Research Article

Global Robust Stabilization Control for Nonlinear Time-Delay Systems with Dead-Zone Input and Complex Dynamics

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This article investigates the robust stabilization problem for nonlinear time-delay systems with dead-zone input and complex dynamics. By flexibly using the inequality technique, the backstepping control method, and skillfully introducing a new Lyapunov–Krasovskii functional, we obtain a stable controller without using unmeasurable signals in the dynamic subsystem. The control system is guaranteed to be stable finally. Two simulation examples are given to verify the control strategy.

1. Introduction

Many practical models in engineering are nonlinear systems such as the flexible-joint robot [1], the wheeled inverted pendulum [2], and the autonomous underwater vehicle [3]. In the past few years, scholars have focused on studying nonlinear systems, such as [4–11]. For numerous nonlinear systems, the time-delay phenomenon which may lead to system instability often exists and is inevitable [12]. Nonlinear systems sometimes involve complex dynamics, in which the information of the states is not available. For more results on nonlinear systems with complex dynamics, we refer the reader to [13]. Besides the time delay, nonlinear conditions, and complex dynamics, specific control inputs such as the dead-zone input [14, 15] can also have a significant impact on the system. Considering the above facts, the nonlinear time-delay systems with dead-zone input and complex dynamics are investigated in this paper.

In recent years, some complicated linear systems have been studied. For example, Xu and Zhang [16] considered the stochastic large population system and presented a novel strategy for linear-quadratic games. However, different from linear systems, control problems of nonlinear systems are often difficult since they have more complicated dynamics. Specially, Guo [17] studied nonlinear chaotic systems and raised a physically implementable controller to solve the projective synchronization problem. In engineering practice, lots of nonlinear systems can be approximated by using linear systems at the origin; thus, the theory of linear systems can be applied. However, some systems may not be linearized at the origin or can be linearized but have uncontrollable Jacobian linearization [18, 19]. So, it is necessary to study nonlinear control design methods for those systems. Besides, time-delay problems cannot be ignored for the system control design, since ignoring it may make the system unstable. Many scholars have studied the associated control design for systems with time delay (for example, see the adaptive control problem [20], the stabilization problem [21], and the tracking problem [22]).

In recent years, the control design for systems with complex dynamics has been one of the interesting topics. Particularly, with the choice of a state observer, the adaptive control problem of systems containing complex dynamics was solved in [23]. By utilizing a new Lyapunov function, the adaptive tracking problem was studied in [24] for systems with input saturation and complex dynamics. On the other hand, time delay may bring negative effects to the stability of systems. Therefore, scholars have studied control problems for nonlinear systems with time delay. For systems involving time delay and complex dynamics, in [25], by the neural network method and the Lyapunov–Krasovskii functional, the tracking control design was studied. In [26], a modified
strategy of adding a power integrator was applied for stochastic delayed nonlinear systems with complex dynamics. Subsequently, this technique was further applied to systems with uncertainty in [27].

In practice, dead zone may exist in the actuator or the control input of the system. There have been some related reports mainly discussing the neural method and the fuzzy method. Specially, the neural control method [28] and the adaptive fuzzy control method [29] were applied to solve adaptive control problems of nonlinear systems. Recently, the Lyapunov–Krasovskii functional control approach has been used to study the nonlinear control problem of systems containing dead-zone input and implies that the approach is important for systems with time delay, see [30]. However, this method is not extended to solve the robust stabilization problem for systems involving time delay, dead-zone input, and complex dynamics. Also, few studies in the literature considered the robust control problem for the system.

The difficulty and the contribution of this paper are as follows:

(i) Considering that the system of this paper involves complicated dynamics, time delay, external disturbances, and input dead zone, the robust stabilization control problem of this paper is more challenging.

\[
\begin{aligned}
\dot{\eta} &= A\eta + \phi_0(x_1, x_1(t - \tau)), \\
\dot{x}_i &= d_i(t)x_{i+1} + \phi_i(\eta, \eta(t - \tau), x_i, x_i(t - \tau)), \quad i = 1, \ldots, n-1, \\
\dot{x}_n &= d_n(t)u(v) + \phi_n(\eta, \eta(t - \tau), x, x(t - \tau)),
\end{aligned}
\]

