Research Article

A Study on a New Class of Backward Stochastic Differential Equation

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1. Introduction

Pardoux and Peng in [1] first provided the famous backward stochastic differential equations (BSDEs). The existence and uniqueness for the BSDEs are proved by them. Since then, BSDEs have been discussed and applied to many fields, e.g., Chen and Epstein [2] and Karoui et al. [3–7]. A lot of research has focused on the assumptions on the generator, such as [8–12]. Recently, Delong and Imkeller in [13, 14] obtained many interesting results about the time-delayed equation in which the generator at time \( t \) only depends on the past solution. Peng and Yang in [15] discussed anticipated BSDEs, in which the generator includes present and future solutions.

Therefore, the natural questions are as follows: can we discuss the backward stochastic differential equations when the generator includes not only the past and the present but also the future solutions? The comparison theorem for it is still true? Indeed, these questions are answered in the affirmative in this paper. The existence and uniqueness for a new type of backward stochastic differential equation when the generator includes the values of solutions of the past, the present, and the future are obtained in this paper. An important comparison theorem for this sort of BSDEs is also proved.

2. Main Notations

Let \((\Omega, \mathcal{F}, P)\) be a probability space equipped with a natural filtration \((\mathcal{F}_t)_{t\geq0}\) and \((B_s)_{s\geq0}\) be a standard Brownian motion. We denote the norm in \(\mathbb{R}^n\) by \(|\cdot|\). Given \(T > 0\), denote the following:

(i) \(L^2(\mathcal{F}_T; \mathbb{R}^n) = \{\xi; \mathcal{F}_T\text{-measurable and } E[|\xi|^2] < \infty\}\).
(ii) \(L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) = \{\varphi; \mathcal{F}_t\text{-adapted and } E[\int_0^T |\varphi|^2 dt] < \infty\}\).
(iii) \(S^2_{\mathcal{F}}(0, T; \mathbb{R}^n) = \{ \chi; \chi \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \text{ and continuous, as well as } E[\sup_{0\leq t\leq T} |\chi|^2] < \infty\}\).

In the case \(n = 1\), they are abbreviated to \(L^2(\mathcal{F}_T), L^2_{\mathcal{F}}(0, T), \text{ and } S^2_{\mathcal{F}}(0, T)\), respectively.

3. Delay and Anticipated Backward Stochastic Differential Equations

We propose a new type of BSDEs as follows:
where $d_i(\cdot), i = 1, \ldots, 4$, are four continuous functions s.t.

(D1) $\exists D \geq 0$ s.t.

\[
0 \leq t - d_1(t) \leq t; 0 \leq t - d_2(t) \leq T; t + d_3(t) \leq T + D; t + d_4(t) \leq T + D, 0 \leq t \leq T.
\]

(D2) $\exists L \geq 0$ s.t., for all nonnegative and integrable $f(\cdot)$,

\[
\int_t^T f(v - d_1(v))dv \leq L \int_t^T f(v)dv; \int_t^T f(v - d_2(v))dv \leq L \int_t^T f(v)dv,
\]

\[
\int_t^T f(v + d_3(v))dv \leq L \int_t^T f(v)dv; \int_t^T f(v + d_4(v))dv \leq L \int_t^T f(v)dv.
\]

Note that the simple examples for $d_i(s)$ satisfying (D1) and (D2) are constant delay and $d_i(s) = s$.

We call system (1) delay and anticipated backward stochastic differential equations (delay and anticipated BSDEs).

Our aim is to search out a pair of processes

\[
(Y, Z) \in S_{\mathcal{F}}^2(0, D + T; R^m) \times L^2_{\mathcal{F}}(0, D + T; R^{mod})
\]

which satisfies the delay and anticipated BSDEs (1).

For all $s \in [0, T]$, we suppose $g(\omega, s, \phi, \mu, y, z, \psi, \nu)$: $\Omega \times [0, T] \times L^2_{\mathcal{F}}(0, s); R^m) \times L^2_{\mathcal{F}}(0, s; R^{mod}) \times R^m \times R^{mod} \times L^2$.

We set a mapping

\[
M[(y, z)] = (Y, Z) : L^2_{\mathcal{F}}(0, D + T; R^m \times R^{mod}) \longrightarrow L^2_{\mathcal{F}}(0, D + T; R^m \times R^{mod}).
\]

Let us show $M$ is a contraction mapping under $\|\|_\beta$.

