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Research Article

Generalization of h-Convex Stochastic Processes and Some Classical Inequalities

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The field of stochastic processes is essentially a branch of probability theory, treating probabilistic models that evolve in time. It is best viewed as a branch of mathematics, starting with the axioms of probability and containing a rich and fascinating set of results following from those axioms. In probability theory, a convex function applied to the expected value of a random variable is always bounded above by the expected value of the convex function of the random variable. In this paper, the concept of generalized h-convex stochastic processes is introduced, and some basic properties concerning generalized h-convex stochastic processes are developed. Furthermore, we establish Jensen and Hermite–Hadamard and Fejér-type inequalities for this generalization.

1. Introduction

Stochastic processes are a branch of probability theory, treating probabilistic models that evolve in time [1–3]. It is a branch of mathematics, starting with the axioms of probability and containing a rich and fascinating set of results following from those axioms [4]. Although the results are applicable to many areas [5], they are best understood initially in terms of their mathematical structure and interrelationships [6].

There are various ways to define stochastic monotonicity and convexity for stochastic processes [7], and they are of great importance in optimization, especially in optimal designs, and also useful for numerical approximations when there exist probabilistic quantities in the literature [8]. We also refer [9–12] for detailed survey about the importance and interesting properties of stochastic models.

The idea of convex functions was put forward, and many generalizations were made in this area [13]. h-Convex [14] and ϕ-convex functions [15] are famous generalizations of convex functions. Later, the theory of stochastic processes has developed very rapidly and has found application in a large number of fields. The study on convex stochastic processes was initiated in [16] by B. Nagy in 1974. After that, Nikodem in 1980 introduced the convex stochastic processes in his article [17]. Following this line of investigation, Skowronski described the properties of Jensen-convex and Wright-convex stochastic processes in [18, 19]. Some interesting properties of convex and Jensen-convex processes are also presented in [20, 21]. Assume a Jensen-convex stochastic process $\xi\colon I\times\Omega\longrightarrow\mathbb{R}$ which is mean-square continuous in I, where $I\subseteq\mathbb{R}$ is an interval. Then, for every $a_1, a_2\in I\ (a_1<a_2)$,

$$\xi\left(\frac{a_1+a_2}{2}\right)\leq\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(u)\mathrm{d}u\leq\frac{\xi(a_1)+\xi(a_2)}{2}\ (a.e.),\quad (1)$$

is known as the Hermite–Hadamard-type inequality for the convex stochastic process [22].

The aim of this paper is to introduce the notion of the generalized h-convex stochastic process and to extend the classical Hermite–Hadamard inequality for convex stochastic processes to generalized h-convex stochastic processes.

2. Novelty and Significance

The study of convex functions makes them special because of their interesting properties as maximum are attained at the boundary point, and moreover, any local minimum is global one. So, this topic of research got the attention of many researchers of different areas because of their enormous applications in optimization theory. As far as convex stochastic processes are concerned, there is a lot of work in the last few decades. In [23], the authors investigated the gradient descent optimality for strongly convex stochastic optimization. A continuous-time financial portfolio selection model with expected utility maximization typically boils down to solve a convex stochastic optimization problem in terms of terminal wealth with budget constraints, see, e.g., [24]. For more details related to this work, see [25–31].

3. Preliminaries

Let (Ω, P) be a probability space. A function $\xi\colon \Omega \longrightarrow \mathbb{R}$ is a random variable if it is -measurable. A function $\xi\colon I \times \Omega \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is a stochastic process if for every $t \in I$, the function $\xi(t)$ is a random variable.

The stochastic process $\xi\colon I \times \Omega \longrightarrow \mathbb{R}$ is known as

(1) Stochastically continuous in I if

$$\mu - \lim_{v \longrightarrow v^{\circ}} \xi(v) = \xi(v^{\circ}), \tag{2}$$

for all $v^{\circ} \in I$, where $\mu - \lim$ represents the limit in the probability.

(2) Mean-square continuous in I if

$$\lim_{v \longrightarrow v^{\circ}} \mathbb{E}\left(\xi(v) - \xi(v^{\circ})\right)^2 = 0, \tag{3}$$

for all $v^{\circ} \in I$, where $\mathbb{E}[\xi(v)]$ represents an expectation value of random variable $\xi(v)$.

Clearly in probability, mean-square continuity implies continuity, but converse is not true.