where \( \eta = [\eta_1, \ldots, \eta_m]^T \in \mathbb{R}^m \) is the unmeasurable dynamics and \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) is the state vector which is measurable. \( \eta = [\eta_1(t - \tau), \ldots, \eta_m(t - \tau)]^T, \eta(t - \tau) = [\eta_1(t - \tau), \ldots, \eta_m(t - \tau)]^T \) and \( \eta = [x_1(t - \tau), \ldots, x_n(t - \tau)]^T, \tau \) is the constant time delay. \( A \in \mathbb{R}^{m \times m} \) is the Hurwitz matrix that satisfies \( A^TP + PA \leq -Q \), where \( P \) and \( Q \) are positive definite symmetric matrices. The control coefficients \( d_i(t) \in \mathbb{R}, i = 1, \ldots, n \) satisfy \( d \leq d_i(t) \leq D \) with \( d \geq 0 \) and \( D \geq 0 \). \( \phi_k \in \mathbb{R}, k = 0, 1, \ldots, n \), are continuous functions. The dead-zone input \( u(v) \) is

\[
u(v) = \begin{cases} 
    m_r(t)(v - b_r(t)), & v > b_r(t), \\
    0, & -b_l(t) < v < b_r(t), \\
    m_l(t)(v + b_l(t)), & v < -b_l(t),
\end{cases}
\]

where \( m_r(t) > 0, m_l(t) > 0, b_r(t) > 0, \) and \( b_l(t) > 0 \) are time-varying functions.

To facilitate the problem simply, let \( \phi = (q/p) \geq 0 \), where \( q \) and \( p \) are the even integer and the odd integer, respectively. Therefore, define the constants \( \varepsilon_1 = 1 \) and \( \varepsilon_j = \varepsilon_{j-1} + \phi, j = 2, \ldots, n \). Next, we need the following assumptions.

Assumption 1. The nonlinear terms satisfy the following:

\[
\begin{aligned}
|\phi_0(x_1, x_1(t - \tau))| &\leq C_0|x_1| + |x_1(t - \tau)|, \\
|\phi_i(\eta, \eta(t - \tau), x_i, x_i(t - \tau))| &\leq C \left( \|\eta\|^{\varepsilon_i} + \|\eta(t - \tau)\|^{\varepsilon_i} + \sum_{j=1}^{i} \|x_j\|^{\varepsilon_i/\varepsilon_j} + \sum_{j=1}^{i} \|x_j(t - \tau)\|^{\varepsilon_i/\varepsilon_j} \right) + M_0(t), \quad i = 1, \ldots, n,
\end{aligned}
\]
where \( C_0 > 0 \) and \( C > 0 \) are constants and \( M_0(t) \) is a bounded disturbance.

**Assumption 2.** There are constants \( m_r > 0, m_0 > 0, \overline{b}_r > 0, \) and \( \overline{b}_0 > 0 \) satisfying

\[
m_r \leq m_i(t), \quad m_0 \leq m_i(t), \quad \overline{b}_r(t) \leq \overline{b}_r, \quad \text{and} \quad \overline{b}_0(t) \leq \overline{b}_0.
\]  

(4)

Next, Lemma 1 is provided for control design.

**Lemma 1** (see [30]). For given \( m > 0 \) and \( n > 0 \) and functions \( a(x, y), c(x, y) > 0 \) and \( f(x, y) \) satisfying

\[
|a(x, y)x^m y^n| \leq c(x, y)|x|^{m+n} + \frac{n}{m+n} a(x, y)^m |y|^n + \frac{m}{m+n} |a(x, y)|^{m+n}|y|^{m+n}. 
\]

(5)

### 3. Main Results

#### Theorem 1.

For system (1), suppose that Assumptions 1-2 hold. Then, under the following transformation

\[
\begin{align*}
\zeta_1 &= x_1, \\
\bar{\zeta}_2 &= x_i - \alpha_{i-1}, \\
\alpha_i &= -g_i \zeta_1^{2\epsilon_i/\rho_i}, & i = 2, \ldots, n,
\end{align*}
\]

there exists a robust controller:

\[
\begin{align*}
\alpha_n &= \frac{m_r}{m_r + \overline{b}_r}, \quad \alpha_n > 0, \\
\nu &= \left\{ \begin{array}{ll}
0, & \alpha_n = 0, \\
\frac{m_r}{m_r - \overline{b}_r}, & \alpha_n < 0,
\end{array} \right.
\end{align*}
\]

(6)

(7)

where \( \alpha_n = -g_i \zeta_1^{2\epsilon_i/\rho_i} \). The controller ensures that the considered system is globally stable.