We set a mapping $\bar{M}[(y, z)]$: $= (\bar{Y}, \bar{Z}) : L^2_{\mathcal{F}}(0, D + T; R^m \times R^{mod})$.

Proof. We choose suitable $\beta$ which satisfies $0 < (e^{\beta T}/\beta) \leq 1/(32C^2(2L + 1))$ and define a norm in $L^2_{\mathcal{F}}(0, D + T; R^m)$:

\[
\|\mu(\cdot)\|_\beta = \sqrt{E \left[ \int_0^{D+T} |\mu(t)|^2 e^{\beta t} dt \right]}.
\]

We set

\[
\begin{align*}
Y_t &= \xi_T + \int_u^T g(v, y_{u-d_1(v)}, z_{u-d_2(v)}, y_v, z_v, y_{u+d_3(v)}, z_{u+d_4(v)}dv - \int_u^T Z_s dB_s, & 0 \leq u \leq T; \\
Y_t &= \xi_T, & T \leq u \leq T + D; \\
Z_u &= \eta_u, & T \leq u \leq T + D.
\end{align*}
\]

Using Itô’s lemma for $|\bar{Y}|^2 e^{\beta u}, 0 \leq u \leq T$, and taking expectation,

\[
(Y, Z) = (\bar{Y}, \bar{Z}), \quad (Y', Z') = (\bar{Y}', \bar{Z}'),
\]

where $d_i(\cdot), i = 1, \ldots, 4$, are four continuous functions s.t.

(D1) $\exists D \geq 0$ s.t.

\[
0 \leq t - d_1(t) \leq t; 0 \leq t - d_2(t) \leq T; t + d_3(t) \leq T + D; t + d_4(t) \leq T + D, 0 \leq t \leq T.
\]

(D2) $\exists L \geq 0$ s.t., for all nonnegative and integrable $f(\cdot)$,

\[
\int_t^T f(v - d_1(v))dv \leq L \int_t^T f(v)dv; \int_t^T f(v - d_2(v))dv \leq L \int_t^T f(v)dv,
\]

\[
\int_t^T f(v + d_3(v))dv \leq L \int_t^T f(v)dv; \int_t^T f(v + d_4(v))dv \leq L \int_t^T f(v)dv.
\]
\[
|\tilde{Y}_0|^2 + \beta E \left[ \int_0^T e^{\beta u} |\tilde{Y}_u|^2 \, du \right] + E \left[ \int_0^T e^{\beta u} |\tilde{Z}_u|^2 \, du \right] \\
= E \left[ 2 \int_0^T e^{\beta u} \left( g(\tilde{u}, Y_{u-d_1(\omega)}, Z_{u-d_2(\omega)}) Y_u, Z_u, Y_{u+d_1(\omega)}, Z_{u+d_1(\omega)}) \right) \, du \right] \\
- \beta \left( u, Y_{u-d_1(\omega)}, Z_{u-d_2(\omega)}) + Y_u, Z_u, Y_{u+d_1(\omega)}, Z_{u+d_1(\omega)}) \right) \, du \right] \\
\leq \frac{\beta}{2} E \left[ \int_0^T e^{\beta u} |\tilde{Y}_u|^2 \, du \right] + \frac{2}{\beta} E \left[ \int_0^T e^{\beta u} \left( g(\tilde{u}, Y_{u-d_1(\omega)}, Z_{u-d_2(\omega)}) Y_u, Z_u, Y_{u+d_1(\omega)}, Z_{u+d_1(\omega)}) \right) \, du \right] \\
- \beta \left( u, Y_{u-d_1(\omega)}, Z_{u-d_2(\omega)}) + Y_u, Z_u, Y_{u+d_1(\omega)}, Z_{u+d_1(\omega)}) \right) \, du \right]. 
\]

Thus,
\[
E \left[ \int_0^T e^{\beta u} \left( \frac{\beta}{2} |\tilde{Y}_u|^2 + |\tilde{Z}_u|^2 \right) \, du \right] \leq \frac{2C^2}{\beta} E \left[ \int_0^T e^{\beta u} \left( \frac{1}{\beta} |\tilde{Y}_{u-d_1(\omega)}| + |\tilde{Z}_{u-d_2(\omega)}| + |\tilde{Y}_u| + |\tilde{Z}_u| \right) + E \left[ |\tilde{Y}_{u+d_1(\omega)}| + |\tilde{Z}_{u+d_1(\omega)}| \right] \, du \right] \\
\leq \frac{8C^2}{\beta} E \left[ \int_0^T e^{\beta u} \left( 2|\tilde{Y}_{u-d_1(\omega)}|^2 + 2|\tilde{Z}_{u-d_2(\omega)}|^2 + |\tilde{Y}_u|^2 + 2|\tilde{Y}_{u+d_1(\omega)}|^2 + 2|\tilde{Z}_{u+d_1(\omega)}|^2 \right) \, du \right]. 
\]