A stochastic process ξ is known as mean-square differentiable in I if there is a stochastic process ξ' (derivative of ξ) such that

$$\lim_{v \longrightarrow v^{\circ}} \mathbb{E}\left[\frac{\xi(v) - \xi(v^{\circ})}{v - v^{\circ}} - \xi'(v^{\circ})\right]^2 = 0, \tag{4}$$

for every $v^{\circ} \in I$.

Now, we would like to recall the concept of the mean-square integral. For the definition and basic properties, see [32, 33].

Assume a stochastic process $\xi\colon I \times \Omega \longrightarrow \mathbb{R}$ with $\mathbb{E}[\xi(t)^2] < \infty$. A random variable $\zeta\colon \Omega \longrightarrow \mathbb{R}$ is said to be the mean-square integral of the process ξ on $[a, b]$ if for every normal sequence of partitions of $[a_1, a_2]$, $a_1 = v^{\circ} < v_1 < \ldots < v_n = a_2$, and for all $\Theta_k \in [v_{k-1}, v_k]$, we have

$$\lim_{n \longrightarrow \infty} \mathbb{E}\left[\left(\sum_{k=1}^{n} \xi(\Theta_k)(v_k - v_{k-1}) - \zeta(.)\right)^2\right] = 0. \tag{5}$$

Then, we write

$$\zeta(.) = \int_a^b \xi(s)\mathrm{d}s\,(a.e). \tag{6}$$

Now, we shall present some definitions and generalizations of the convex stochastic process.

Definition 1 (see [34]). A stochastic process $\xi\colon I \times \Omega \longrightarrow \mathbb{R}$ is known as generalized convex with respect to a bifunction $\eta\colon \xi(I) \times \xi(I) \longrightarrow \mathbb{R}$ if

$$\xi(\alpha u + (1-\alpha)v) \le \xi(v) + \alpha\eta(\xi(u), \xi(v))\,(a.e), \tag{7}$$

for all $u, v \in I$ and $\alpha \in [0, 1]$.

Definition 2 (see [35]). Let $h\colon (0, 1) \longrightarrow \mathbb{R}$ be a function which is nonnegative and $h \neq 0$. A stochastic process $\xi\colon I \times \Omega \longrightarrow \mathbb{R}$ is said to be h-convex if

$$\xi(\alpha u + (1-\alpha)v) \le h(\alpha)\xi(u) + h(1-\alpha)\xi(v)\,(a.e), \tag{8}$$

for every $u, v \in I$ and $\alpha \in (0, 1)$.

Definition 3 (see [15]). The function $\eta\colon \xi(I) \times \xi(I) \longrightarrow \mathbb{R}$ is said to be

(1) Nonnegatively homogeneous if $\eta(\gamma u, \gamma v) = \gamma\eta(u, v)$ for all $u, v \in \mathbb{R}$ and $\gamma \ge 0$

(2) Additive if $\eta(u_1, v_1) + \eta(u_2, v_2) = \eta(u_1 + u_2, v_1 + v_2)$ for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$

Definition 4 (see [14]). A function $h\colon J \longrightarrow \mathbb{R}$, $J \subseteq \mathbb{R}$, is said to be a supermultiplicative function if

$$h(uv) \ge h(u)h(v), \tag{9}$$

for all $u, v \in J$.

Now, we are ready to give our main definition.

Definition 5 (generalized h-convex stochastic process). h and η are fixed like above. We say that a stochastic process $\xi\colon I \times \Omega \longrightarrow$ is called generalized h-convex if

$$\xi(\alpha u + (1-\alpha)v) \le \xi(v) + h(\alpha)\eta(\xi(u), \xi(v))\,(a.e), \tag{10}$$

for all $u, v \in I$ and $\alpha \in [0, 1]$.

In (10), if we take $\eta(u, v) = u - v$ and $h(\alpha) = \alpha$, we obtain the convex stochastic process. We observe that, by taking $u = v$ in (10), we get

$$h(\alpha)\eta(\xi(u), \quad \xi(u)) \ge 0, \tag{11}$$

for any $u \in I$ and $\alpha \in [0, 1]$, which implies that

$$\eta(\xi(u), \quad \xi(u)) \ge 0, \tag{12}$$

for any $u \in I$. Also, if we take $\alpha = 1$ and $h(1) = 1$ in (10), we get

$$\xi(u) - \xi(v) \leq \eta(\xi(u), \xi(v)), \tag{13}$$

for any $u, v \in I$. (13) obviously implies (12).

