Research Article

Generalization of $h$-Convex Stochastic Processes and Some Classical Inequalities

Hao Zhou,1 Muhammad Shoaib Saleem2, Mamoona Ghafoor,2 and Jingjing Li3

1Transportation School of Wuhan University of Technology, Wuhan University of Technology, Wuhan 430070, China
2Department of Mathematics, University of Okara, Okara, Pakistan
3China Ship Development and Design Center, Wuhan 430064, China

Correspondence should be addressed to Muhammad Shoaib Saleem; shaby455@yahoo.com

Received 30 April 2020; Revised 21 June 2020; Accepted 30 June 2020; Published 8 August 2020

1. Introduction

Stochastic processes are a branch of probability theory, treating probabilistic models that evolve in time [1–3]. It is a branch of mathematics, starting with the axioms of probability and containing a rich and fascinating set of results following from those axioms [4]. Although the results are applicable to many areas [5], they are best understood initially in terms of their mathematical structure and inter-relationships [6].

There are various ways to define stochastic monotonicity and convexity for stochastic processes [7], and they are of great importance in optimization, especially in optimal designs, and also useful for numerical approximations when there exist probabilistic quantities in the literature [8]. We also refer [9–12] for detailed survey about the importance and interesting properties of stochastic models.

The idea of convex functions was put forward, and many generalizations were made in this area [13]. $h$-Convex [14] and $\phi$-convex functions [15] are famous generalizations of convex functions. Later, the theory of stochastic processes has developed very rapidly and has found application in a large number of fields. The study on convex stochastic processes was initiated in [16] by B. Nagy in 1974. After that, Nikodem in 1980 introduced the convex stochastic processes in his article [17]. Following this line of investigation, Skowronski described the properties of Jensen-convex and Wright-convex stochastic processes in [18, 19]. Some interesting properties of convex and Jensen-convex processes are also presented in [20, 21]. Assume a Jensen-convex stochastic process $\xi : I \times \Omega \rightarrow \mathbb{R}$ which is mean-square continuous in $I$, where $I \subseteq \mathbb{R}$ is an interval. Then, for every $a_1, a_2 \in I$ ($a_1 < a_2$),

$$\xi\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \xi(u)du \leq \frac{\xi(a_1) + \xi(a_2)}{2} \text{ (a.e.)}, \quad (1)$$

is known as the Hermite–Hadamard-type inequality for the convex stochastic process [22].

The aim of this paper is to introduce the notion of the generalized $h$-convex stochastic process and to extend the classical Hermite–Hadamard inequality for convex stochastic processes to generalized $h$-convex stochastic processes.
2. Novelty and Significance

The study of convex functions makes them special because of their interesting properties as maximum are attained at the boundary point, and moreover, any local minimum is global one. So, this topic of research got the attention of many researchers of different areas because of their enormous applications in optimization theory. As far as convex stochastic processes are concerned, there is a lot of work in the last few decades. In [23], the authors investigated the gradient descent optimality for strongly convex stochastic processes. We observe that, by taking optimization typically boils down to solve a convex stochastic optimization problem in terms of terminal wealth with budget constraints, see, e.g., [24]. For more details related to this work, see [25–31].

3. Preliminaries

Let \((\Omega, P)\) be a probability space. A function \(\xi: \Omega \to \mathbb{R}\) is a random variable if it is \(\mathcal{F}\)-measurable. A function \(\xi: \mathbb{R} \times \Omega \to \mathbb{R}\), where \(\Omega \subseteq \mathbb{R}\) is an interval, is a stochastic process if for every \(t \in I\), the function \(\xi(t)\) is a random variable.

The stochastic process \(\xi: \mathbb{R} \times \Omega \to \mathbb{R}\) is known as

1. Stochastically continuous in \(I\) if

\[
\mu - \lim_{v \to v'} \xi(v) = \xi(v'),
\]

for all \(v' \in I\), where \(\mu - \lim v\) represents the limit in the probability.

2. Mean-square continuous in \(I\) if

\[
\lim_{v \to v'} \mathbb{E}\left[ (\xi(v) - \xi(v'))^2 \right] = 0,
\]

for all \(v' \in I\), where \(\mathbb{E}[\xi(v)]\) represents an expectation value of random variable \(\xi(v)\).

Clearly in probability, mean-square continuity implies continuity, but converse is not true.