First, we note that
\[
\int_0^T e^{\beta u} |\tilde{Y}_{u-d_1(\omega)}|^2 \, du = \int_0^T e^{\beta u} |\tilde{Y}_{e^G(\omega)}|^2 \, du \\
\leq e^{GT} \int_0^T e^{\beta u} |\tilde{Y}_{u}|^2 \, du. 
\]

Because \(d_i(s), i = 1, \ldots, 4,\) satisfy (D2) and \(g\) satisfies (A1), we have

\[
E \left[ \int_0^T e^{\beta u} \left( \frac{\beta}{2} |\tilde{Y}_u|^2 + |\tilde{Z}_u|^2 \right) \, du \right] \leq \frac{8C^2 (2L+1) e^{GT}}{\beta} E \left[ \int_0^T e^{\beta u} (|\tilde{Y}_u|^2 + |\tilde{Z}_u|^2) \, du \right]. 
\]

Since \(\beta\) satisfies \((e^{GT} / \beta) \leq 1/(32C^2 (2L + 1))\), then
\[
E \left[ \int_0^{T+D} e^{\beta u} (|\tilde{Y}_u|^2 + |\tilde{Z}_u|^2) \, du \right] \leq E \left[ \int_0^{T+D} e^{\beta u} \left( \frac{\beta}{2} |\tilde{Y}_u|^2 + |\tilde{Z}_u|^2 \right) \, du \right] \\
\leq \frac{1}{4} E \left[ \int_0^{T+D} e^{\beta u} (|\tilde{Y}_u|^2 + |\tilde{Z}_u|^2) \, du \right]. 
\]

Therefore,
\[
\left\| \begin{pmatrix} \tilde{Y} \\ \tilde{Z} \end{pmatrix} \right\|_\beta \leq \frac{1}{2} \left\| \begin{pmatrix} \tilde{Y} \\ \tilde{Z} \end{pmatrix} \right\|_\beta. 
\]

Consequently, \(M\) is a strict contraction mapping. From the fixed point theorem, the DABSDE (1) has a unique solution.

\[\text{Example 1. Consider a typical delay and anticipated backward stochastic differential equation}\]
\[
\begin{align*}
-dY_u &= (Z_{u-(\pi/2)} + Z_{u+\pi})du - Z_u dB_u, \quad 0 \leq u \leq T, \\
Y_u &= \int_0^T \sin t dB_t, \quad T \leq u \leq T + D, \\
Z_u &= \sin T, \quad T \leq u \leq T + D,
\end{align*}
\]

with \( Y_T = \int_0^T \sin t dB_t \). We can get the unique solution of DABSDE (15) which is
\[
\begin{align*}
Y_u &= \int_0^u \sin s dB_s - (\cos u + \sin u) + \cos T + \sin T, \quad 0 \leq u \leq T, \\
Z_u &= \sin u, \quad 0 \leq u \leq T.
\end{align*}
\]

(16)

\[
\begin{align*}
-dY^{(j)}_s &= g_j(s, Y^{(j)}_{s-d_1(s)}), Y^{(j)}_s, Z^{(j)}_s, Y^{(j)}_{s+d_2(s)})ds - Z^{(j)}_s dB_s, \quad s \in [0, T], \\
Y^{(j)}_t &= \xi^{(j)}_t,
\end{align*}
\]

(17)

\[
Y^{(2)}_t \leq Y^{(1)}_t, \quad \text{a.e., a.s.}
\]

(18)

4. **Comparison Theorem for Delay and Anticipated BSDEs**

Next, we deduce the comparison theorem for one-dimensional DABSDEs. Denote \((i)^{(j)} := Y^{(i)}_t, i = 1, 2, 3\). Let \((Y^{(1)}, Z^{(1)}), (Y^{(2)}, Z^{(2)}), \ldots, (Y^{(j)}, Z^{(j)}) (j = 1, 2)\) be the solution of the following one-dimensional delay and anticipated BSDEs:

