nanofluid are defined as
4. Results and Discussion

Numerical results for CNTs with ethylene glycol nanofluids flow due to a stretchable rotating disk with Cattaneo–Christov heat flux mode are stated here. The spectral quasilinearization method (SQLM) is used for the numerical computation with the number of collocation points $N = 40$ in space $\xi$ and the scaled parameter $L = 10$. The physical properties of ethylene glycol ($C_2H_6O_2$), SWCNTs, and MWCNTs are employed from Table 1. The convergence of the SQLM solution and the stability of the results are tested. Tables 2 and 3 reveal that the convergence of solutions for both SWCNTs and MWCNTs with ethylene glycol nanofluids is achieved at $5^{th}$ order of approximations for all radial wall stress $-f''(0)$, tangential wall stress $-h''(0)$, normalized skin friction coefficient $\sqrt{f''(0)^2 + h''(0)^2}$, and local Nusselt number $-\theta'(0)$.

Table 4 encompasses the computations of normalized skin friction coefficient and local Nusselt number for changing values of important involved parameters such as nanoparticle volume fraction, scaled stretching parameter, Casson fluid parameter, velocity slip parameter, and thermal slip parameter for both SWCNTs and MWCNTs with ethylene glycol nanofluids. Here, it is shown that the normalized skin friction coefficient grows as nanoparticle volume fraction $\phi_s$ increases from 0.01 to 0.1 for SWCNT nanofluid. For both SWCNTs and MWCNT nanofluids, normalized skin friction coefficient enhances as scaled stretching parameter and Casson fluid parameter grow. Further, a decreasing trend in normalized skin friction coefficient is perceived for increasing value of velocity slip parameter. Furthermore, from Table 4, we perceived that for both SWCNT and MWCNT nanofluids, the local Nusselt number shows an increasing tendency for higher values of scaled stretching parameter and Casson fluid parameter. However, it shows a decreasing tendency if the value of Casson fluid parameter is greater than 0.5. It is also noted that the local Nusselt number is reduced for greater values of...
Table 2: SQLM Solutions Convergence for SWCNTs – C₂H₄O₂ when ϕᵣ = 0.05, A = 0.5, M = 5, δ = 0.3, γ = 0.5, Pr = 7.3, β = 0.2, α = 0.3 are used.

<table>
<thead>
<tr>
<th>Order</th>
<th>−fʺ(0)</th>
<th>−hʺ(0)</th>
<th>\sqrt{fʺ(0)^2 + hʺ(0)^2}</th>
<th>−θʺ(0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>0.23689710</td>
<td>0.89154598</td>
<td>0.922482783</td>
<td>0.67685416</td>
</tr>
<tr>
<td>02</td>
<td>0.23359837</td>
<td>0.89107446</td>
<td>0.92519082</td>
<td>0.67655878</td>
</tr>
<tr>
<td>03</td>
<td>0.23359687</td>
<td>0.89107467</td>
<td>0.92118505</td>
<td>0.67679460</td>
</tr>
<tr>
<td>04</td>
<td>0.23359687</td>
<td>0.89107467</td>
<td>0.92118487</td>
<td>0.67679460</td>
</tr>
<tr>
<td>05</td>
<td>0.23359687</td>
<td>0.89107467</td>
<td>0.92118487</td>
<td>0.67679460</td>
</tr>
</tbody>
</table>

Table 3: SQLM Solutions Convergence for MWCNTs – C₂H₄O₂ when ϕᵣ = 0.05, A = 0.5, M = 5, δ = 0.3, γ = 0.5, Pr = 7.3, β = 0.2, α = 0.3 are used.

<table>
<thead>
<tr>
<th>Order</th>
<th>−fʺ(0)</th>
<th>−hʺ(0)</th>
<th>\sqrt{fʺ(0)^2 + hʺ(0)^2}</th>
<th>−θʺ(0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>0.22142446</td>
<td>0.80709704</td>
<td>0.83691960</td>
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</tr>
<tr>
<td>02</td>
<td>0.21820148</td>
<td>0.80799870</td>
<td>0.83694119</td>
<td>0.74087932</td>
</tr>
<tr>
<td>03</td>
<td>0.21819376</td>
<td>0.80799811</td>
<td>0.83694054</td>
<td>0.74082968</td>
</tr>
<tr>
<td>04</td>
<td>0.21819374</td>
<td>0.80799812</td>
<td>0.83694054</td>
<td>0.74082974</td>
</tr>
<tr>
<td>05</td>
<td>0.21819374</td>
<td>0.80799812</td>
<td>0.83694054</td>
<td>0.74082974</td>
</tr>
</tbody>
</table>

nanoparticle volume fraction, velocity slip, and thermal slip parameters for both SWCNT and MWCNT nanofluids. For some parameters, a comparable result was established in the literatures of Aziz et al. [4] and Mushtaq and Mustafa [29].