**Proof.** We divide the proof into several parts.

#### 3.1. Part I: Robust Control Design. We construct the controller by using the modified backstepping technique.

Step 1: defining \( \sigma = \varepsilon_t \) and choosing \( U_0 = 1/\sigma (\eta^T P \eta)^{\sigma} \), we have

\[
\begin{align*}
\dot{U}_0 &= (\eta^T P \eta)^{-1} (\eta^T P (A\eta + \phi_0) + (A\eta + \phi_0)^T P \eta) \\
&= (\eta^T P \eta)^{-1} (\eta^T (PA + A^T P) \eta + 2\eta^T P \phi_0).
\end{align*}
\]

(8)

Noting that \( \lambda_p \|\eta\|^2 \leq \eta^T P \eta \leq \lambda_p \|\eta\|^2 \) and \( \eta^T(AP + A^T P)\eta \leq -\lambda_p \|\eta\|^2 \), where \( \lambda_p \) is the minimal eigenvalue of \( P \) and \( \lambda_p \) is the maximal eigenvalue of \( P \). \( \lambda_q \) is the minimal eigenvalue of \( Q \). Then, introduce \( \xi_1 = x_1 \) and define \( \lambda = \min\{\lambda_p, \lambda_q\} \). From Assumption 1 and Lemma 1, we get

\[
U_0 \leq -\lambda^q \|\eta\|^2 + 2(\eta^T P \eta)^{\sigma - 1} \eta^T P \phi_0 = -\lambda^q \|\eta\|^2 + \Delta,
\]

(9)

\[
\Delta = 2(\eta^T P \eta)^{\sigma - 1} \eta^T P \phi_0 \leq 2\lambda_p \|\eta\|^{2\sigma - 2} \cdot \|\eta\| \cdot \|P\| \cdot \|\phi_0\| \\
\leq \frac{2\lambda_p^{\sigma - 1} \|\eta\|^2}{\|P\| \|\eta\|^{2\sigma - 2}} \cdot C_{\zeta_1} \cdot \|\zeta_1(t - r)\| \\
\leq \frac{1}{n + 2} \lambda^q \|\eta\|^2 + \gamma_01(C_0, \lambda_p) \zeta_1^{2\sigma/\rho_1} + \gamma_02(C_0, \lambda_p) e^{-\mu \tau} \zeta_1^{2\sigma/\rho_1}(t - r),
\]

(10)

where \( \gamma_01(C_0, \lambda_p) \) and \( \gamma_02(C_0, \lambda_p) \) are the positive constants depended on \( C_0 \) and \( \lambda_p \). Substituting (10) into (9), it yields

\[
\dot{U}_0 \leq -n + 1 \lambda^q \|\eta\|^2 + \gamma_01(C_0, \lambda_p) \zeta_1^{2\sigma/\rho_1} + \gamma_02(C_0, \lambda_p) e^{-\mu \tau} \zeta_1^{2\sigma/\rho_1}(t - r).
\]

(11)

By Lemma 1 and Assumption 1, there are constants \( \gamma_{11} > 0 \) and \( M_1 > 0 \) such that

\[
|\chi_1^{2\sigma - \epsilon_i/\rho_i} \phi_1| \leq C|\chi_1^{2\sigma - \epsilon_i/\rho_i} (\|\eta\|^{\epsilon_i} + \|\eta\| (t - \epsilon_i)) + |\chi_1^{2\sigma - \epsilon_i/\rho_i} + |M_1| |\chi_1^{2\sigma - \epsilon_i/\rho_i} (t - \epsilon_i) + |\gamma_{11} \chi_1^{2\sigma/\rho_1} + \frac{1}{n + 2} \lambda^q \|\eta\|^2 \\
+ e^{-\mu \tau} \chi_1^{2\sigma/\rho_1}(t - \epsilon_i) + \frac{M_1^{2\sigma/\rho_1}}{M_1}.
\]

(12)