A stochastic process \(\xi\) is known as mean-square differentiable in \(I\) if there is a stochastic process \(\xi'\) (derivative of \(\xi\)) such that

\[
\lim_{v \to v'} \mathbb{E}\left[ \frac{\xi(v) - \xi(v')}{v - v'} - \xi'(v') \right]^2 = 0,
\]

for every \(v' \in I\).

Now, we would like to recall the concept of the mean-square integral. For the definition and basic properties, see [32, 33].

Assume a stochastic process \(\xi: \mathbb{R} \times \Omega \to \mathbb{R}\), with \(\mathbb{E}[(\xi(t))^2] < \infty\). A random variable \(\xi: \Omega \to \mathbb{R}\) is said to be the mean-square integral of the process \(\xi\) on \([a, b]\) if for every normal sequence of partitions of \([a, b]\), \(a_i = v' < v_1 < \ldots < v_n = a_2\), and for all \(\Theta_k \in [v_{k-1}, v_k]\), we have

\[
\lim_{n \to \infty} \mathbb{E}\left[ \left( \sum_{k=1}^{n} \xi(\Theta_k)(v_k - v_{k-1}) - \xi(.) \right)^2 \right] = 0.
\]

Then, we write

\[
\xi(.) = \int_a^b \xi(s)ds(a.e).
\]

Now, we shall present some definitions and generalizations of the convex stochastic process.

**Definition 1** (see [34]). A stochastic process \(\xi: \mathbb{R} \times \Omega \to \mathbb{R}\) is known as generalized convex with respect to a bifunction \(\eta: \xi(I) \times \xi(I) \to \mathbb{R}\) if

\[
\xi(au + (1 - a)v) \leq (a \xi(u) + h(1-a)\xi(v))(a.e),
\]

for all \(u, v \in I\) and \(a \in [0, 1]\).

**Definition 2** (see [35]). Let \(h: (0, 1) \to \mathbb{R}\) be a function which is nonnegative and \(h \neq 0\). A stochastic process \(\xi: \mathbb{R} \times \Omega \to \mathbb{R}\) is said to be \(h\)-convex if

\[
\xi(au + (1 - a)v) \leq h(a)\xi(u) + h(1-a)\xi(v)(a.e),
\]

for every \(u, v \in I\) and \(a \in (0, 1)\).

**Definition 3** (see [15]). The function \(\eta: \xi(I) \times \xi(I) \to \mathbb{R}\) is said to be

(1) Nonnegatively homogeneous if \(\eta(yu, yv) = y\eta(u, v)\) for all \(u, v \in \mathbb{R}\) and \(y \geq 0\)

(2) Additive if \(\eta(u_1, v_1) + \eta(u_2, v_2) = \eta(u_1 + u_2, v_1 + v_2)\) for all \(u_1, u_2, v_1, v_2 \in \mathbb{R}\).

**Definition 4** (see [14]). A function \(h: I \to \mathbb{R}, J \subseteq \mathbb{R}\), is said to be a supermultiplicative function if

\[
h(\lambda v) \geq h(u)h(v),
\]

for all \(u, v \in J\).

Now, we are ready to give our main definition.

**Definition 5** (generalized \(h\)-convex stochastic process). \(h\) and \(\eta\) are fixed like above. We say that a stochastic process \(\xi: \mathbb{R} \times \Omega \to \mathbb{R}\) is called generalized \(h\)-convex if

\[
\xi(au + (1 - a)v) \leq \eta(\xi(u), \xi(v))(a.e),
\]

for all \(u, v \in I\) and \(a \in [0, 1]\).

In (10), if we take \(\eta(u, v) = u - v\) and \(h(a) = a\), we obtain the convex stochastic process. We observe that, by taking \(u = v\) in (10), we get

\[
h(a)\eta(\xi(u), \xi(u)) \geq 0,
\]

for any \(u \in I\) and \(a \in [0, 1]\), which implies that

\[
\eta(\xi(u), \xi(u)) \geq 0,
\]

for any \(u \in I\). Also, if we take \(a = 1\) and \(h(1) = 1\) in (10), we get
the following inequality is satisfied:

\[ \xi(u) - \xi(v) \leq \eta(\xi(u), \xi(v)), \]  

(13)

for any \( u, v \in I \). (13) obviously implies (12).