The study of important embedded parameters on radial velocity \( f'(\xi) \), tangential velocity \( h(\xi) \), and temperature \( \theta(\xi) \) profiles is plotted in Figures 2–11 for \( \phi_r = 0.05 \), \( A = 0.5 \), \( M = 5 \), \( \delta = 0.3 \), \( \gamma = 0.5 \), \( Pr = 7.3 \), \( \beta = 0.2 \), and \( \alpha = 0.3 \). Figures 2 and 3 represent the effects of nanoparticle volume fraction \( (0 < \phi_r < 0.1) \) on the radial velocity and temperature profiles, respectively, for the ethylene glycol- \( (C_2H_4O_2) \) based nanofluid with SWCNTs and MWCNTs. It is shown that, for both SWCNTs and MWCNTs, as the value of \( \phi_r \) increases, the radial velocity and temperature profiles of the nanofluid increase. It is also recognized that there is higher velocity distribution in the MWCNT nanofluid than SWCNT nanofluid, whereas the opposite trend is observed for temperature distribution. This is due to the fact that SWCNTs have higher density than MWCNTs. An analogous result was also presented in Figures 12–15 for embedded parameters \( \phi_r = 0.05 \), \( A = 0.5 \), \( M = 5 \), \( \delta = 0.3 \), \( \gamma = 0.5 \), \( Pr = 7.3 \), \( \beta = 0.2 \), and \( \alpha = 0.3 \). Figure 12 depicts the variation of normalized skin friction coefficient \( C_f \) and the local heat transfer rate \( Nu \) are presented in Figures 12–15 for embedded parameters \( \phi_r = 0.05 \), \( A = 0.5 \), \( M = 5 \), \( \delta = 0.3 \), \( \gamma = 0.5 \), \( Pr = 7.3 \), \( \beta = 0.2 \), and \( \alpha = 0.3 \). Figure 12 depicts the variation of normalized skin friction coefficient with magnetic field parameter \( M \) and nanoparticle volume fraction \( \phi_r \). The normalized skin friction coefficient is improved with increasing values of both \( M \) and \( \phi_r \). From Figure 13, it is also observed that the normalized skin friction coefficient is increased with higher values of scaled stretching parameter \( A \) and Casson fluid parameter \( \gamma \). Form both Figures, it is also recognized that there is a higher normalized skin friction coefficient in the SWCNT nanofluid than in the MWCNT nanofluid. The variation of local heat transfer rate with \( M \) and \( \phi_r \) is demonstrated in Figure 14 for both SWCNT and MWCNT nanofluids. The local heat transfer rate is decreased with the greater value of \( M \). On the other hand, the local heat transfer rate improves with rise values of \( M \) and \( \phi_r \). Further, acting in the axial direction provides resistance force which accounts for dropping the radial and tangential fluid velocities. In Figure 7, an enhancing tendency of temperature profile is perceived as \( M \) increases. Hence, incidence of magnetic field raises the thermal boundary layer thickness as the temperature increases. It is also noticed that there are lower velocity and higher temperature distributions in the SWCNT nanofluid than in the MWCNT nanofluid. This is due to the fact that SWCNTs have higher electrical conductivity than MWCNTs. An analogous result was also found by Aziz et al. [4] and Mushtaq and Mustafa [29]. Figure 8 shows the influence of Casson fluid parameter \( \gamma \) on radial velocity profile. Near the boundary surface, the velocity profile shows an increasing tendency as \( \gamma \) increases. This is anticipated since increasing the Casson fluid parameter \( \gamma \) tends to reduce in yield stress, and the fluid flows easily. However, as far the radial velocity is away from the boundary surface, the reverse effect resulting in the thinning of the momentum boundary layer is due to the presence of the magnetic field effect.