Introduce the transformation \( \zeta_2 = x_2 - \alpha_1 \), and choose \( V_1 = U_0 + W_1 + T_1 \), where \( U_1 = \eta/2(n + 2)\lambda^q \int_{t-r}^t e^{\mu(t-s)} \|\eta\|^{\epsilon_i} ds + W_1 = (\epsilon_i/2\alpha - \omega) \zeta_1^{2\sigma - \epsilon_i/\rho_i} \), and \( T_1 = (n + \gamma_02) \int_{t-r}^t e^{\mu(t-s)} \zeta_1^{2\sigma/\rho_1} ds. \mu > 0 \) is a constant. Then, it follows from (11) and (12) that
\[
\dot{V}_1 = U_0 + \zeta_1^{20-\epsilon_t} (d_1 x_2 + \phi_1) + (n + y_{01}) \zeta_1^{20-\epsilon_t} - (n + y_{02}) e^{-\mu \tau} \zeta_1^{20-\epsilon_t} (t - \tau) + \frac{n}{2(n + 2)} \lambda^\sigma \eta \| \eta \|^{2\sigma}
\]
\[
- \frac{n}{2(n + 2)} e^{-\mu \tau} \zeta_1^{20-\epsilon_t} (t - \tau) - \mu (U_1 + T_1)
\]
\[
\leq - \frac{n}{2(n + 2)} \lambda^\sigma \eta \| \eta \|^{2\sigma} + d_1 \zeta_2 \zeta_1^{20-\epsilon_t} + d_1 \zeta_2 \zeta_1^{20-\epsilon_t} + (n + y_{01} + y_{02} + y_{11}) \zeta_1^{20-\epsilon_t} + M_0 \zeta_1^{20-\epsilon_t} (t - \tau) - \frac{n - 1}{2(n + 2)} e^{-\mu \tau} \zeta_1^{20-\epsilon_t} (t - \tau) - \mu (U_1 + T_1). \tag{13}
\]

Selecting the virtual control \(\alpha_t = -1/(2n + y_{01} + y_{02} + y_{11}) \zeta_1^{(20-\epsilon_t)}\), and substituting it into (13), it yields that
\[
V_1 \leq - \frac{n}{2(n + 2)} \lambda^\sigma \eta \| \eta \|^{2\sigma} - M_1 \zeta_1^{20-\epsilon_t} + d_1 \zeta_2 \zeta_1^{20-\epsilon_t} + M_0 \zeta_1^{20-\epsilon_t} \left( \frac{\lambda^\sigma}{\mu} \right) \eta \| \eta \|^{2\sigma} - \mu (U_1 + T_1) - (n - 1) e^{-\mu \tau} \zeta_1^{20-\epsilon_t} (t - \tau) \\
- \frac{n - 1}{2(n + 2)} e^{-\mu \tau} \zeta_1^{20-\epsilon_t} (t - \tau). \tag{14}
\]

Step \(k (k = 2, \ldots, n - 1)\): for step \(k - 1\), suppose that there exist transformations (6) and a Lyapunov–Krasovskii functional \(V_{k-1}\) such that there exist the following inequality:
\[
\dot{V}_{k-1} \leq - \frac{n - k + 2}{2(n + 2)} \lambda^\sigma \eta \| \eta \|^{2\sigma} - (n - k + 2) \sum_{j=1}^{k-1} \zeta_j^{20-\epsilon_t} + d_{k-1} \zeta_k \zeta_{k-1}^{20-\epsilon_t} + \sum_{j=1}^{k-1} \frac{M_0 \zeta_{j+1}}{M_j} \left( \frac{\lambda^\sigma}{\mu} \right) \eta \| \eta \|^{2\sigma} - \mu (U_1 + \sum_{j=1}^{k-1} T_j) \\
- (n - k + 1) e^{-\mu \tau} \sum_{j=1}^{k-1} \zeta_j^{20-\epsilon_t} (t - \tau) \\
- \frac{n - k + 1}{2(n + 2)} e^{-\mu \tau} \zeta_k \zeta_{k-1}^{20-\epsilon_t} + \sum_{j=1}^{k-1} \frac{M_0 \zeta_{j+1}}{M_j} \left( \frac{\lambda^\sigma}{\mu} \right) \eta \| \eta \|^{2\sigma} + d_{k-1} \zeta_k \zeta_{k-1}^{20-\epsilon_t} + d_k \zeta_k \zeta_{k-1}^{20-\epsilon_t} + d_k \zeta_k \zeta_{k-1}^{20-\epsilon_t}. \tag{17}
\]

On the basis of \(d_{k-1} \leq \overline{d}\) and Lemma 1, there is a constant \(y_{11} > 0\) such that
\[
d_{k-1} \zeta_k \zeta_{k-1}^{20-\epsilon_t} \leq \frac{1}{3} y_{k-1} \zeta_k^{20-\epsilon_t} + y_{k-1} \zeta_k^{20-\epsilon_t}. \tag{18}
\]

