Research Article

Expressions for the Autocorrelation Function and Power Spectral Density of BOC Modulation Based on Convolution Operation

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We present universal expressions for the autocorrelation functions (ACFs) of the Binary Offset Carrier (BOC), Multiplexed BOC (MBOC), and Alternative BOC (AltBOC) modulations based on convolution operations. We also derive the expressions for the power spectrum densities (PSDs) of these modulations using the Fourier transform of their ACFs. The results obtained in this contribution are useful for Global Navigation Satellite System (GNSS) signal simulation, performance evaluation, and high-performance acquisition and tracking algorithm design. The derivation methods of the expressions for the ACFs are common and can be used to derive expressions for the ACFs of other BOC-based modulations.

1. Introduction

The initial modulation scheme for the Global Navigation Satellite System (GNSS) signal is binary phase shift keying (BPSK) modulation, which enjoys considerable success as the first-generation signal scheme in the Global Positioning System (GPS). With the development of GNSS applications, there are continuing expectations for improved accuracy. BPSK modulation with a faster-spreading code rate improves accuracy but also requires larger bandwidth and has limited improvement in multipath performance [1]. To better satisfy increasing application demands, Binary Offset Carrier (BOC) modulation was proposed because of its higher accuracy, better spectral isolation from heritage signals, and better multipath interference resistance and flexibility in implementation compared with BPSK modulation [2]. The BOC modulation has been adopted in the modernized GPS [3], European Galileo System [4], and BeiDou Navigation System [5]. The GPS L1M and L2M signals adopt sine-phased BOC BOC\text{sin}(10,5), while the Galileo E6 Public Regulated Service (PRS) signal uses cosine-phased BOC BOC\text{cos}(10,5) and E1 PRS uses BOC\text{cos}(15,2.5). Some new BOC-based modulations have also been adopted by GNSS signals. A Time Multiplexed BOC (TMBOC) modulation, TMBOC(6,1,1/11), was selected for the GPS L1C signal, and a Composite BOC (CBOC), CBOC(6,1,1/11), was selected for the Galileo E1 Open Service (OS) signal [6]. An Alternative BOC (AltBOC) modulation, AltBOC(15,10), was used to transmit Galileo E5a and E5b signals [7].

The BOC modulation uses a square-wave subcarrier to create separated spectra on each side of the transmitted spectrum, which provides spectral isolation from heritage signals and leads to significant improvements in terms of tracking, interference, and multipath mitigation. Fundamentally, these excellent characteristics of the BOC modulation are determined by its autocorrelation function (ACF) and power spectrum density (PSD). Therefore, investigations into the ACF and PSD properties of the BOC modulations are important. Knowing the analytical expressions for their ACFs, it is possible, in principle, to quantitatively calculate the potential code tracking accuracy and estimate the signal resolution under multipath propagation and interference conditions. For example, the analytic expressions for the ACFs of the BOC modulations are useful for GNSS signal simulation [8, 9] and performance evaluation [10, 11]. They can also enable designers to develop near-optimal receiver discriminators, which would ensure...
unambiguous tracking of the main peak of the ACFs wherever possible and minimize the probability of capture of their false peaks [12–16]. The deduced explicit formulas for the ACFs of the BOC modulations play an important role in GNSS signal research. However, there are currently no universal expressions for the ACFs of the BOC and BOC-based modulations. The expressions for the ACFs of BOCsin(pn,n), BOCos(pn,n), MBOC(6, 1, 1/11), and constant and nonconstant envelope AltBOC(15, 10) are presented in [8, 16–18]. Those expressions available for the ACFs are typically given for particular cases. There is continuing expectation for universal expressions for the ACFs of these modulations. The PSDs of the BOC modulations determine the bandwidth consumption and the spectral separation with legacy signals sharing the same frequency band, which can be used to evaluate the tracking and demodulation performance of the signal in thermal noise and interference environments. The expressions for the PSDs of the BOC and BOC-based signals have been presented in [19].

In this paper, we present universal expressions for the ACFs of the BOC, MBOC, and AltBOC modulations based on convolution operations. In [8], Sousa and Nunes derived the expressions for the ACFs of BOCsin(pn,n) and BOCos(pn,n) by using the characteristic of piecewise linearity of the ACFs. We utilize the conversion relationship between the convolution operation and the correlation function calculation, combined with the characteristics of randomness and symmetry of these modulations, to derive the expressions for the ACFs of the BOC, MBOC, and AltBOC modulations. In the next section, we first establish the mathematical models of these modulations and then present details about the expressions for the ACFs of these modulations. We finally derive the expressions for the PSDs of these modulations using the Fourier transform of their ACFs.

1.1. Signal Model. Considering a BOC signal as the product of a pseudorandom noise (PRN) spreading code modulated data with a square-wave subcarrier, we assume that the PRN spreading code is random, infinite, aperiodic, identically distributed, and independent, and the signal bandwidth is infinite.