We observe that if $\xi: I \longrightarrow \mathbb{R}$ is a convex stochastic process and $\eta: \xi(I) \times \xi(I) \longrightarrow \mathbb{R}$ is an arbitrary bifunction that satisfies

$$\eta(a, b) \geq a - b, \tag{14}$$

for any $a, b \in I$ and $h(\alpha) \geq \alpha$, then for any $u, v \in I$ and $\alpha \in [0, 1]$, we have

$$\begin{aligned} \xi(\alpha u + (1-\alpha)v) &\leq \xi(v) + \alpha(\xi(u) - \xi(v)) \\ &\leq \xi(v) + h(\alpha)\eta(\xi(u), \xi(v)), \end{aligned} \tag{15}$$

showing that ξ is a generalized h-convex stochastic process.

Example 1. Every generalized h-convex function gives an example of a generalized h-convex stochastic process.

Example 2. Let $\xi: I \times \Omega \longrightarrow \mathbb{R}$ be a convex stochastic process. For every $k \leq 1$, consider the function

$$\begin{aligned} h_k: (0, 1) &\longrightarrow \mathbb{R}, \\ x &\longmapsto x^k. \end{aligned} \tag{16}$$

Note that $h_k(\alpha) \geq \alpha$ for all $\alpha \in (0, 1)$. Also, take $\eta(x, y) = x - y$. Moreover, for every $u, v \in I$ and $\alpha \in (0, 1)$, the following inequality is satisfied:

$$\begin{aligned} \xi(\alpha u + (1-\alpha)v) &\leq \alpha\xi(u) + (1-\alpha)\xi(v) \\ &\leq \xi(v) + h_k(\alpha)\eta(\xi(u), \xi(v))\,(a.e). \end{aligned} \tag{17}$$

Then, ξ is a generalized h_k-convex stochastic process.

The paper is organized as follows: in Section 4, we will derive some basic results for the generalized h-convex stochastic process. In Section 5, Jensen-type inequality will be proved, whereas in Section 6, Hermite–Hadamard and Fejér-type inequalities will be established. Finally, in Section 7, we will derive Ostrowski-type inequality for generalized h-convex stochastic processes.

4. Basic Results

Proposition 1. *Consider two generalized h-convex stochastic processes $\xi_1, \xi_2: I \times \Omega \longrightarrow \mathbb{R}$ such that*

(1) If η is additive, then $\xi_1 + \xi_2: I \longrightarrow \mathbb{R}$ is η_h-convex stochastic process

(2) If η is nonnegatively homogeneous, then for any $\gamma \geq 0$, $\gamma\xi_1: I \times \Omega \longrightarrow \mathbb{R}$ is a generalized h-convex stochastic process

(3) If ξ_2 is linear, then $\xi_1 \circ \xi_2$ is a generalized h-convex stochastic process

Proof. The proof of the proposition is straightforward. □

Proposition 2. *If $\xi: [a, b] \longrightarrow \mathbb{R}$ is a generalized h-convex stochastic process and $h(\alpha) \leq k$, then we have almost everywhere*

$$\max_{u \in [a,b]} \xi(u) \leq \max\{\xi(b), \xi(b) + k\eta(\xi(a), \xi(b))\}. \tag{18}$$

Proof. For any $u \in [a, b]$, we have $u = \alpha a + (1-\alpha)b$, for some $\alpha \in [0, 1]$, and

$$\xi(u) = \xi(\alpha a + (1-\alpha)b). \tag{19}$$

since ξ is a generalized h-convex stochastic process, so

$$\begin{aligned} \xi(u) &\leq \xi(b) + h(\alpha)\eta(\xi(a), \xi(b)) \\ &\leq \max\{\xi(b), \xi(b) + k\eta(\xi(a), \xi(b))\}. \end{aligned} \tag{20}$$

Since u is arbitrary,

$$\max_{u \in [a,b]} \xi(u) \leq \max\{\xi(b), \xi(b) + k\eta(\xi(a), \xi(b))\}. \tag{21}$$

□

5. Jensen-Type Inequality

We will use the next lemma to derive the Jensen-type inequality for the generalized h-convex stochastic process.