We observe that if \( \xi: I \rightarrow \mathbb{R} \) is a convex stochastic process and \( \eta: I \times I \rightarrow \mathbb{R} \) is an arbitrary bifunction that satisfies

\[ \eta(a, b) \geq a - b, \]  

(14)

for any \( a, b \in I \) and \( h(a) \geq a \), then for any \( u, v \in I \) and \( a \in [0, 1] \), we have

\[ \xi(au + (1 - a)v) \leq \xi(v) + a(\xi(u) - \xi(v)) \leq \xi(v) + h(a)\eta(\xi(u), \xi(v)), \]  

(15)

showing that \( \xi \) is a generalized \( h \)-convex stochastic process.

Example 1. Every generalized \( h \)-convex function gives an example of a generalized \( h \)-convex stochastic process.

Example 2. Let \( \xi: I \times \Omega \rightarrow \mathbb{R} \) be a convex stochastic process. For every \( k \geq 1 \), consider the function

\[ h_k: (0, 1) \rightarrow \mathbb{R}, \]  

\[ x \mapsto x^k. \]  

(16)

Note that \( h_k(a) \geq a \) for all \( a \in (0, 1) \). Also, take \( \eta(x, y) = x - y \). Moreover, for every \( u, v \in I \) and \( a \in (0, 1) \), the following inequality is satisfied:

\[ \xi(au + (1 - a)v) \leq a\xi(u) + (1 - a)\xi(v) \leq \xi(v) + h_k(a)\eta(\xi(u), \xi(v)) (a.e). \]  

(17)

Then, \( \xi \) is a generalized \( h_k \)-convex stochastic process.

The paper is organized as follows: in Section 4, we will derive some basic results for the generalized \( h \)-convex stochastic process. In Section 5, Jensen-type inequality will be proved, whereas in Section 6, Hermite–Hadamard and Fejér-type inequalities will be established. Finally, in Section 7, we will derive Ostrowski-type inequality for generalized \( h \)-convex stochastic processes.

4. Basic Results

Proposition 1. Consider two generalized \( h \)-convex stochastic processes \( \xi_1, \xi_2: I \times \Omega \rightarrow \mathbb{R} \) such that

1. If \( \eta \) is additive, then \( \xi_1 + \xi_2: I \rightarrow \mathbb{R} \) is \( h_k \)-convex stochastic process
2. If \( \eta \) is nonnegatively homogeneous, then for any \( \gamma \geq 0 \), \( \gamma \xi_1: I \times \Omega \rightarrow \mathbb{R} \) is a generalized \( h \)-convex stochastic process
3. If \( \xi_2 \) is linear, then \( \xi_1 \circ \xi_2 \) is a generalized \( h \)-convex stochastic process

Proof. The proof of the proposition is straightforward. □

Proposition 2. If \( \xi: [a, b] \rightarrow \mathbb{R} \) is a generalized \( h \)-convex stochastic process and \( h(a) \leq k \), then we have almost everywhere

\[ \max_{u \in [a, b]} \xi(u) \leq \max\{ \xi(b), \xi(b) + k\eta(\xi(a), \xi(b)) \}. \]  

(18)

Proof. For any \( u \in [a, b] \), we have \( u = aa + (1 - a)b \), for some \( a \in [0, 1] \), and

\[ \xi(u) = \xi(aa + (1 - a)b). \]  

(19)

since \( \xi \) is a generalized \( h \)-convex stochastic process, so

\[ \xi(u) \leq \xi(b) + h(a)\eta(\xi(a), \xi(b)) \leq \max\{ \xi(b), \xi(b) + k\eta(\xi(a), \xi(b)) \}. \]  

(20)

Since \( u \) is arbitrary,

\[ \max_{u \in [a, b]} \xi(u) \leq \max\{ \xi(b), \xi(b) + k\eta(\xi(a), \xi(b)) \}. \]  

(21)

5. Jensen-Type Inequality

We will use the next lemma to derive the Jensen-type inequality for the generalized \( h \)-convex stochastic process.