Similarly to the proof of (12), one obtains
where $\gamma_k > 0$ and $M_k > 0$ are constants. From Assumption 1, we have

$$
|d_{j}x_{j+1} + \phi| \leq d\left(\xi_{j+1} + g_j\xi_j\right)^{\varepsilon_{j+1}/\varepsilon_j} + C\left(\|\eta\|^{\varepsilon_j} + \|\eta(t-\tau)\|^{\varepsilon_j} + \sum_{i=1}^{j} |\xi_i - g_{i-1}\xi_{i-1}|^{\varepsilon_i/\varepsilon_j} + \sum_{i=1}^{j} |\xi_j(t-\tau) - g_{i-1}\xi_{i-1}(t-\tau)|^{\varepsilon_i/\varepsilon_j}\right)
$$

Utilizing (20) and Lemma 1 and noting that $\|d\xi_k / dx_j\| = g_j g_{j+1} \cdots g_k (\varepsilon_k / \varepsilon_j) [\xi_j]^{\varepsilon_j/\varepsilon_i} \cdots [\xi_{i-1}]^{\varepsilon_{i-1}/\varepsilon_i}$, it follows that

$$
\frac{d\xi_k}{dx_j} |d_{j}x_{j+1} + \phi| \leq \left(d\left(\xi_{j+1} + g_j\xi_j\right)^{\varepsilon_{j+1}/\varepsilon_j} + C\left(\|\eta\|^{\varepsilon_j} + \|\eta(t-\tau)\|^{\varepsilon_j} + \sum_{i=1}^{j} |\xi_i - g_{i-1}\xi_{i-1}|^{\varepsilon_i/\varepsilon_j} + |M_0|\right)g_j g_{j+1} \cdots g_k \varepsilon_k / \varepsilon_j [\xi_j]^{\varepsilon_j/\varepsilon_i} \cdots [\xi_{i-1}]^{\varepsilon_{i-1}/\varepsilon_i}\right)
$$

$$
\leq C_{j-1} \sum_{i=1}^{j} |\xi_i|^{\varepsilon_i/\varepsilon_j} + C_{j-1} \sum_{i=1}^{j} |\xi_i(t-\tau)|^{\varepsilon_i/\varepsilon_j} + C_{j-1} \|\eta\|^{\varepsilon_j} + C_{j-1} \|\eta(t-\tau)\|^{\varepsilon_j} + |M_0| g_j g_{j+1} \cdots g_k \varepsilon_k / \varepsilon_j [\xi_j]^{\varepsilon_j/\varepsilon_i} \cdots [\xi_{i-1}]^{\varepsilon_{i-1}/\varepsilon_i},
$$

(21)
where \( C_{jk} > 0, k = 1, \ldots, 4 \) are constants. It can be deduced from (21) and Lemma 1 that

\[
-\varepsilon^{2\sigma-\varepsilon_n, \varepsilon_n} \cdot \left( d_j x_{j+1} + \phi_j \right) \leq \left( \gamma_k + \gamma_{k+1} \right) \varepsilon^{2\sigma} + \frac{1}{2(n+2)} \lambda^2 \| \eta \|^2 + \frac{1}{2(n+2)} e^{-\mu t} \lambda^2 \| \eta(t-t') \|^2 \\
+ \frac{1}{2} e^{-\mu t} \sum_{j=1}^{k} \zeta^{2\sigma} j (t-t') + \frac{1}{3} \sum_{j=1}^{k-1} \zeta^{2\sigma} j,
\]

(22)

where \( \gamma_k > 0 \) and \( M_k > 0 \) are constants. Choosing \( M_k \) such that \( 1/M_k = (1/M_{k1}) + (1/M_{k2}) \) and using (19) and (22), it yields that

\[
\varepsilon^{2\sigma-\varepsilon_n, \varepsilon_n} \left( \phi_k - \sum_{j=1}^{k-1} \frac{\partial \phi_k}{\partial x_j} (d_j x_{j+1} + \phi_j) \right) \leq \left( \gamma_k + \gamma_{k+1} \right) \varepsilon^{2\sigma} + \frac{1}{2(n+2)} \lambda^2 \| \eta \|^2 + \frac{1}{2(n+2)} e^{-\mu t} \lambda^2 \| \eta(t-t') \|^2 \\
+ e^{-\mu t} \sum_{j=1}^{k} \zeta^{2\sigma} j (t-t') + \frac{2}{3} \sum_{j=1}^{k-1} \zeta^{2\sigma} j.
\]

(23)

Now, we construct the virtual control \( \alpha_k = -1/(2n-2k+2+\gamma_k + \gamma_{k+1} + \gamma_k) \varepsilon^{2\sigma} \phi_k - \varepsilon^{2\sigma} \varepsilon_n \). With the help of (17), (18), and (23), it is deduced that

\[
\dot{V}_k \leq -\frac{n-k+1}{2(n+2)} \lambda^2 \| \eta \|^2 - (n-k+1) \sum_{j=1}^{k} \zeta_j^{2\sigma} + \frac{1}{2} \sum_{j=1}^{k} \zeta_j^{2\sigma} (t-t') - \mu \left( U_k + \sum_{j=1}^{k} T_j \right)
\]

(24)

This completes the control design for step \( k = 2, \ldots, n-1 \).