Lemma 1. *Assume a generalized h-convex stochastic process $\xi: I \times \Omega \longrightarrow \mathbb{R}$. For $u_1, u_2 \in I$ and $\alpha_1 + \alpha_2 = 1$, we have*

$$\xi(\alpha_1 u_1 + \alpha_2 u_2) \leq \xi(u_2) + h(\alpha_1)\eta(\xi(u_2), \xi(u_2))\,(a.e). \tag{22}$$

Also, when $n > 2$ for $u_1, u_2, \ldots, u_n \in I$, $\sum_{i=1}^{n} \alpha_i = 1$, and $T_i = \sum_{j=1} \alpha_j$, the following inequality holds almost everywhere:

$$\begin{aligned} \xi\left(\sum_{i=1}^{n} \alpha_i u_i\right) &= \xi\left(\left(T_{n-1} \sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} u_i\right) + \alpha_n u_n\right) \\ &\leq \xi(u_n) + h(T_{n-1})\eta\left(\xi\left(\sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} u_i\right), \xi(u_n)\right). \end{aligned} \tag{23}$$

Theorem 1 *(Jensen-type inequality). Assume a generalized h-convex stochastic process $\xi: I \times \Omega \longrightarrow \mathbb{R}$, and let $\eta: A \times B \longrightarrow \mathbb{R}$ be nondecreasing, nonnegatively sublinear in the first variable. If $T_i = \sum_{j=1}^{i} \alpha_j$ for $i = 1, 2, \ldots, n$ such that $T_n = 1$, then we have almost everywhere*

$$\xi\left(\sum_{i=1}^{n} \alpha_i u_i\right) \leq \xi(u_n) + \sum_{i=1}^{n-1} h(T)_i \eta_\xi(u_i, u_{i+1}, \ldots, u_n), \tag{24}$$

where

$$\begin{aligned} \eta_\xi(u_i, u_{i+1}, \ldots, u_n) &= \eta\big(\eta_\xi(u_i, u_{i+1}, \ldots, u_{n-1}), \\ &\quad \xi(u_n)\big) \text{ and } \eta_\xi(u) = \xi(u), \end{aligned} \tag{25}$$

for all $u \in I$.

Proof. Since η is nondecreasing, nonnegatively sublinear in the first variable, from Lemma 1, we have

$$\begin{aligned}
\xi\left(\sum_{i=1}^{n}\alpha_i u_i\right) &\le \xi(u_n) + h(T_{n-1})\eta\left(\xi\left(\sum_{i=1}^{n-1}\frac{\alpha_i}{T_{n-1}}u_i\right),\xi(u_n)\right) = \xi(u_n)\\
&\quad + h(T_{n-1})\eta\left(\xi\left(\frac{T_{n-2}}{T_{n-1}}\sum_{i=1}^{n-2}\frac{\alpha_i}{T_{n-2}}u_i + \frac{\alpha_{n-1}}{T_{n-1}}u_{n-1}\right),\xi(u_n)\right)\\
&\le \xi(u_n) + h(T_{n-1})\eta\left(\xi(u_{n-1}) + h\left(\frac{T_{n-2}}{T_{n-1}}\right)\times\eta\left(\xi\left(\sum_{i=1}^{n-2}\frac{\alpha_i}{T_{n-2}}u_i\right),\xi(u_{n-1})\right),\xi(u_n)\right)\\
&\le \xi(u_n) + h(T_{n-1})\eta(\xi(u_{n-1}),\xi(u_n))\\
&\quad + h(T_{n-2})\eta\left(\eta\left(\xi\left(\sum_{i=1}^{n-2}\frac{\alpha_i}{T_{n-2}}u_i\right),\xi(u_{n-1})\right),\xi(u_n)\right) \le \dots\\
&\le \xi(u_n) + h(T_{n-1})\eta(\xi(u_{n-1}),\xi(u_n))\\
&\quad + h(T_{n-2})\eta(\eta(\xi(u_{n-2}),\xi(u_{n-1})),\xi(u_n)) + \dots\\
&\quad + h(T_1)\eta(\eta(\dots\eta(\eta(\xi(u_1),\xi(u_2)),\xi(u_3))\dots)),(\xi(u_{n-1})),(\xi(u_n))\\
&= \xi(u_n) + h(T_{n-1})\eta_\xi(u_{n-1},u_n) + h(T_{n-2})\eta_\xi(u_{n-2},u_{n-1},u_n) + \dots + h(T_1)\eta_\xi(u_1,u_2,\dots,u_{n-1},u_n).
\end{aligned} \tag{26}$$

Finally, small calculations yield (26). □

6. Hermite–Hadamard and Fejér-Type Inequalities

Theorem 2. *Assume a mean-square integrable generalized h-convex stochastic process ξ: $[a_1,a_2]\times\Omega\longrightarrow\mathbb{R}$. Then, for any $a_1,a_2\in I$ $(a_1<a_2)$, the following inequality holds almost everywhere:*

$$\begin{aligned}
&\xi\left(\frac{a_1+a_2}{2}\right) - \frac{h(1/2)}{a_2-a_1}\int_{a_1}^{a_2}\eta(\xi(a_1\\
&\quad + a_2 - u),\xi(u))\mathrm{d}u\\
&\le \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(u)\mathrm{d}u \le \frac{\xi(a_1)+\xi(a_2)}{2}\\
&\quad + \frac{1}{2}[\eta(\xi(a_1),\xi(a_2))\\
&\quad + \eta(\xi(a_2),\xi(a_1))]\int_0^1 h(\alpha)\mathrm{d}\alpha.
\end{aligned} \tag{27}$$