Lemma 1. Assume a generalized \( h \)-convex stochastic process \( \xi: I \times \Omega \rightarrow \mathbb{R} \). For \( u_1, u_2 \in I \) and \( a_1, a_2 = 1 \), we have

\[ \xi(a_1u_1 + a_2u_2) \leq \xi(u_1) + h(a_1)\eta(\xi(u_2), \xi(u_2)) (a.e). \]  

(22)

Also, when \( n \geq 2 \) for \( u_1, u_2, \ldots, u_n \in I \), \( \sum_{i=1}^{n} a_i = 1 \), and \( T_i = \sum_{j=1}^{i} a_j \), the following inequality holds almost everywhere:

\[ \xi\left( \sum_{i=1}^{n} a_i u_i \right) \leq \xi\left( \sum_{i=1}^{n-1} a_i u_i \right) + h(T_{n-1})\eta\left( \xi\left( \sum_{i=1}^{n} a_i u_i \right), \xi(u_n) \right). \]  

(23)

Theorem 1 (Jensen-type inequality). Assume a generalized \( h \)-convex stochastic process \( \xi: I \times \Omega \rightarrow \mathbb{R} \), and let \( \eta: A \times B \rightarrow \mathbb{R} \) be nondecreasing, nonnegatively sublinear in the first variable. If \( T_i = \sum_{j=1}^{i} a_j \) for \( i = 1, \ldots, n \) such that \( T_n = 1 \), then we have almost everywhere

\[ \xi\left( \sum_{i=1}^{n} a_i u_i \right) \leq \sum_{i=1}^{n} a_i h(T_i)\eta_{|T_i|}(u_i, u_{i+1}, \ldots, u_n), \]  

(24)

where

\[ \eta_{|T_i|}(u_i, u_{i+1}, \ldots, u_n) = \eta\left( \eta(u_i, u_{i+1}, \ldots, u_{n-1}) \right), \]  

(25)

for all \( u \in I \).

Proof. Since \( \eta \) is nondecreasing, nonnegatively sublinear in the first variable, from Lemma 1, we have
\[
\xi\left(\frac{1}{2} \int_{a_1}^{a_2} \xi(\alpha) d\alpha \right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \xi(u) du \leq \frac{\xi(a_1) + \xi(a_2)}{2}
\]

Proof. Let \( x = a_1 + (1 - a)a_2 \) and \( y = (1 - a)a_1 + aa_2 \); then,

\[
\xi\left(\frac{1}{2} \int_{a_1}^{a_2} \xi(\alpha) d\alpha \right) \leq \xi(x) + \xi(y) \leq \frac{\xi(a_1) + \xi(a_2)}{2}.
\]

Finally, small calculations yield (26).

6. Hermite–Hadamard and Fejér-Type Inequalities

**Theorem 2.** Assume a mean-square integrable generalized \( h \)-convex stochastic process \( \xi : [a_1, a_2] \times \Omega \rightarrow \mathbb{R} \). Then, for any \( a_1, a_2 \in I \) \((a_1 < a_2)\), the following inequality holds almost everywhere:

\[
\xi\left(\frac{a_1 + a_2}{2}\right) \leq \xi\left(\frac{1}{2} \int_{a_1}^{a_2} \xi(\alpha) d\alpha \right) \leq \frac{\xi(a_1) + \xi(a_2)}{2}.
\]

Integrating the above inequality with respect to \( \alpha \) over \([0, 1] \),

\[
\xi\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \xi(u) du + \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \eta(1/2) d\alpha.
\]

Now,
\[
\int_{a_1}^{a_2} \xi(u)du = (a_2 - a_1) \int_0^1 \xi(aa_1 + (1 - \alpha)a_2) d\alpha \\
\leq (a_2 - a_1) \left[ \xi(a_2) + \int_0^1 h(\alpha) \eta(\xi(a_1), \xi(a_2)) d\alpha \right] \\
+ \int_0^1 h(\alpha) \eta(\xi(a_1), \xi(a_2)) d\alpha.
\]

Combining (29) and (32), we get (27).

**Remark 1.** If we take \( \eta(x, y) = x - y \) and \( h(\alpha) = \alpha \), then Theorem 1 reduces to the Hermite–Hadamard inequality for stochastic convexity in [22].