Step \( n \): in this step, we choose \( V_n = V_{n-1} + W_n \)

\[
T_n = \int_{t-T}^{t} e^{\mu(t-t')} \varepsilon^{2\sigma} \varepsilon_n (s) ds,
\]

and using (24), some simple computations lead to

\[
\dot{V}_n \leq -\frac{2}{2(n+2)} \lambda^2 \| \eta \|^2 - 2 \sum_{j=1}^{n} \zeta_j^{2\sigma} + \frac{1}{2} \sum_{j=1}^{n} \zeta_j^{2\sigma} (t-t') - \mu \left( U_k + \sum_{j=1}^{n} T_j \right)
\]

(25)
Similar to (18) and (23), by Lemma 1, we obtain that

\[
d_{n-1}^\alpha \frac{d^{2\alpha - \epsilon_n/\epsilon_{n-1}}}{ds^{2\alpha - \epsilon_n/\epsilon_{n-1}}}(\phi - \frac{\sum_{j=1}^n \partial \alpha_n}{\partial x_j}(dx_{j+1} + \phi_j)) \leq \frac{1}{3} \frac{d^{2\alpha - \epsilon_n/\epsilon_{n-1}}}{ds^{2\alpha - \epsilon_n/\epsilon_{n-1}}} + \gamma_{n1} \frac{d^{2\alpha - \epsilon_n/\epsilon_{n-1}}}{ds^{2\alpha - \epsilon_n/\epsilon_{n-1}}},
\]

(26)

where the constants \( \gamma_{n1} \geq 0, \gamma_{n2} \geq 0, \gamma_{n3} \geq 0, \) and \( M_n \geq 0. \)

Now, we choose the virtual control \( \alpha_n = -d/(2 + \gamma_{n1} + \gamma_{n2} + \gamma_{n3})\frac{\lambda_n}{\sigma_{n1}^{2\alpha - \epsilon_n/\epsilon_{n-1}}} \) defined by \( g_{a_n}^n \), which, and (25)–(27), give that

\[
V_n \leq -\frac{1}{2(n + 2)} \frac{\lambda}{m} \| \eta \|^2 - \sum_{j=1}^n \lambda_j \frac{\lambda}{m} \| \eta \|^2 - \mu \left( U_1 + \sum_{j=1}^n T_j \right) + \sum_{j=1}^n \frac{M_0^{2\alpha - \epsilon_n/\epsilon_{n-1}}}{M_j} + d_n (u - \alpha_n) \frac{d^{2\alpha - \epsilon_n/\epsilon_{n-1}}}{ds^{2\alpha - \epsilon_n/\epsilon_{n-1}}}
\]

(28)

Finally, we choose the control input \( v \) as (7). Then, by using (2), it follows that

\[
u (v) = \begin{cases} m_n \frac{\alpha_n}{m} + \tilde{b}_n - b_i, & \alpha_n > 0, \\ m_n \frac{\alpha_n}{m} + \tilde{b}_n - b_i, & \alpha_n = 0, \\ m_n \frac{\alpha_n}{m} + \tilde{b}_n - b_i - \alpha_n, & \alpha_n < 0, \end{cases}
\]

(29)

which further renders that \( d_n (u - \alpha_n) \frac{d^{2\alpha - \epsilon_n/\epsilon_{n-1}}}{ds^{2\alpha - \epsilon_n/\epsilon_{n-1}}} \leq 0. \)

Then, using (28), we get

\[
V_n \leq -\frac{1}{2(n + 2)} \frac{\lambda}{m} \| \eta \|^2 - \sum_{j=1}^n \lambda_j \frac{\lambda}{m} \| \eta \|^2 - \mu \left( U_1 + \sum_{j=1}^n T_j \right) + \sum_{j=1}^n \frac{M_0^{2\alpha - \epsilon_n/\epsilon_{n-1}}}{M_j}
\]

(30)