Proof. Let $x=\alpha a_1+(1-\alpha)a_2$ and $y=(1-\alpha)a_1+\alpha a_2$; then,

$$\begin{aligned}
\xi\left(\frac{a_1+a_2}{2}\right) &= \xi\left(\frac{x+y}{2}\right)\\
&= \xi\left(\frac{1}{2}(\alpha a_1+(1-\alpha)a_2) + \frac{1}{2}((1-\alpha)a_1+\alpha a_2)\right)\\
&\le \xi((1-\alpha)a_1+\alpha a_2) + h\left(\frac{1}{2}\right)\\
&\quad\times\eta(\xi(\alpha a_1+(1-\alpha)a_2),\xi((1-\alpha)a_1+\alpha a_2)).
\end{aligned} \tag{28}$$

Integrating the above inequality with respect to'α' over $[0,1]$,

$$\begin{aligned}
\xi\left(\frac{a_1+a_2}{2}\right) &\le \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(u)\mathrm{d}u + \frac{h(1/2)}{a_2-a_1}\int_{a_1}^{a_2}\eta\\
&\quad\cdot(\xi(a_1+a_2-u),\xi(u))\mathrm{d}u\\
\xi\left(\frac{a_1+a_2}{2}\right) &- \frac{h(1/2)}{a_2-a_1}\int_{a_1}^{a_2}\eta(\xi(a_1+a_2-u),\\
&\xi(u))\mathrm{d}u \le \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(u)\mathrm{d}u.
\end{aligned} \tag{29}$$

Now,

$$\begin{aligned}\int_{a_1}^{a_2}\xi(u)\mathrm{d}u &= (a_2-a_1)\int_0^1\xi(\alpha a_1+(1-\alpha)a_2)\mathrm{d}\alpha\\ &\le (a_2-a_1)\left[\xi(a_2)+\int_0^1 h(\alpha)\eta(\xi(a_1),\xi(a_2))d\alpha\right]\\ \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(u)\mathrm{d}u &\le \xi(a_2)\\ &+\int_0^1 h(\alpha)\eta(\xi(a_1),\xi(a_2))\mathrm{d}\alpha.\end{aligned} \tag{30}$$

Similarly,

$$\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(u)\mathrm{d}u \le \xi(a_1)+\int_0^1 h(\alpha)\eta(\xi(a_2),\xi(a_1))\mathrm{d}\alpha. \tag{31}$$

Adding (30) and (31),

$$\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(u)\mathrm{d}u \le \frac{\xi(a_1)+\xi(a_2)}{2}+\frac{1}{2}\left[\eta(\xi(a_1),\xi(a_2))+\eta(\xi(a_2),\xi(a_1))\right]\int_0^1 h(\alpha)\mathrm{d}\alpha. \tag{32}$$

Combining (29) and (32), we get (27). □

Remark 1. If we take $\eta(x,y)=x-y$ and $h(\alpha)=\alpha$, then Theorem 1 reduces to the Hermite–Hadamard inequality for stochastic convexity in [22].

Definition 6 (see [36]). A stochatic process $\xi\colon [a,b]\longrightarrow\mathbb{R}$ is said to be symmetric with respect to $a+b/2$ on $[a,b]$ if

$$\xi(x)=\xi(a+b-x), \tag{33}$$

for any $a\le x\le b$.

Theorem 3. *Assume a generalized h-convex stochastic process $\xi_1\colon [a_1,a_2]\times\Omega\longrightarrow\mathbb{R}$ with η bounded above on $\xi_1([a_1,a_2])\times\xi_1([a_1,a_2])$ and a nonnegative function $h\colon (0,1)\longrightarrow\mathbb{R}$. Also, suppose that $\xi_2\colon [a_1,a_2]\longrightarrow\mathbb{R}^+$ is integrable and symmetric about $a_1+a_2/2$. Then,*

$$\begin{aligned}\int_{a_1}^{a_2}\xi_1(u)\xi_2(u)\mathrm{d}u &\le \frac{\xi_1(a_1)+\xi_1(a_2)}{2}\int_{a_1}^{a_2}\xi_2(u)\mathrm{d}u\\ &+\frac{\eta(\xi_1(a_1),\xi_1(a_2))+\eta(\xi_1(a_2),\xi_1(a_1))}{2h(a_2-a_1)}\times\int_{a_1}^{a_2}h(a_2-u)\xi_2(u)\mathrm{d}u.\end{aligned} \tag{34}$$