Theorem 2. Assume that \(g_1, g_2\) satisfy (A1) and (A2), \(\xi^{(1)}, \xi^{(2)} \in S^2_1(T, D + T), d_k(t), k = 1, 3, \) satisfy (D1), (D2), and \(\forall y \in R, z \in R^4, g_3(t, y, z, \cdot, \cdot)\) is strictly increasing. If \(\xi^{(2)} \leq \xi^{(1)}, s \in [T, D + T]\) and \(g_3(\cdot, \cdot, y, z, \cdot, \cdot, r) \geq g_2(\cdot, \cdot, y, z, \cdot, \cdot, r), \phi_r \in L^2_\mathbb{F}(0, T), \varphi_r \in L^2_\mathbb{F}(t, T + D), r \in [0, T], f \in [t, D + T],\) then

\[
\begin{align*}
Y^{(1)}_0 &= Y^{(2)}_0, \\
\xi^{(1)}_t &= \xi^{(2)}_t,
\end{align*}
\]

(19)

Proof. Set

\[
\begin{align*}
Y^{(3)}_u &= \xi^{(2)}_u + \int_u^T g_3(s, Y^{(1)}_{s-d_1(s)}, Y^{(3)}_s, Z^{(3)}_s, Y^{(1)}_{s+d_2(s)})ds - \int_u^T Z^{(3)}_s dB_s, \quad u \in [0, T], \\
Y^{(3)}_u &= \xi^{(2)}_u,
\end{align*}
\]

(20)

Denote \(\tilde{\gamma}_u := g_1(u, Y^{(1)}_{u-d_1(\cdot)}, Y^{(1)}_{u}, Z^{(1)}_{u}, Y^{(1)}_{u+d_2(\cdot)}) - g_2(u, Y^{(1)}_{u-d_1(\cdot)}, Y^{(1)}_{u}, Z^{(1)}_{u}, Y^{(1)}_{u+d_2(\cdot)}) \) and \(\tilde{\xi} = (\tilde{\xi}_u)^{(1)}, \tilde{\gamma}_u = Y^{(1)}_u - Y^{(3)}_u, \tilde{\xi}_u = Z^{(1)}_u - Z^{(3)}_u\). Then, \((\tilde{\gamma}, \tilde{\xi})\) is the solution of the following DABSDEs:
where

\[
a_v = \begin{cases} 
\frac{g_1(v, Y_{v^{-d_1(v)}}, Y_{v}, Z_{v^{-d_2(v)}}, Y_{v}^{-d_2(v)}) - g_2(v, Y_{v^{-d_1(v)}}, Y_{v}, Z_{v^{-d_2(v)}}, Y_{v}^{-d_2(v)})}{Y_{v}^{-d_2(v)} - Y_{v}^{-d_2(v)}}, & Y_{v}^{-d_2(v)} \neq Y_{v}^{-d_2(v)}, \\
0, & \text{otherwise}.
\end{cases}
\]

\[
b_v = \begin{cases} 
\frac{g_1(v, Y_{v^{-d_1(v)}}, Y_{v}^{-d_2(v)}), Z_{v^{-d_2(v)}}, Y_{v}^{-d_2(v)}) - g_2(v, Y_{v^{-d_1(v)}}, Y_{v}, Z_{v^{-d_2(v)}}, Y_{v}^{-d_2(v)})}{Z_{v}^{-d_2(v)} - Z_{v}^{-d_2(v)}}, & Z_{v}^{-d_2(v)} \neq Z_{v}^{-d_2(v)}, \\
0, & \text{otherwise}.
\end{cases}
\]

Since \( g_2 \) follows assumption (A1), then \(|a_v| \leq C \) and \(|b_v| \leq C\). Denote

\[
q_s = \exp\left[-\frac{1}{2} \int_0^s |a_v|^2 \, dv + \int_0^s a_v \, dv + \int_0^s b_v \, dB_v \right].
\]

Using Itô’s lemma for \( q_s \bar{Y}_s \) and taking expectation, then

\[
\begin{align*}
Y^{(4)}_{u} &= \int_u^T g_1(s, Y_{s^{-d_1(s)}}, Y_{s}, Z_{s^{-d_2(s)}}, Y_{s}^{-d_2(s)}) \, ds - \int_u^T Z^{(4)}_{s} \, dB_s + \bar{\xi}_u^{(2)}, \quad 0 \leq u \leq T, \\
Y^{(4)}_u &= \bar{\xi}_u^{(2)},
\end{align*}
\]