3.2. Part II: Stability Analysis. Since \( U_0 = 1/(\sigma \| \eta \|^2 \| P \| \eta \| \leq \sigma \| \eta \|^2 \| \eta \|^2 \), we have \( \| \eta \|^2 \geq (\sigma \| \lambda \| P \| \eta \| \leq \gamma_{n1} \| \eta \|^2 \), which leads to

\[
\frac{1}{2(n + 2)} \frac{\lambda}{m} \| \eta \|^2 \leq -\frac{\sigma}{2(n + 2)} \lambda \| \eta \|^2 \leq 0.
\]

(31)

Defining \( \Delta_j = (2\sigma - \omega/2\sigma)(\delta \omega/2\sigma)^{2\alpha - \epsilon_n/\epsilon_{n-1}} \frac{\lambda}{m} \), it follows from Lemma 1 that \( W_j = (\epsilon/2\sigma - \omega) \lambda_j \leq (1/\delta_j) + \Delta_j \lambda_j \), which renders that

\[
-\frac{\sigma}{2(n + 2)} \lambda \| \eta \|^2 \leq -\Delta m \sum_{j=1}^n W_j + \frac{1}{\delta_j} \lambda \| \eta \|^2
\]

(32)

Substituting (31) and (32) into (28), it yields that

\[
V_n \leq -\frac{\sigma}{2(n + 2)} \lambda \| \eta \|^2 U_0 - \Delta m \sum_{j=1}^n W_j + \frac{1}{\delta_j} \lambda \| \eta \|^2
\]

(33)

where \( \rho_1 = \min[(\sigma/2(n + 2)\lambda \| \eta \|^2 \| \eta \|^2)] \) and \( \rho_2 = \sum_{j=1}^n (1/\delta_j) \lambda_j \). By the definition of \( U_0, U_1, W_i, \) and \( T_i, \) it follows that

\[
U_0 + U_1 = \frac{1}{\sigma} (\| \eta \|^2 \| P \| \eta \| \leq \frac{n}{n + 2}) \int_{t \in} e^{\mu (1-i)} \| \eta \| \| \eta \|^2 dt
\]

\[
\leq \frac{\lambda}{m} \| \eta \|^2 + \frac{n}{n + 2} \sup_{t \in} \| \eta (k + t) \|^2
\]

\[
\leq \left( \frac{\lambda}{m} + \frac{n}{n + 2} \right) \sup_{t \in} \| \eta (k + t) \|^2
\]

(34)

\[
\sum_{i=1}^n \frac{\epsilon_i}{\sigma - \omega} \lambda_i \leq \epsilon_n \sum_{i=1}^n \sup_{t \in} \lambda_i \leq (\sigma \| \lambda \| P \| \eta \| \leq \gamma_{n1} \| \eta \|^2 \leq \sum_{i=1}^n \sup_{t \in} \lambda_i \leq \sum_{i=1}^n \sup_{t \in} \lambda_i \leq (n - i + 1)
\]

\[
\sum_{i=1}^n \frac{\epsilon_i}{\sigma - \omega} \lambda_i \leq (n - i + 1)
\]

(35)

which lead to

\[
V_n \leq U_0 + U_1 + \frac{\sum_{i=1}^n W_i + \sum_{i=1}^n T_i \leq \varphi_1 \left( \sup_{t \in} \| \eta (k + t) \|^2 \right)
\]

(35)
Figure 1: The trajectory of $\eta$.

Figure 2: The trajectory of $x_1$.

Figure 3: The trajectory of $x_2$.

Figure 4: The trajectory of $v$.

Figure 5: The trajectory of $\eta$.

Figure 6: The trajectory of $x_1$.

Figure 7: The trajectory of $x_2$.

Figure 8: The trajectory of $x_3$. 
where the function \( \varphi_1 \in K_\infty \) and \( \Xi(t) = [\eta(t)^T, \zeta(t)^T]^T \). On the other hand, there exists a function \( \varphi_2 \in K_\infty \) such that
\[
V_n = U_0 + U_1 + \sum_{i=1}^{n} W_i + \sum_{i=1}^{n} T_i \geq \frac{\lambda}{2} \eta^2 + \frac{\lambda}{2} \zeta^2 \geq \varphi_2(\Xi(t)).
\]

(36)

In view of (28), (35), and (36), and using Lemma 3.2.4 in [33], we obtain
\[
\dot{V}_n \leq \left( V_n(0) - \frac{\rho_3}{\rho_1} \right) e^{-\rho_1 t} + \frac{\rho_2}{\rho_1} \eta^2 + \frac{\rho_2}{\rho_1} \zeta^2.
\]

(37)

which indicates that \( \eta \) and \( \zeta \) are bounded. From \( \xi_1 = x_1, \xi_i = x_i - \alpha_{e,i}, \alpha_1 = -\theta[s_i^{1/2}], \) and \( \alpha_i = -\theta[s_i^{1/2}], i = 2, \ldots, n, \) the states of \( \eta, x_1, \ldots, x_n \) are bounded, which indicates the system composed of (1) and (7) is globally stable.