Proof. By using the definition of the generalized h-stochastic convexity of ξ_1, change of variable, and the assumption that ξ_2 is symmetric about $a_1+a_2/2$, we have

$$\begin{aligned}&\int_{a_1}^{a_2}\xi_1(u)\xi_2(u,)\mathrm{d}u \le (a_2-a_1)\int_0^1\left[\xi_1(a_2)+h(\alpha)\eta(\xi_1(a_1),\xi_1(a_2))\right]\\ &\times\xi_2(\alpha a_1+(1-\alpha)a_2)\mathrm{d}\alpha\\ &=(a_2-a_1)\left[\int_0^1\xi_1(a_2)\xi_2(\alpha a_1+(1-\alpha)a_2)\mathrm{d}\alpha+\eta(\xi(a_1),\xi(a_2))\int_0^1 h(\alpha)\xi_2(\alpha a_1+(1-\alpha)a_2)\mathrm{d}\alpha\right],\end{aligned} \tag{35}$$

$$\begin{aligned}\int_{a_1}^{a_2}\xi_1(u,)\xi_2(u,)\mathrm{d}u &\le (a_2-a_1)\int_0^1\left[\xi_1(a_1)+h(\alpha)\eta(\xi_1(a_2),\xi_1(a_1))\right]\\ &\times\xi_2((1-\alpha)a_1+\alpha a_2)\mathrm{d}\alpha=(a_2-a_1)\left[\int_0^1\xi_1(a_1)\xi_2(\alpha a_1+(1-\alpha)a_2)\mathrm{d}\alpha\right.\\ &\left.+\eta(\xi_1(a_2),\xi_1(a_1))\int_0^1 h(\alpha)\xi_2(\alpha a_1+(1-\alpha)a_2)\mathrm{d}\alpha\right].\end{aligned} \tag{36}$$

Adding (35) and (36), we get

$$2\int_{a_1}^{a_2}\xi_1(u)\xi_2(u)\mathrm{d}u \le (a_2-a_1)(\xi_1(a_1)+\xi_1(a_2))$$
$$\times\int_0^1\xi_2(\alpha a_1+(1-\alpha)a_2)\mathrm{d}\alpha+(a_2-a_1)(\eta(\xi_1(a_1),\xi_1(a_2)))$$
$$+\eta((\xi_1(a_2),\xi_1(a_1)))\int_0^1 h(\alpha)\xi_2(\alpha a_1+(1-\alpha)a_2)\mathrm{d}\alpha, \tag{37}$$

and by changing the variable $u=\alpha a_1+(1-\alpha)a_2$, we obtain

$$\int_{a_1}^{a_2}\xi_1(u)\xi_2(u)\mathrm{d}u \le \frac{\xi_1(a_1)+\xi_1(a_2)}{2}$$
$$\cdot\int_{a_1}^{a_2}\xi_2(u)\mathrm{d}u$$
$$+\frac{\eta(\xi_1(a_1),\xi_1(a_2))+\eta(\xi_1(a_2),\xi_1(a_1))}{2h(a_2-a_1)} \tag{38}$$
$$\times\int_{a_1}^{a_2}h(a_2-u)\xi_2(u)\mathrm{d}u.$$

□

Theorem 4. *Assume a generalized h-convex stochastic process ξ_1: $[a_1,a_2]\times\Omega\longrightarrow\mathbb{R}$ with η bounded above on $\xi_1([a_1,a_2])\times\xi_1([a_1,a_2])$ and a nonnegative function h: $(0,1)\longrightarrow\mathbb{R}$. Also, suppose that ξ_2: $[a_1,a_2]\longrightarrow\mathbb{R}^+$ is integrable and symmetric about $a_1+a_2/2$. Then,*

$$\xi_1\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}\xi_2(u)-h\left(\frac{1}{2}\right)$$
$$\cdot\int_{a_1}^{a_2}\eta(\xi_1(a_1+a_2-u),\xi(u))\xi_2(u)\mathrm{d}u \tag{39}$$
$$\le\int_{a_1}^{a_2}\xi(u)\xi_2(u)\mathrm{d}u.$$