Because \( Y^{(1)}_t \geq Y^{(3)}_t \) a.e., a.s., and \( g_2 (\cdot) \) is strictly increasing, from the classical comparison theorem, then \( Y^{(4)} \leq Y^{(3)} \). When \( n = 5, 6, \ldots \), we investigate the DABSDEs:

\[
\begin{align*}
Y^{(n)}_{u} &= \bar{\xi}_u^{(2)} + \int_u^T g_1(s, Y^{(n-1)}_{s^{-d_1(s)}}, Y_{s}, Z^{(n-1)}_{s^{-d_2(s)}}, Y_{s}^{-d_2(s)}) \, dv - \int_u^T Z^{(n)}_{s} \, dB_s, \quad 0 \leq u \leq T, \\
Y^{(n)}_u &= \bar{\xi}_u^{(2)},
\end{align*}
\]

Since \( g_2 (t, v, y, z, \cdot) \) is strictly increasing, we have \( Y^{(4)}_t \geq Y^{(3)}_t \geq \cdots \geq Y^{(5)}_t \geq \cdots \). From the proving method of Theorem 1, \( (Y^{(n)}_u) \) and \( (Z^{(n)}_u)_{n \geq 4} \) are Cauchy sequences in \( L^2_{\bar{F}} (0, D + T) \) and in \( L^2_{\bar{G}} (0, T) \), \( n \geq 4 \). Write their limits as \( Y \) and \( Z \); then, \( (Y, Z) \in L^2_{\bar{F}} (0, D + T) \times L^2_{\bar{G}} (0, T) \), and when \( n \to \infty \),

\[
E \left[ \int_t^T \left| g_2(s, Y^{(n-1)}_{s^{-d_1(s)}}, Y_{s}, Z^{(n-1)}_{s^{-d_2(s)}}, Y_{s}^{-d_2(s)}) - g_2(s, Y_{s^{-d_1(s)}}, Y_{s}, Z_{s}, Y_{s}^{-d_2(s)}) \right|^2 \, dB_s \right] \leq 4C^2E \left[ \int_t^T \left( |Y^{(n)}_v - Y_v|^2 + 2L|Y^{(n-1)}_v - Y_v|^2 + |Z^{(n)}_v - Z_v|^2 \right) \, dB_v \right] \to 0.
\]
Therefore, \((Y_t, Z_t)\) satisfies the following delay and anticipated BSDEs:

\[
\begin{align*}
Y_t &= \xi^{(2)}_t + \int_t^T g_2(s, Y_{s-d_t(s)}, Y_{s}, Z_{s}, Y_{s+d_t(s)}) \, ds - \int_t^T Z_{s} \, dB_{s}, \quad 0 \leq t \leq T, \\
Y_t^{(n)} &= \xi^{(2)}_t, \\
T &\leq t \leq T + D.
\end{align*}
\]

By Theorem 1, \(Y_t = Y^{(2)}_t\). Since \(Y^{(1)}_t \geq Y^{(3)}_t \geq \cdots \geq Y^{(n)}_t \geq \cdots\), it holds immediately \(Y^{(1)}_t \geq Y^{(2)}_t\).

Similar to the deducing technique of Peng and Yang [15], next we prove the strict comparison theorem.

If \(Y^{(1)}_0 = Y^{(2)}_0\), then we have

\[
\begin{align*}
g_1(u, Y^{(1)}_{t-d_t(u)}, Y^{(1)}_t, Z^{(1)}_t, Y^{(1)}_{t+d_t(u)}) \\
&= g_2(u, Y^{(2)}_{t-d_t(u)}, Y^{(2)}_t, Z^{(2)}_t, Y^{(2)}_{t+d_t(u)}),
\end{align*}
\]

Since \(Y^{(1)}_0 \geq Y^{(3)}_0 \geq Y^{(2)}_0\), then \(Y^{(1)}_0 = Y^{(3)}_0\) and

\[
g_1(\tilde{t}, Y^{(1)}_{\tilde{t} - d_{\tilde{t}}(\tilde{t})}, Y^{(1)}_{\tilde{t}}, Z^{(1)}_t, Y^{(1)}_{\tilde{t} + d_{\tilde{t}}(\tilde{t})})
\]

\[
= g_2(\tilde{t}, Y^{(2)}_{\tilde{t} - d_{\tilde{t}}(\tilde{t})}, Y^{(2)}_{\tilde{t}}, Z^{(2)}_t, Y^{(2)}_{\tilde{t} + d_{\tilde{t}}(\tilde{t})}),
\]
as well as \(g_0(\cdot)\) is strictly increasing, and we can easily get satisfying results.