**Definition 6 (see [36]).** A stochastic process \( \xi: [a, b] \rightarrow \mathbb{R} \) is said to be symmetric with respect to \( a + b/2 \) on \([a, b]\) if

\[
\int_{a_1}^{a_2} \xi_1(u) \xi_2(u) du \leq \frac{\xi_1(a_1) + \xi_1(a_2)}{2} \int_{a_1}^{a_2} \xi_2(u) du \\
+ \frac{\eta(\xi_1(a_1), \xi_1(a_2)) + \eta(\xi_1(a_2), \xi_1(a_1))}{2h(a_2 - a_1)} \times \int_{a_1}^{a_2} h(a_2 - u) \xi_2(u) du.
\]

**Proof.** By using the definition of the generalized h-stochastic convexity of \( \xi_1 \), change of variable, and the assumption that \( \xi_2 \) is symmetric about \( a_1 + a_2/2 \), we have

\[
\int_{a_1}^{a_2} \xi_1(u) \xi_2(u) du \leq (a_2 - a_1) \int_0^1 \left[ \xi_1(a_2) + h(\alpha) \eta(\xi_1(a_1), \xi_1(a_2)) \right] \\
\times \xi_2(aa_1 + (1 - \alpha)a_2) d\alpha \\
= (a_2 - a_1) \left[ \int_0^1 \xi_1(a_2) \xi_2(aa_1 + (1 - \alpha)a_2) d\alpha + \eta(\xi_1(a_1), \xi_1(a_2)) \int_0^1 h(\alpha) \xi_2(aa_1 + (1 - \alpha)a_2) d\alpha \right].
\]

\[
\int_{a_1}^{a_2} \xi_1(u) \xi_2(u) du \leq (a_2 - a_1) \int_0^1 \left[ \xi_1(a_1) + h(\alpha) \eta(\xi_1(a_2), \xi_1(a_1)) \right] \\
\times \xi_2((1 - \alpha)a_1 + aa_2) d\alpha = (a_2 - a_1) \left[ \int_0^1 \xi_1(a_1) \xi_2(aa_1 + (1 - \alpha)a_2) d\alpha \\
+ \eta(\xi_1(a_2), \xi_1(a_1)) \int_0^1 h(\alpha) \xi_2(aa_1 + (1 - \alpha)a_2) d\alpha \right].
\]

Adding (35) and (36), we get

\[
\int_{a_1}^{a_2} \xi(u)du \leq (a_2 - a_1) \int_0^1 \xi(a_1) + \int_0^1 h(\alpha) \eta(\xi(a_2), \xi(a_1)) d\alpha. \quad (31)
\]

Adding (30) and (31),

\[
\xi(x) = \xi(a + b - x), \quad (33)
\]

for any \( a \leq x \leq b \).

**Theorem 3.** Assume a generalized h-convex stochastic process \( \xi: [a_1, a_2] \times \Omega \rightarrow \mathbb{R} \) with \( h(\cdot) \) bounded above on \( [a_1, a_2] \times [a_1, a_2] \) and a nonnegative function \( h: (0, 1) \rightarrow \mathbb{R} \). Also, suppose that \( \xi_2: [a_1, a_2] \rightarrow \mathbb{R}^+ \) is integrable and symmetric about \( a_1 + a_2/2 \). Then,
2 \int_{a_1}^{a_2} \xi_1(u)\xi_2(u)du \leq (a_2 - a_1)(\xi_1(a_1) + \xi_1(a_2))

\times \int_{0}^{1} \xi_2(aa_1 + (1 - a)a_2)da + (a_2 - a_1)(\eta(\xi_1(a_1), \xi_1(a_2)))

+ \eta((\xi_1(a_2), \xi_1(a_1))) \int_{0}^{1} h(a)\xi_2(aa_1 + (1 - a)a_2)da,

(37)

and by changing the variable \( u = aa_1 + (1 - a)a_2 \), we obtain

\int_{a_1}^{a_2} \xi_1(u)\xi_2(u)du \leq \frac{\xi_1(a_1) + \xi_1(a_2)}{2}

\cdot \int_{a_1}^{a_2} \xi_2(u)du

+ \eta(\xi_1(a_1), \xi_1(a_2)) + \eta(\xi_1(a_2), \xi_1(a_1))

2h(a_2 - a_1)

\times \int_{a_1}^{a_2} h(a_2 - u)\xi_2(u)du.