Proof. Using the definition of the generalized h-convex stochastic process, change of variable, and the assumption that ξ_2 is symmetric about $a_1+a_2/2$, we get

$$\xi_1\left(\frac{a_1+a_2}{2}\right)=\xi_1\left(\frac{\alpha a_1-\alpha a_1+a_1+a_2+\alpha a_2-\alpha a_2}{2}\right)$$
$$=\xi_1\left(\frac{\alpha a_1+(1-\alpha)a_2+\alpha a_2+(1-\alpha)a_1}{2}\right)$$
$$\le\xi_1(\alpha a_2+(1-\alpha)a_1)+h\left(\frac{1}{2}\right)$$
$$\times\eta(\xi_1(\alpha a_2+(1-\alpha)a_1),\xi_1(\alpha a_2+(1-\alpha)a_1)). \tag{40}$$

By changing the variable $u=\alpha a_2+(1-\alpha)a_1$, we obtain

$$\xi_1\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}\xi_2(u)\mathrm{d}u=\xi_1\left(\frac{a_1+a_2}{2}\right)$$
$$\cdot\int_0^1\xi_2(\alpha a_2+(1-\alpha)a_1)(a_2-a_1)\mathrm{d}\alpha$$
$$\le\int_0^1\xi_1(\alpha a_2+(1-\alpha)a_1)\xi_2(\alpha a_2$$
$$+(1-\alpha)a_1)(a_2-a_1)\mathrm{d}\alpha$$
$$+h\left(\frac{1}{2}\right)\int_0^1\eta(\xi_1(\alpha a_1+(1-\alpha)a_2),\xi_1(\alpha a_2+(1-\alpha)a_1))$$
$$\times\xi_2(\alpha a_2+(1-\alpha)a_1)(a_2-a_1)\mathrm{d}\alpha$$
$$=\int_{a_1}^{a_2}\xi_1(u)\xi_2(u)\mathrm{d}u+h\left(\frac{1}{2}\right)$$
$$\times\int_{a_1}^{a_2}\eta(\xi_1(a_1+a_2-u),\xi_1(u))\xi_2(u)\mathrm{d}u. \tag{41}$$

□

Corollary 1. *By setting $\eta(x,y)=x-y$ and $h(\alpha)=\alpha$ in (34 and 39) and then combining, we obtain the classical Hermite–Hadamard–Fejér-type inequality as*

$$\xi_1\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}\xi_2(u)\mathrm{d}u\le\int_{a_1}^{a_2}\xi_1(u)\xi_2(u)\mathrm{d}u$$
$$\le\frac{\xi_1(a_1)+\xi_1(a_2)}{2} \tag{42}$$
$$\cdot\int_{a_1}^{a_2}\xi_2(u)\mathrm{d}u.$$

7. Ostrowski-Type Inequality

In order to prove Ostrowski-type inequality for the generalized h-convex stochastic process, the following lemma is required.

Lemma 2 (see [37]). *Let ξ: $I\times\Omega\longrightarrow\mathbb{R}$ be a stochastic process which is mean-square differentiable on I° and its derivative ξ' be mean-square integrable on $[a_1,a_2]$, where a_1, $a_2\in I$ with $a_1<a_2$; then, the following equality holds:*

$$\xi(u)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(v)\mathrm{d}v=\frac{(u-a_1)^2}{a_2-a_1}$$
$$\cdot\int_0^1\alpha\xi'(\alpha u+(1-\alpha)a_1)\mathrm{d}\alpha \tag{43}$$
$$-\frac{(a_2-u)^2}{a_2-a_1}\int_0^1\alpha\xi'(\alpha u+(1-\alpha)b_1)\mathrm{d}\alpha\,(a.e.),$$

for each $u\in[a,b]$.

Theorem 5. *Assume a mean-square stochastic process ξ: $I \times \Omega \longrightarrow \mathbb{R}$ such that $\xi\prime$ (the derivative of ξ) is mean-square integrable on $[a_1, a_2]$, where a_1, $a_2 \in I$ with $a_1 < a_2$, and consider a nonnegative function h: $(0,1) \longrightarrow \mathbb{R}$ which is supermultiplicative such that, for every α, $h(\alpha) > \alpha$. If $|\xi\prime|$ is a generalized h-convex stochastic process on I and $|\xi\prime(u)| \leq M$ for every u, then*

$$\left|\xi(u) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2}\xi(v)\mathrm{d}v\right| \leq M\left[\frac{(u-a_1)^2 + (a_2-u)^2}{a_2 - a_1}\right]\int_0^1 h(y)\mathrm{d}y + \left(\frac{(u-a_1)^2}{a_2-a_1}\eta\left(|\xi\prime(u)|, \left|\xi'(a_1)\right|\right) + \frac{(a_2-u)^2}{a_2-a_1}\eta\left(|\xi'(u)|, \left|\xi'(a_2)\right|\right)\right)\int_0^1 h^2(y)\mathrm{d}y. \tag{44}$$