Figure 9 illustrates the effect of velocity slip parameter \( \delta \) on tangential velocity profile for both SWCNT and MWCNT nanofluids. The tangential velocity profile shows diminishing trend for increasing value of \( \delta \). In slip condition, since the fluid velocity near the surface of the sheet is no more equal to the velocity of stretchable rotating disk, the momentum boundary layer thickness slows down as \( \delta \) grows. Figure 10 represents greater thermal slip parameter \( \beta \) leads to the drop of temperature profile for both SWCNT and MWCNT nanofluids with the highest effect recognized at the wall of the disk. Thus, an increases in the thermal slip factor \( \beta \) reduces the heat transfer from the surface of the disk to the fluid. Figure 11 shows the effect of the thermal relaxation parameter \( \alpha \) on temperature distribution. It reveals that the temperature profile reduces with increasing value of \( \alpha \). Physically, thermal relaxation time is the time required by the fluid particles to transfer heat energy to its adjacent particles. Consequently, as \( \alpha \) raises, the material particles need extra time to transfer heat to its adjacent particles and this leads to less transfer of heat from the disk to the fluid.

The important physical quantities, the normalized skin friction coefficient \( C_f \) and the local heat transfer rate \( Nu \) are presented in Figures 12–15 for embedded parameters \( \phi_r = 0.05 \), \( A = 0.5 \), \( M = 5 \), \( \delta = 0.3 \), \( \gamma = 0.5 \), \( Pr = 7.3 \), \( \beta = 0.2 \), and \( \alpha = 0.3 \). Figure 12 depicts the variation of normalized skin friction coefficient with magnetic field parameter \( M \) and nanoparticle volume fraction \( \phi_r \). The normalized skin friction coefficient is improved with increasing values of both \( M \) and \( \phi_r \). From Figure 13, it is also observed that the normalized skin friction coefficient is increased with higher values of scaled stretching parameter \( A \) and Casson fluid parameter \( \gamma \). Form both Figures, it is also recognized that there is a higher normalized skin friction coefficient in the SWCNT nanofluid than in the MWCNT nanofluid. The variation of local heat transfer rate with \( M \) and \( \phi_r \) is demonstrated in Figure 14 for both SWCNT and MWCNT nanofluids. The local heat transfer rate is decreased with the greater value of \( M \). On the other hand, the local heat transfer rate improves with rise values of \( M \) and \( \phi_r \). Further, acting in the axial direction provides resistance force which accounts for dropping the radial and tangential fluid velocities. In Figure 7, an enhancing tendency of temperature profile is perceived as \( M \) increases. Hence, incidence of magnetic field raises the thermal boundary layer thickness as the temperature increases. It is also noticed that there are lower velocity and higher temperature distributions in the SWCNT nanofluid than in the MWCNT nanofluid. This is due to the fact that SWCNTs have higher electrical conductivity than MWCNTs. An analogous result was also found by Aziz et al. [4] and Mushtaq and Mustafa [29]. Figure 8 shows the influence of Casson fluid parameter \( \gamma \) on radial velocity profile. Near the boundary surface, the velocity profile shows an increasing tendency as \( \gamma \) increases. This is anticipated since increasing the Casson fluid parameter \( \gamma \) tends to reduce in yield stress, and the fluid flows easily. However, as far the radial velocity is away from the boundary surface, the reverse effect resulting in the thinning of the momentum boundary layer is due to the presence of the magnetic field effect.
Table 4: Values of the normalized skin friction coefficient \( \sqrt{f''(0)^2 + h'(0)^2} \) and local heat transfer \(-\theta'(0)\) for SWCNTs – C\(_2\)H\(_6\)O\(_2\) and MWCNTs – C\(_2\)H\(_6\)O\(_2\) nanofluid, with \(M = 5, \text{Pr} = 7.3, \alpha = 0.3\) for various physical parameters.

<table>
<thead>
<tr>
<th>(\phi_v)</th>
<th>(A)</th>
<th>(\beta)</th>
<th>(\gamma)</th>
<th>(\delta)</th>
<th>SWCNTs (f''(0)^2 + h'(0)^2)</th>
<th>MWCNTs (f''(0)^2 + h'(0)^2)</th>
<th>SWCNTs (-\theta'(0))</th>
<th>MWCNTs (-\theta'(0))</th>
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<td>0.1</td>
<td>0.921185</td>
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<td>0.675679</td>
<td>0.740830</td>
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</tr>
</tbody>
</table>

Figure 2: Radial Velocity profiles for various values of \(\phi_v\).