### 4. Simulation Example

**Example 1.** Consider the following nonlinear time-delay system:
\[
\begin{align*}
\dot{\eta} &= -\eta + 0.5x_1, \\
\dot{x}_1 &= x_2 + x_1^{5/3}(t - 0.2), \\
\dot{x}_2 &= u(\nu) + x_2^{7/5} + 0.3 \sin(t),
\end{align*}
\]

(38)

where \( \eta \) is the unmeasurable state, \( x_1 \) and \( x_2 \) are the measurable states, \( u(\nu) \) is the dead-zone input given in (2) with \( b_r = 0.4 + 0.1 \sin(t), b_l = 0.4 - 0.1 \sin(t), \) and \( \nu \) is the input of the system. We see that Assumptions 1-2 are satisfied for system (32) with \( \omega = 2/3, \epsilon_1 = 1, \epsilon_2 = 5/3, \epsilon_3 = 7/3, \) and \( \bar{b}_r = \bar{b}_l = 0.5, \) and \( m_r = m_l = 1. \) Applying the above control method, we choose \( \alpha_2 = -15(x_2 + 9x_1^{5/3})^{7/5}. \) Then, the actual controller is constructed as
\[
\nu = \begin{cases} 
\alpha_1 + \bar{b}_r, & \alpha_1 > 0, \\
0, & \alpha_1 = 0, \\
\alpha_1 - \bar{b}_l, & \alpha_1 < 0.
\end{cases}
\]

(39)

In the simulation, choose \( \eta(0) = 2, x_1(0) = 0.5, \) and \( x_2(0) = -2. \) Figures 1–4 give the trajectories of \( \eta, x_1, \) and \( x_2 \) and the control input \( \nu. \) All signals in systems (38) and (39) are bounded. Hence, the validity of the presented control method is verified.

**Example 2.** Consider the following system:
\[
\begin{align*}
\dot{\eta} &= -2\eta + \frac{1}{1 + \eta^2} \eta x_1, \\
\dot{x}_1 &= x_2 + x_1^{1/3} x_1^{5/3}(t - 0.5), \\
\dot{x}_2 &= x_3 - 5 \sin(x_1) - x_2 + \sin(x_1(t - 0.5)), \\
\dot{x}_3 &= u(\nu) + 10x_2 - 5x_3,
\end{align*}
\]

(40)

where \( \eta \) is the unmeasurable state, \( x_1, x_2, x_3 \) are the system states, \( \nu \) is the control input, and \( u(\nu) \) is the dead-zone input defined in (2) with \( b_r = 0.5 + \sin(t), b_l = 0.5 - \sin(t). \) It can be deduced that \( |x_1|, |x_1^{1/3}|, |x_1^{5/3}| \) are bounded. Using the control design method in Section 3, we can design the actual controller as follows:
\[
\nu = \begin{cases} 
\alpha_4 + \bar{b}_r, & \alpha_4 > 0, \\
0, & \alpha_4 = 0, \\
\alpha_4 - \bar{b}_l, & \alpha_4 < 0.
\end{cases}
\]

(41)

where \( \alpha_4 = -15(x_3 + 10(x_2 + 6x_1)). \)

In the simulation, the initial conditions are selected as \( \eta(0) = 1, x_1(0) = 0.5, x_2(0) = -0.5, \) and \( x_3(0) = 0. \) Figures 5–9 show the trajectories of \( \eta, x_1, x_2, x_3, \) and the control input \( \nu. \) It can be seen that all signals in systems (34) and (35) are bounded. Hence, the presented control method is effective.

### 5. Conclusions

The robust stabilization for nonlinear systems with dead-zone input and time delay has been studied. Because the system involves the dead-zone input, time-delay, disturbance, and unmeasurable states, the stabilization control in this work is more challenging. A robust stable controller has been designed via the Lyapunov–Krasovskii functional and the backstepping technique. Another interesting problem is as follows: When the considered system includes uncertainty parameters, and only the system output is measurable, how can we design the adaptive controller via the output feedback control method?

**Data Availability**

The data used to support the findings are included within this article.

**Conflicts of Interest**

The authors declare no conflicts of interest in preparing this article.
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