Research Article

On Zero Left Prime Factorizations for Matrices over Unique Factorization Domains

Jinwang Liu, Tao Wu, Dongmei Li, and Jiancheng Guan

School of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan, Hunan 410081, China

Correspondence should be addressed to Jiancheng Guan; jiancheng_guan@aliyun.com

Received 17 February 2020; Accepted 26 March 2020; Published 22 April 2020

1. Introduction

Multidimensional linear systems theory has a wide range of applications in circuits, systems, control of networked systems, signal processing, and other areas (see, e.g., [1, 2]). Multivariate polynomial matrix theory is a well-established tool for these systems, since many problems in the analysis and synthesis of control systems can be well solved using multivariate polynomial matrix techniques [1–3].

In recent years, n-D polynomial matrix factorizations have been widely studied [4–10]. In [11, 12], the zero left prime factorization problem was raised. This problem has been solved in [4–6]. The minor left prime factorization problem has been solved in [7, 10]. In the algorithms given in [7, 10], a fitting ideal of some module over the multivariate (n-D) polynomial ring needs to be computed. It is a little complicated.

It is well known that a multivariate polynomial ring over a field is a unique factorization domain. Then, the following problem is interesting.

Problem 1. How to decide if a matrix with full row rank over a unique factorization domain has a zero left prime factorization?

In this paper, we will give a partial solution to this problem.

2. Preliminaries

Let $R$ be a unique factorization domain. The set of all $l \times m$ matrices with entries from $R$ is denoted by $R^{l \times m}$. Let $F \in R^{l \times m} (l < m)$. We denote the greatest common divisor of all $l \times l$ minors of $F$ by $d(F)$. Let $C \in R^{l \times l}$ be a submatrix of $F$. By deleting $C$ from $F$, we get a submatrix of $F$. This submatrix is denoted by $F \setminus C$.

Let $C \in R^{m \times m}$. $\text{adj}(C)$ denotes the adjoint matrix of $C$. $\text{acof}_{i,j}(C)$ denotes the $i,j$th algebraic cofactor of $C$.

Definition 1. Let $F \in R^{l \times m} (l < m)$, and let $C \in R^{l \times l}$ be a submatrix of $F$. A minor of $F$ consisting of $l - 1$ columns from $C$ and one column from $F \setminus C$ is said to be a related minor of $C$.

Definition 2. Let $F \in R^{l \times m}$ be of full row rank. Then, $F$ is said to be zero left prime (ZLP) if the $l \times l$ minors of $F$ generate the unit ideal $R$. Suppose $F$ has a factorization $F = CF_1$, where $C \in R^{l \times l}$ and $F_1 \in R^{l \times m}$. If $F_1$ is ZLP, then this factorization is said to be a zero left prime factorization.

3. Main Results

First, we need a lemma.

Lemma 1. Let $F = (C, \overline{C}) \in R^{l \times m} (l < m)$, where $C \in R^{l \times l}$ and $\overline{C} \in R^{l \times (m-l)}$. Then, the elements of $\text{adj}C \cdot \overline{C}$ are just all related minors of $C$ (up to a sign).
Corollary 2. Let \( C = (c_{ij})_{l \times l} \) and \( \bar{C} = (\bar{c}_{ij})_{l \times (m-l)} \). Let \( \text{adj} C \cdot \bar{C} = (b_{ij})_{l \times (m-l)} \). Then, 
\[
b_{ij} = \text{acof}_{ii}(C) \bar{c}_{ij} + \ldots + \text{acof}_{ii}(C) \bar{c}_{ij}
\]
\[
= \det \begin{pmatrix}
  c_{i1} & \cdots & c_{i-1} & c_{i+1} & \cdots & c_{il} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  c_{i1} & \cdots & c_{i-1} & c_{i+1} & \cdots & c_{il}
\end{pmatrix},
\]
(1)
by Laplace Theorem. Thus, \( b_{ij} \) is a related minor of \( C \) (up to a sign). It is clear that they are just all related minors of \( C \) (up to a sign).

Now, we prove the main theorem of this paper. □

**Theorem 1.** Let \( F \in \mathbb{R}^{lm} (l < m) \). If there exists an \( l \times l \) submatrix \( C \) of \( F \) such that \( \text{det} C \) is a common factor of all related minors of \( C \), then there exists \( F_1 \in \mathbb{R}^{lm} \) such that \( F = CF_1 \) and \( F_1 \) is ZLP; i.e., \( F \) has a ZLP factorization. Proof.