Let \(\xi \in S^2(T, D + T), d^{(1)}_1\) and \(d^{(2)}_1\) satisfy (D1) and (D2), and the function \(g(\cdot)\) satisfies (A1) and (A2). If \(d^{(1)}_1(v) \geq d^{(2)}_1(v), d^{(2)}_1(v) \geq d^{(1)}_1(v), 0 \leq v \leq T,\) then \(\exists \lambda: = \lambda(C, L, T) > 0\) s.t.

\[
\begin{align*}
\left|Y^{(1)}_t - Y^{(2)}_t \right|^2 &\leq \lambda \int_t^T \left( d^{(1)}_1(v) - d^{(2)}_1(v) + d^{(2)}_1(v) - d^{(1)}_1(v) \right) \, dv \\
&\times E \left[ \int_T^{T+D} \left| \xi_t \right|^2 \, dt + \left| \xi_t \right|^2 + \int_t^T |g(r, 0, 0, 0, 0)|^2 \, dr \right] \mathbb{F}_t.
\end{align*}
\]

Proof. Denote \(y = Y^{(1)}_t - Y^{(2)}_t, z = Z^{(1)}_t - Z^{(2)}_t\), and then apply Itô’s lemma for \(|y_s|^2 e^{\beta t}\):

\[
\begin{align*}
|y_s|^2 &+ E \left[ \int_u^T \left( |y_s|^2 + |z_s|^2 \right) e^{\beta(v-u)} \, ds \right] \mathbb{F}_u \\
&= E \left[ \int_u^T 2y_s \left( g\left( y^{(1)}_v, Y^{(1)}_v, Z^{(1)}_v, Y^{(1)}_{v+d^{(1)}_1(v)} \right) \right. \\
&\left. - g\left( y^{(2)}_v, Y^{(2)}_v, Z^{(2)}_v, Y^{(2)}_{v+d^{(2)}_1(v)} \right) \right) |y_s|^2 e^{\beta(v-u)} \, dv \right] \mathbb{F}_u.
\end{align*}
\]
namely,

\[ |y_i|^2 + E \left[ \int_t^T \exp(\beta (s-t)) \left( |z_s|^2 + 0.5 \beta |y_s|^2 \right) ds \mid \mathcal{F}_t \right] \]
\[ \leq 2 \beta^{-1} E \left[ \int_t^T g \left( s, Y_s^{(1)}(a), Y_s^{(1)}(a), Z_s^{(1)} + Y_s^{(1)}(a) \right) \right. \]
\[ \left. - g \left( s, Y_s^{(2)}(a), Y_s^{(2)}(a), Z_s^{(2)} + Y_s^{(2)}(a) \right) \left| 2 \delta(s-t) \right| ds \mid \mathcal{F}_t \right] \]
\[ \leq \frac{8C_2^2}{\beta} E \left[ \int_t^T \left( |Y_s^{(1)} - Y_s^{(2)}(a)|^2 \right) + |y_s|^2 + |z_s|^2 \right] \]
\[ + E \left[ \left| Y_s^{(1)} - Y_s^{(2)}(a) \right|^2 \mid \mathcal{F}_s \right] \]
\[ \leq \frac{8C_2^2 + 24C_2L}{\beta} E \left[ \int_t^T \exp(\beta (s-t)) |y_s|^2 ds \mid \mathcal{F}_t \right] + \frac{8C_2^2}{\beta} E \left[ \int_t^T \exp(\beta (s-t)) |z_s|^2 ds \mid \mathcal{F}_t \right] \]
\[ + \frac{16C_2^2}{\beta} E \left[ \int_t^T \left( \int_{s-d_s^{(1)}}^{s+d_s^{(1)}} g(r, Y_{r-d_s^{(1)}}, Y_{r+d_s^{(1)}}, Z_{r+d_s^{(1)}}, Y_{r+d_s^{(1)}}) dr \right)^2 \mid \mathcal{F}_s \right] e^{\beta(s-t)} ds \mid \mathcal{F}_t \]
\[ + \frac{16C_2^2}{\beta} E \left[ \int_t^T \left( \int_{s-d_s^{(1)}}^{s+d_s^{(1)}} g(r, Y_{r-d_s^{(1)}}, Y_{r+d_s^{(1)}}, Z_{r+d_s^{(1)}}, Y_{r+d_s^{(1)}}) dr \right)^2 e^{\beta(s-t)} ds \mid \mathcal{F}_t \right]. \]