Figure 3: Temperature profiles for various values of \(\phi_v\).

Figure 4: Radial Velocity profiles for various values of \(A\).

Figure 5: Temperature profiles for various values of \(A\).
**Figure 6:** Tangential velocity profiles for various values of $M$.

**Figure 7:** Temperature profiles for various values of $M$.

**Figure 8:** Radial velocity profiles for various values of $\gamma$.

**Figure 9:** Tangential Velocity profiles for various values of $\delta$.

**Figure 10:** Temperature profiles for various values of $\beta$.

**Figure 11:** Temperature profiles for various values of $\alpha$. 
Figure 15 clearly illustrates that the local heat transfer rate is enhanced with greater values of both Prandtl number Pr and thermal relaxation parameter $\alpha$. In addition, lower local heat transfer rate in the SWCNT nanofluid than MWCNT nanofluid is reported.

5. Conclusion

In this study, CNTs with ethylene glycol nanofluid flow and heat transfer due to stretchable rotating disk with Cattaneo–Christov heat flux model are examined. The effects of normally applied magnetic field, wall velocity, and thermal slip conditions are considered. The governing equations of the flow are solved numerically employing the spectral quasilinearization method (SQRM). From the main outcomes of the present observation, the following conclusions are drawn:

- Higher radial and tangential velocities distributions are observed in the MWCNT nanofluid than in the SWCNT nanofluid, whereas the trend is opposite for temperature distribution.
- The normalized skin friction coefficient enhances for both SWCNT and MWCNT nanofluids as scaled stretching parameter, magnetic parameter, and Casson fluid parameter grow.
- Local heat transfer rate enhances for both SWCNT and MWCNT nanofluids for higher values of scaled stretching parameter, Prandtl number, and thermal relaxation parameter.
- The higher incidence of magnetic field raises the temperature filed, and it improves the thermal boundary layer thickness.
- The momentum boundary layer thickness slows down as the velocity slip factor grows.
- Increases in the thermal slip factor tend to drop the temperature distribution at the wall of the disk, and it reduces the heat transfer from the surface of the disk to the fluid.
- Increase in the scaled stretching parameter thickens the momentum boundary layer while it thins down the thermal boundary layer.
- Radial stretching of the disk is helpful to improve the cooling process of the rotating disk in practical applications.
5.1. Future Directions. The present study results of CNTs with ethylene glycol nanofluid flow and heat transfer due to a stretchable rotating disk would have contributed valuable information to the uses of CNT nanofluids in various fields of nanotechnology. However, the weak magnetic properties and weak solubility of CNTs limit their applications in various technological and biomedical fields. Consequently, nowadays, the hybrid nanoparticles of magnetite metals with CNTs will be receiving attention and experimentally investigating by researchers. Therefore, numerical analysis of hybrid nanofluid flow and heat transfer of magnetite metals with CNTs' hybrid nanoparticles with various base fluids will be the future work of the researchers.

Appendix

A. A Numerical Scheme

Applying the quasilinearization method, the system of nonlinear ordinary differential equations (15)–(17) gives the following iterative scheme of linear differential equations:

\[ a_{1,j} f^{(n)}_{j+1} + a_{2,j} f_{j+1} + a_{3,j} f^{(n)}_{j+1} + a_{4,j} f_{j+1} + a_{5,j} h_{j+1} = a_{6,j}, \]  
(A.1)

\[ b_{1,j} h^{(n)}_{j+1} + b_{2,j} h_{j+1} + (b_{3,j} + b_{4,j}) h_{j+1} + b_{5,j} f_{j+1} \]

\[ + b_{6,j} f_{j+1} = b_{7,j}, \]  
(A.2)

\[ (c_{1,j} - c_{2,j}) \theta^{(n)}_{j+1} + c_{3,j} \theta^{(n)}_{j+1} - c_{4,j} f_{j+1} + c_{5,j} f_{j+1} = c_{6,j}, \]  
(A.3)

with boundary conditions:

\[ f_{j+1}(0) = A + \left(1 + \frac{1}{\gamma}\right) \delta f^{(n)}_{j+1}(0), \]

\[ h_{j+1}(0) = 1 + \left(1 + \frac{1}{\gamma}\right) \delta h_{j+1}(0), \]  
(A.4)