Proof. By using Lemma 2 and the definition of generalized h-stochastic convexity of $|\xi'|$, we get

$$\begin{aligned}
&\left|\xi(u) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(v)\mathrm{d}v\right| \leq \frac{(u-a_1)^2}{a_2-a_1}\int_0^1 y\left|\xi\prime(yu + (1-y)a_1)\right|\mathrm{d}y \\
&\quad + \frac{(a_2-u)^2}{a_2-a_1}\int_0^1 y\left|\xi\prime(yu + (1-y)a_2)\right|\mathrm{d}y \\
&\leq \frac{(u-a_1)^2}{a_2-a_1}\int_0^1 y\left[\left|\xi\prime(a_1)\right| + h(y)\eta\left(|\xi\prime(u)|, \left|\xi\prime(a_1)\right|\right)\right]\mathrm{d}y \\
&\quad + \frac{(a_2-u)^2}{a_2-a_1}\int_0^1 y\left[\left|\xi\prime(a_2)\right| + h(y)\eta\left(|\xi\prime(u)|, \left|\xi\prime(a_2)\right|\right)\right]\mathrm{d}y \leq M\frac{(u-a_1)^2}{a_2-a_1}\int_0^1 h(y)\mathrm{d}y \\
&\quad + \frac{(u-a_1)^2}{a_2-a_1}\int_0^1 h^2(y)\eta\left(|\xi\prime(u)|, \left|\xi\prime(a_1)\right|\right)\mathrm{d}y + M\frac{(a_2-u)^2}{a_2-a_1}\int_0^1 h(y)\mathrm{d}y \\
&\quad + \frac{(a_2-u)^2}{a_2-a_1}\int_0^1 h^2(y)\eta\left(|\xi\prime(u)|, \left|\xi\prime(a_2)\right|\right)\mathrm{d}y \leq M\left[\frac{(u-a_1)^2 + (a_2-u)^2}{a_2-a_1}\right]\int_0^1 h(y)\mathrm{d}y \\
&\quad + \frac{(u-a_1)^2}{a_2-a_1}\eta\left(|\xi\prime(u)|, \left|\xi\prime(a_1)\right|\right) \times \int_0^1 h^2(y)\mathrm{d}y + \frac{(a_2-u)^2}{a_2-a_1}\eta\left(|\xi\prime(u)|, \left|\xi\prime(a_2)\right|\right)\int_0^1 h^2(y)\mathrm{d}y \\
&= M\left[\frac{(u-a_1)^2 + (a_2-u)^2}{a_2-a_1}\right]\int_0^1 h(y)\mathrm{d}y + \left(\frac{(u-a_1)^2}{a_2-a_1}\eta\left(|\xi\prime(u)|, \left|\xi\prime(a_1)\right|\right)\right. \\
&\quad \left. + \frac{(a_2-u)^2}{a_2-a_1}\eta\left(|\xi\prime(u)|, \left|\xi\prime(a_2)\right|\right)\right)\int_0^1 h^2(y)\mathrm{d}y.
\end{aligned} \tag{45}$$

□

8. Conclusion

Stochastic processes have many applications in statistics, which obviously lead to lots of other domains, for example, Kolmogorov–Smirnoff test on the equality of distributions [38–40] (the test statistic is derived from a Brownian bridge, which is a Brownian motion conditioned to have certain values at the endpoints of an interval of time). The other applications include sequential analysis [41, 42] (this is the rigorous way you can stop an A/B test dynamically. Stopping rules are obtained by approximating discrete problems with their continuous time analogs, in which the sufficient statistic process follows a stochastic differential equation) and quickest detection [43, 44] (my stock story is that you are using a gold mine which gives you a random amount of gold per day, but at some point, it will get depleted, and you want to know when this happens as quickly as possible. Decision rules are again based on properties of hitting times of

random processes). In this paper, we have introduced generalized h-conevex stochastic processes and proved Jensen, Hermite–Hadamard, and Fejér-type inequalities. Our results are applicable because applying the convex function to the expected value of a random variable is always bounded above by the expected value of the convex function of the random variable.

Data Availability

All data required for this research are included within this paper.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors' Contributions

All authors contributed equally to this paper.

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