Proof. We can change the order of the columns of \( F \) such that the submatrix \( C \) consists of the left \( l \) columns of \( F \). Thus, there exists an invertible matrix \( Q \in \mathbb{R}^{m \times m} \) such that \( FQ = (C, \bar{C}) \), where \( C \in \mathbb{R}^{l \times l} \) and \( \bar{C} \in \mathbb{R}^{l \times (m-l)} \). Since \( \text{det} C \) is a common factor of all related minors of \( C \), by Lemma 1, we have \( C^{-1} \bar{C} = \text{adj} C \cdot \bar{C} / \text{det} C \in \mathbb{R}^{l \times (m-l)} \). Let 
\[
Q_1 = \begin{pmatrix}
  I_l & -C^{-1} \bar{C} \\
  0 & I_{m-l}
\end{pmatrix}.
\]
(2)
Then, \( Q_1 \in \mathbb{R}^{m \times m} \). We have 
\[
FQQ_1 = (C, \bar{C})Q_1 = (C, \bar{C}) \begin{pmatrix}
  I_l & -C^{-1} \bar{C} \\
  0 & I_{m-l}
\end{pmatrix} = (C, C) = (C, O).
\]

Then, 
\[
F = (C, O)Q_1^{-1}Q_1^{-1} = (C, O), \quad \text{(by (3))}
\]
\[
= C(I_l, O)Q_1^{-1}Q_1^{-1}.
\]
(4)
Let \( F_1 = (I_l, O)Q_1^{-1}Q_1^{-1} \in \mathbb{R}^{l \times m} \). Then, \( F = CF_1 \). Since \( F_1 \) consists of the upper \( l \) rows of invertible matrix \( Q_1^{-1}Q_1^{-1} \), we have \( F_1 \) is ZLP. □

**Corollary 1.** Let \( F \in \mathbb{R}^{lm} (l < m) \). If there exists an \( l \times l \) submatrix \( C \) of \( F \) such that \( \text{det} C \) is a common factor of all related minors of \( C \), then \( \text{det} C = d(F) \).

Proof. Clearly, \( d(F) \mid \text{det} C \). By Theorem 1, there exists \( F_1 \in \mathbb{R}^{lm} \) such that \( F = CF_1 \). By Cauchy–Binet formula, we have \( \text{det} C \mid d(F) \). Therefore, \( \text{det} C = d(F) \). □

**Corollary 2.** Let \( F \in \mathbb{R}^{lm} (l < m) \). If there exists an \( l \times l \) submatrix \( C \) of \( F \) such that \( \text{det} C \) is a common factor of all related minors of \( C \), then \( F \) is equivalent to \( (C, O) \).

Proof. By Theorem 1, there exists \( F_1 \in \mathbb{R}^{lm} \) such that \( F = CF_1 \) and \( F_1 \) is ZLP. By Quillen–Suslin theorem, there exists \( F_2 \in \mathbb{R}^{(m-l) \times m} \) such that \( (F^T, F^T)^T \) is an invertible matrix. Since \( F = CF_1 = (C, O)(F_1^T, F_1^T)^T \), we have \( F \) being equivalent to \( (C, O) \).

Now, let \( F \in \mathbb{R}^{lm} (l < m) \). Suppose there exists an \( l \times l \) submatrix \( C \) of \( F \) such that \( \text{det} C = d(F) \). We can give an algorithm to directly compute the ZLP factorization of \( F \). □

**Algorithm 1**

(i) Compute all \( l \times l \) minors of \( F \) and \( d(F) \).
(ii) Find an \( l \times l \) submatrix \( C \) of \( F \) such that \( \text{det} C = d(F) \).
(iii) Compute invertible matrix \( Q \) such that 
\[
FQ = (C, \bar{C}).
\]
(iv) Let 
\[
Q_1 = \begin{pmatrix}
  I_l & -C^{-1} \bar{C} \\
  0 & I_{m-l}
\end{pmatrix}
\]
and \( F_1 = (I_l, O)Q_1^{-1}Q_1^{-1} \).
Then, \( F = CF_1 \).

Now, we give an example to illustrate this algorithm.

**Example 1.** Let \( R = \mathbb{Z}[x, y] \), and let 
\[
F = \begin{pmatrix}
  6x^2y + 2xy & 2x & 2xy \\
  6x^2y + 6x^2y + 2xy^2 + 5xy & 2xy + 2x & 2xy^2 + 2xy + y
\end{pmatrix}.
\]
(5)
Then, \( d(F) = 2xy \). Let 
\[
C = \begin{pmatrix}
  2x & 2xy \\
  2xy + 2x & 2xy^2 + 2xy + y
\end{pmatrix}.
\]
(6)
Then, \( C \) is a \( 2 \times 2 \) submatrix of \( F \) and \( \text{det} C = d(F) \). Let 
\[
Q = \begin{pmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix}.
\]
(7)
Then, \( FQ = (C, \bar{C}) \), where 
\[
\bar{C} = \begin{pmatrix}
  6x^2y + 2xy \\
  6x^2y + 6x^2y + 2xy^2 + 5xy
\end{pmatrix}.
\]
(8)
Thus, 
\[
-C^{-1} \bar{C} = \begin{pmatrix}
  -y \\
  -3x
\end{pmatrix}.
\]
(9)
Let 
\[
Q_1 = \begin{pmatrix}
  y & 1 & 0 \\
  3x & 0 & 1 \\
  1 & 0 & 0
\end{pmatrix}.
\]
(10)
Then, 
\[
Q_1^{-1}Q_1^{-1} = \begin{pmatrix}
  y & 1 & 0 \\
  3x & 0 & 1
\end{pmatrix}.
\]
(11)
Then, \( F = CF_1 \).
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments
This paper was supported by the National Science Foundation of China (11971161 and 11871207).

References