(38)

**Theorem 4.** Assume a generalized \( h \)-convex stochastic process \( \xi_1: [a_1, a_2] \times \Omega \rightarrow \mathbb{R} \) with \( \eta \) bounded above on \( \xi_1([a_1, a_2]) \times \xi_1([a_1, a_2]) \) and a nonnegative function \( h: (0, 1) \rightarrow \mathbb{R} \). Also, suppose that \( \xi_2: [a_1, a_2] \rightarrow \mathbb{R}^+ \) is integrable and symmetric about \( a_1 + a_2/2 \). Then,

\[ \xi_1(\frac{a_1 + a_2}{2}) \int_{a_1}^{a_2} \xi_2(u)du - h(\frac{1}{2}) \]

\[ \cdot \int_{a_1}^{a_2} \eta(\xi_1(a_1 + a_2 - u), \xi(u))\xi_2(u)du \]

\leq \int_{a_1}^{a_2} \xi(u)\xi_2(u)du.

(39)

**Proof.** Using the definition of the generalized \( h \)-convex stochastic process, change of variable, and the assumption that \( \xi_2 \) is symmetric about \( a_1 + a_2/2 \), we get

\[ \xi_1(\frac{a_1 + a_2}{2}) = \xi_1(\frac{aa_1 - aa_1 + a_1 + a_2 + aa_2 - aa_2}{2}) \]

\[ = \xi_1(\frac{aa_1 + (1 - a)a_2 + aa_2 + (1 - a)a_1}{2}) \]

\[ \leq \xi_1(aa_2 + (1 - a)a_1) + h(\frac{1}{2}) \]

\[ \times \eta(\xi_1(aa_2 + (1 - a)a_1), \xi_1(aa_2 + (1 - a)a_1)). \]

(40)

By changing the variable \( u = aa_2 + (1 - a)a_1 \), we obtain

\[ \xi_1(\frac{a_1 + a_2}{2}) \int_{a_1}^{a_2} \xi_2(u)du = \xi_1(\frac{a_1 + a_2}{2}) \]

\[ \cdot \int_{0}^{1} \xi_2(aa_2 + (1 - a)a_1)(a_2 - a_1)da \]

\[ \leq \int_{0}^{1} \xi_1(aa_2 + (1 - a)a_1)\xi_2(aa_2 + (1 - a)a_1) \]

\[ + (1 - a)a_1)(a_2 - a_1)da \]

\[ + h(\frac{1}{2}) \int_{0}^{1} \eta(\xi_1(aa_1 + (1 - a)a_2), \xi_1(aa_2 + (1 - a)a_1)) \]

\[ \times \xi_2(aa_2 + (1 - a)a_1)(a_2 - a_1)da \]

\[ = \int_{a_1}^{a_2} \xi_1(u)\xi_2(u)du + h(\frac{1}{2}) \]

\[ \times \int_{a_1}^{a_2} \eta(\xi_1(a_1 + a_2 - u), \xi_1(u))\xi_2(u)du. \]

(41)

**Corollary 1.** By setting \( \eta(x, y) = x - y \) and \( h(a) = a \) in (34 and 39) and then combining, we obtain the classical Hermite–Hadamard–Fejér-type inequality as

\[ \xi_1(\frac{a_1 + a_2}{2}) \int_{a_1}^{a_2} \xi_2(u)du \leq \int_{a_1}^{a_2} \xi_1(u)\xi_2(u)du \]

\[ \leq \frac{\xi_1(a_1) + \xi_1(a_2)}{2} \]

\[ \cdot \int_{a_1}^{a_2} \xi_2(u)du. \]

(42)

**7. Ostrowski-Type Inequality**

In order to prove Ostrowski-type inequality for the generalized \( h \)-convex stochastic process, the following lemma is required.

**Lemma 2** [37]. Let \( \xi: I \times \Omega \rightarrow \mathbb{R} \) be a stochastic process which is mean-square differentiable on \( I \) and its derivative \( \xi' \) be mean-square integrable on \( [a_1, a_2] \), where \( a_1, a_2 \in I \) with \( a_1 < a_2 \); then, the following equality holds:

\[ \xi(u) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \xi(v)dv = (u - a_1)^2 \]

\[ \cdot \int_{0}^{1} a \xi''(au + (1 - a)a_1)da \]

\[ - (a_2 - u)^2 \]

\[ \cdot \int_{0}^{1} a \xi''(au + (1 - a)b_1)da (a.e.), \]

for each \( u \in [a, b] \).
Theorem 5. Assume a mean-square stochastic process $\xi: I \times \Omega \rightarrow \mathbb{R}$ such that $\xi'$ (the derivative of $\xi$) is mean-square integrable on $[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$, and consider a nonnegative function $h: (0, 1) \rightarrow \mathbb{R}$ which is supermultiplicative such that, for every $a, h(a) > a$. If $|\xi'|$ is a generalized $h$-convex stochastic process on $I$ and $|\xi'(u)| \leq M$ for every $u$, then