We set \( \beta = 8C_2^2 \); then,

\[ |y_i|^2 \leq (1 + 3L)e^{\beta(T-t)} E \left[ \int_t^T |y_s|^2 ds \mid \mathcal{F}_t \right] \]
\[ + 2E \left[ \int_t^T E \left[ \int_{s-d_s^{(1)}}^{s+d_s^{(1)}} g(r, Y_{r-d_s^{(1)}}, Y_{r+d_s^{(1)}}, Z_{r+d_s^{(1)}}, Y_{r+d_s^{(1)}}) dr \mid \mathcal{F}_s \right] e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \]
\[ + 2E \left[ \int_t^T \left( \int_{s-d_s^{(1)}}^{s+d_s^{(1)}} g(r, Y_{r-d_s^{(1)}}, Y_{r+d_s^{(1)}}, Z_{r+d_s^{(1)}}, Y_{r+d_s^{(1)}}) dr \right)^2 e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \]
\[ \leq e^{\beta(T-t)} (1 + 3L)e^{\beta(T-t)} E \left[ \int_t^T |y_s|^2 ds \mid \mathcal{F}_t \right] \]
\[ + 8 \int_t^T \left( (d_1^{(1)}(s) - d_1^{(2)}(s)) + (d_3^{(2)}(s) - d_3^{(1)}(s)) \right) e^{\beta(s-t)} ds \]
\[ \cdot E \left[ \int_t^T \left( C^2 |Y_{r-d_s^{(1)}}|^2 + C^2 |Y_{r+d_s^{(1)}}|^2 + C^2 |Z_{r+d_s^{(1)}}|^2 + |g(r, 0, 0, 0, 0)|^2 \right) dr \mid \mathcal{F}_t \right] \]
\[ \leq (1 + 3L)e^{\beta(T-t)} E \left[ \int_t^T |y_s|^2 ds \mid \mathcal{F}_t \right] + 8 \int_t^T e^{\beta(s-t)} \left( (d_1^{(1)}(s) - d_1^{(2)}(s)) + (d_3^{(2)}(s) - d_3^{(1)}(s)) \right) ds \]
\[ \cdot E \left[ \int_t^T \left( C^2 (2L + 1) |Y_{r-d_s^{(1)}}|^2 + C^2 |Z_{r+d_s^{(1)}}|^2 + |g(r, 0, 0, 0, 0)|^2 \right) dr \mid \mathcal{F}_t \right], \]
Therefore, there exists $\lambda^* = \lambda' (C, L, K) > 0$ s.t.
\[
|y_t|^2 \leq \lambda' E \int_t^T |y_s|^2 ds \bigg|_{\mathcal{F}_T} + \lambda' \int_t^T \left( |d_1^{(1)}(s) - d_1^{(2)}(s)| + |d_2^{(1)}(s) - d_2^{(2)}(s)| \right) ds \\
\times E \left[ |\xi_T|^2 + \int_T^D |g(r, 0)|^2 dr \bigg|_{\mathcal{F}_T} \right].
\]
(37)

Thus, from Gronwall inequality,
\[
|y_t|^2 \leq \lambda \int_t^T (d_1^{(1)}(s) - d_1^{(2)}(s) + d_2^{(1)}(s) - d_2^{(2)}(s)) ds \\
\times E \left[ |\xi_T|^2 + \int_T^D |g(r, 0)|^2 dr + \int_T^D |\xi_T|^2 dr \bigg|_{\mathcal{F}_T} \right].
\]
(38)

In fact, the delay in (1) can go below the result of Peng and Yang in [15].

Remark 2. If $g(v, Y_{v-d_1(v)}, Z_{v-d_1(v)}, Y_{v+d_2(v)}, Z_{v+d_2(v)}) = f(v, Y_{v}, Z_{v}, Y_{v-d_1(v)}, Z_{v-d_1(v)}, Y_{v+d_2(v)}, Z_{v+d_2(v)})$, then the delay and anticipated BSDE (1) is anticipated BSDEs in [15], and it is the result of Peng and Yang in [15].

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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