\[ \theta_{j+1}(0) = 1 + \beta \theta^{(n)}_{j+1}(0), \]

\[ f^{(n)}_{j+1}(\infty) = h^{(n)}_{j+1}(\infty) = \theta_{j+1}(\infty) \longrightarrow 0, \]

where the terms \( j+1 \) and \( j \) are at the current and previous iteration levels, respectively, and the coefficients are given as follows:

\[ a_{1,j} = 2 \left(1 + \frac{1}{\gamma}\right), \]

\[ a_{2,j} = 2 \phi_{1} f_{j}, \]

\[ a_{3,j} = 2 \phi_{1} f^{(n)}_{j}, \]

\[ a_{4,j} = 2 \phi_{1} f^{(n)}_{j} + \phi_{3} M, \]

\[ a_{5,j} = 2 \phi_{1} h_{j}, \]

\[ a_{6,j} = 2 f_{j} f^{(n)}_{j} - f^{(n)}_{j} + h^{(n)}_{j}, \]

\[ b_{1,j} = 2 \left(1 + \frac{1}{\gamma}\right), \]

\[ b_{2,j} = 2 \phi_{1} f_{j}, \]

\[ b_{3,j} = 2 \phi_{1} f^{(n)}_{j}, \]

\[ b_{4,j} = \phi_{3} M, \]

\[ a_{5,j} = -2 \phi_{1} h_{j}, \]

\[ a_{6,j} = 2 \phi_{1} h_{j}, \]

\[ a_{7,j} = 2 \phi_{1} (f_{j} h^{(n)}_{j} - f^{(n)}_{j} h_{j}), \]  
(A.5)

The semi-infinite domain is truncated to a domain \([0, L_{\infty}]\). Then, the interval \([0, L_{\infty}]\) is transformed to the interval \([-1, 1]\) using the linear transformation \(\xi = 1/2L_{\infty}(\chi + 1)\), where \(L_{\infty}\) is large but a finite number is chosen to represent the behavior of the flow properties of the boundary condition value at infinity. The Gauss–Lobatto points are selected to define the nodes in \([-1, 1]\) as follows:

\[ \chi_{i} = \cos\left(\frac{m}{N}\right), \quad i = 0, 1, 2, \ldots, N; -1 \leq \chi \leq 1, \]  
(A.6)

where \(N\) is the number of collocation points used.

The Chebyshev differentiation matrix used to approximate the derivatives of the unknown variables is given as

\[ \frac{df}{d\xi} = \sum_{p=0}^{N} D_{ik} f(\chi_{i}) = D F, \quad i = 1, 2, 3, \ldots, N, \]  
(A.7)

where \(D = 2 D/L_{\infty}\), and the higher-order derivatives are found as powers of \(D\) as follows:

\[ F^{n}(\xi) = D^{n} F, \]  
(A.8)

where \(n\) is the order of derivative and \(D\) is the matrix of size \((N+1) \times (N+1)\).

Applying the spectral method to the system of equations (A.1)–(A.3), it can be solved as a coupled matrix:

\[ \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{bmatrix} F_{j+1} \\ H_{j+1} \\ \Theta_{j+1} \end{bmatrix} = \begin{bmatrix} B_{1,j} \\ B_{2,j} \\ B_{3,j} \end{bmatrix}, \]  
(A.9)

with boundary condition
where

\[
    \begin{align*}
    A_{11} &= a_{1,j}D^3 + [a_{2,j}]_dD^2 - [a_{3,j}]_dD + [a_{4,j}]_d, \\
    A_{12} &= [a_{5,j}]_d, \\
    A_{13} &= 0; \\
    B_{1,j} &= a_{6,j}, \\
    A_{21} &= [b_{5,j}]_dD + [b_{6,j}]_d, \\
    A_{22} &= b_{1,j}D^3 + [b_{2,j}]_dD - [b_{3,j}]_d - b_{4,j}, \\
    A_{23} &= 0; \\
    B_{5,j} &= b_{7,j}, \\
    A_{31} &= [c_{4,j}]_dD + [c_{5,j}]_d, \\
    A_{32} &= 0, \\
    A_{33} &= (c_{1,j}I - [c_{2,j}]_d)D^2 + [c_{3,j}]_dD; \\
    B_{6,j} &= c_{6,j}, \\
    \end{align*}
\]

are vectors of size \((N+1)\times1\), \([\ldots]_d\) denotes a diagonal matrix, and \(I\) and \(O\) are, respectively, identity and zero matrices of size \((N+1)\times(N+1)\).