$$
\left\lvert \xi(u) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \xi(v)dv \right\rvert \leq M \left[ \frac{(u - a_1)^2 + (a_2 - u)^2}{a_2 - a_1} \right] \int_0^1 h(y)dy
$$

$$
+ \left( \frac{(u - a_1)^2}{a_2 - a_1} \right) \eta(|\xi'(u)|, |\xi'(a_1)|) + \left( \frac{(a_2 - u)^2}{a_2 - a_1} \right) \eta(|\xi'(u)|, |\xi'(a_2)|) \int_0^1 h^2(y)dy.
$$

Proof. By using Lemma 2 and the definition of generalized $h$-stochastic convexity of $|\xi'|$, we get

$$
\left\lvert \xi(u) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \xi(v)dv \right\rvert \leq \frac{(u - a_1)^2}{a_2 - a_1} \int_0^1 y|\xi'(yu + (1 - y)a_1)|dy
$$

$$
+ \frac{(a_2 - u)^2}{a_2 - a_1} \int_0^1 y|\xi'(yu + (1 - y)a_2)|dy
$$

$$
\leq \frac{(u - a_1)^2}{a_2 - a_1} \int_0^1 y \left[ |\xi'(a_1)| + h(y)\eta\left(|\xi'(u)|, |\xi'(a_1)|\right) \right] dy
$$

$$
+ \frac{(a_2 - u)^2}{a_2 - a_1} \int_0^1 y \left[ |\xi'(a_2)| + h(y)\eta\left(|\xi'(u)|, |\xi'(a_2)|\right) \right] dy \leq M \frac{(u - a_1)^2}{a_2 - a_1} \int_0^1 h(y)dy
$$

$$
+ \frac{(a_2 - u)^2}{a_2 - a_1} \int_0^1 h(y)\eta\left(|\xi'(u)|, |\xi'(a_1)|\right) dy + M \frac{(a_2 - u)^2}{a_2 - a_1} \int_0^1 h(y)dy
$$

$$
+ \frac{(u - a_1)^2}{a_2 - a_1} \eta\left(|\xi'(u)|, |\xi'(a_1)|\right) \times \int_0^1 h^2(y)dy + \frac{(a_2 - u)^2}{a_2 - a_1} \eta\left(|\xi'(u)|, |\xi'(a_2)|\right) \int_0^1 h^2(y)dy
$$

$$
= M \left[ \frac{(u - a_1)^2 + (a_2 - u)^2}{a_2 - a_1} \right] \int_0^1 h(y)dy + \left( \frac{(u - a_1)^2}{a_2 - a_1} \right) \eta\left(|\xi'(u)|, |\xi'(a_1)|\right)
$$

$$
+ \left( \frac{(a_2 - u)^2}{a_2 - a_1} \right) \eta\left(|\xi'(u)|, |\xi'(a_2)|\right) \int_0^1 h^2(y)dy.
$$

8. Conclusion

Stochastic processes have many applications in statistics, which obviously lead to lots of other domains, for example, Kolmogorov–Smirnoff test on the equality of distributions [38–40] (the test statistic is derived from a Brownian bridge, which is a Brownian motion conditioned to have certain values at the endpoints of an interval of time). The other applications include sequential analysis [41, 42] (this is the rigorous way you can stop an A/B test dynamically. Stopping rules are obtained by approximating discrete problems with their continuous time analogs, in which the sufficient statistic is derived from a stochastic differential equation) and quickest detection [43, 44] (my stock story is that you are using a gold mine which gives you a random amount of gold per day, but at some point, it will get depleted, and you want to know when this happens as quickly as possible. Decision rules are again based on properties of hitting times of
random processes). In this paper, we have introduced generalized \(h\)-convex stochastic processes and proved Jensen, Hermite–Hadamard, and Fejer-type inequalities. Our results are applicable because applying the convex function to the expected value of a random variable is always bounded above by the expected value of the convex function of the random variable.

Data Availability

All data required for this research are included within this paper.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors’ Contributions

All authors contributed equally to this paper.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant nos. 11971142, 11871202, 61673169, 11701176, 11626101, and 11601485).

References