Starting with the following suitable initial approximations, the iteration schemes are used iteratively to find \(f_{j+1}\), \(h_{j+1}\), and \(\theta_{j+1}\) when \(j = 0, 1, 2, \ldots\)

\[
    \begin{align*}
    f_0(\xi) &= \frac{1 - e^{\alpha\xi}}{1 + (1/(\gamma))\delta}, \\
    h_0(\xi) &= \frac{e^{-\xi}}{1 + (1/(\gamma))\delta}, \\
    \theta_0(\xi) &= \frac{e^{-\xi}}{1 + \beta}, \\
    \end{align*}
\]

**Nomenclature**

- **\(A\)**: Scaled stretching parameters
- **\(B_0\)**: Magnetic field of uniform strength
- **\(C_f\)**: Local skin friction coefficient
- **\((C_p)_f\)**: Specific heat capacity of the base fluid (J/kg-K)
- **\((C_p)_\text{CNT}\)**: Specific heat capacity of CNTs (J/kg-K)
- **\(f\)**: Dimensionless radial stream function
- **\(h\)**: Dimensionless tangential velocity
- **\(k_f\)**: Thermal conductivity of base fluid (W/mK)
- **\((k_f)_\text{CNT}\)**: Thermal conductivity of CNTs (W/mK)
- **\(k_{nf}\)**: Thermal conductivity of nanofluid (W/mK)
- **\(L\)**: Characteristic length
- **\(L_0\)**: Velocity slip factor
- **\(M\)**: Magnetic field parameter
- **\(N\)**: Number of collocation points
- **\(N_u\)**: Local Nusselt number
- **\(N_u\)**: Local Nusselt number
- **\(Pr\)**: Prandtl number
- **\(q\)**: Heat flux vector
- **\(q_w\)**: Wall heat flux (W/m²)
- **\(Re\)**: Local Reynolds number
- **\(s\)**: Radial stretching rate
- **\(T_w\)**: Nanofluid temperature at the wall (K)
- **\(T_\text{amb}\)**: Ambient temperature (K)
- **\(v_r\)**: Nanofluid velocity in \(r\) direction (m/s)
- **\(v_\theta\)**: Nanofluid velocity in \(\theta\) direction (m/s)
- **\(v_z\)**: Nanofluid velocity in \(z\) direction (m/s)

**Greek symbols**

- **\(\alpha\)**: Thermal relaxation parameter
- **\(\alpha_{nf}\)**: Nanofluid thermal diffusivity (m²/s)
- **\(\beta\)**: Thermal slip parameter
- **\(\gamma\)**: Non-Newtonian Casson parameter
- **\(\delta\)**: Velocity slip parameter
- **\(\Omega\)**: Angular velocity
- **\(\theta\)**: Dimensionless temperature
- **\(\theta\)**: Tangential direction
- **\(\lambda_f\)**: Relaxation time of heat flux
- **\(\mu\)**: Dynamic viscosity of base fluid (kg/ms)
- **\(\mu_{nf}\)**: Dynamic viscosity of nanofluid (kg/ms)
- **\(\xi\)**: Dimensionless similarity variable
- **\(\sigma_f\)**: Electrical diffusivity of base fluid (S/m)
- **\(\sigma_{nf}\)**: Electrical diffusivity of nanofluid (S/m)
- **\(\sigma_{\text{CNTs}}\)**: Electrical diffusivity of CNTs (S/m)
- **\(\rho\)**: Density of the base fluid (kg/m³)
- **\(\rho_{nf}\)**: Density of nanofluid (kg/m³)
- **\(\rho_{\text{CNTs}}\)**: Density of CNTs (kg/m³)
- **\(\tau_w\)**: Wall shear stress
- **\(\phi\)**: CNT nanoparticle volume fraction

**Subscripts**

- **\(\in\)**: Condition at the free stream
- **\(w\)**: Condition at the surface
- **\(f\)**: Base fluid
- **\(nf\)**: Nanofluid
- **\(\text{CNTs}\)**: Carbon nanotubes.
Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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[27] T. Hayat, T. Muhammad, S. A. Shehzad, and A. Alsaedi, "On magneto-hydrodynamic flow of nanofluid due to a rotating