1.1.1. BOC Signal. The BOC signal can be expressed as the following equation:

\[ s(t) = \sum_{k=-\infty}^{\infty} c_k p(t - kT_c), \]

where \( c_k \) denotes the data-modulated PRN spreading code chip and \( p(\cdot) \) denotes the square-wave subcarrier symbol with support time \( T_c \), which is defined in either sine-phased or cosine-phased form as follows:

\[
\begin{align*}
    p_{\text{sin}}(t) &= \begin{cases} 
    \text{sign}(\sin(2\pi f_s t)), & 0 \leq t \leq T_c, \\
    0, & \text{otherwise},
    \end{cases} \\
    p_{\text{cos}}(t) &= \begin{cases} 
    \text{sign}(\cos(2\pi f_s t)), & 0 \leq t \leq T_c, \\
    0, & \text{otherwise}.
    \end{cases}
\end{align*}
\]

where \( f_s \) denotes the subcarrier frequency and \( \text{sign}(\cdot) \) is the sign function. The BOC signal is typically denoted by \( \text{BOC}(f_s, f_c) \), \( f_c \) which equals to \( 1/T_c \) is the PRN spreading code chip rate. The designation \( \text{BOC}(\alpha, \beta) \) with reference frequency \( f_{\text{ref}} \) which equals to 1.023 MHz is used as the abbreviation of \( \text{BOC}(f_s, f_c) \), the subcarrier frequency \( f_s = \alpha \times f_{\text{ref}}, \) and the PRN spreading code rate \( f_c = \beta \times f_{\text{ref}} \).

There is also another model defining the BOC signal with \( p(\cdot) \) broken up into rectangular pulses with amplitudes of \( \pm 1 \). For the sine-phased BOC signal, \( p_{\text{sin}}(t) \) is broken up into \( N = 2\alpha/\beta \) rectangular pulses of duration \( t_s = T_c/N \), which can be expressed as

\[ p_{\text{sin}}(t) = \mu(t) \otimes \sum_{m=0}^{N-1} (-1)^m \delta(t - mt_s), \]

where \( \otimes \) denotes the convolution operation, \( \delta(\cdot) \) denotes the impulse function, and \( \mu(t) \) is a rectangular pulse with support \( T \) defined as

\[ \mu(t) = \begin{cases} 
1, & 0 \leq t \leq T, \\
0, & \text{otherwise}.
\end{cases} \]

Figure 1 shows the sine-phased BOC subcarrier wave. For the cosine-phased BOC signal, \( p_{\text{cos}}(t) \) is broken up into \( 2N \) rectangular pulses of duration \( t_s/2 \), so \( p_{\text{cos}}(t) \) can also be expressed as

\[ p_{\text{cos}}(t) = \mu_{t/2}(t) \otimes \sum_{m=0}^{2N-1} (-1)^{(m+1)/2} \delta(t - mt_s/2), \]

where \( \lfloor \cdot \rfloor \) denotes the floor function. Figure 2 shows the cosine-phased BOC subcarrier wave.

1.1.2. MBOC Signal. The MBOC signal is defined in the frequency domain, and a specific case of MBOC modulations, denoted MBOC(6, 1, 1/11), is recommended based on extensive work to meet technical constraints [6], whose PSD (pilot and data components together) is given by

\[ G_{\text{MBOC}(6,1,11)}(f) = \frac{10}{11} G_{\text{BOC}(1,1)}(f) + \frac{1}{11} G_{\text{BOC}(6,1)}(f), \]

where \( G_{\text{BOC}(\alpha,\beta)}(f) \) is the PSD of the BOC signal. There are a variety of methods producing MBOC(6,1,11) signals, which are typically produced by two different methods, CBOC and TMBOC.

(1) CBOC: the subcarrier of a CBOC signal comprises four-level symbols formed by the weighted sum of different BOC subcarrier symbols, and the model of the CBOC signal can be defined as

\[ \begin{align*}
    s_{\text{CBOC}}(t) &= \sum_{k=-\infty}^{\infty} c_k p_{\text{CBOC}}(t - kT_c), \\
    p_{\text{CBOC}}(t) &= w_1 p_{\text{BOC}(1,1)}(t) + w_2 p_{\text{BOC}(6,1)}(t),
\end{align*} \]

where \( w_1 \) and \( w_2 \) denote amplitude weighting factors satisfying \( w_1^2 + w_2^2 = 1 \). Furthermore, \( p_{\text{BOC}(1,1)}(t) \) and \( p_{\text{BOC}(6,1)}(t) \) can also be expressed as...
Figure 3 shows the CBOC subcarrier waves; we assume that the signal power is split into 50%/50% between data and pilot components, and the CBOC subcarrier is used on both pilot and data components. Then, the MBOC implementation using the CBOC method, denoted CBOC(6,1,1/11), can be expressed as

\[
P_{\text{BOC}(6,1)}(t) = \mu T_c/12(t) \otimes \sum_{m=0}^{11} (-1)^{m/6} \delta \left( t - \frac{mT_c}{12} \right).
\]

Figure 3 shows the CBOC subcarrier waves; we assume that the signal power is split into 50%/50% between data and pilot components, and the CBOC subcarrier is used on both pilot and data components. Then, the MBOC implementation using the CBOC method, denoted CBOC(6,1,1/11), can be expressed as

\[
P_{\text{BOC}(6,1)}(t) = \mu T_c/12(t) \otimes \sum_{m=0}^{11} (-1)^{m/6} \delta \left( t - \frac{mT_c}{12} \right).
\]

\[\text{(8)}\]

(2) TMBOC: the model of a TMBOC signal is defined as

\[
s_{\text{TMBOC}}(t) = \sum_{k=-\infty}^{+\infty} c_k p_k(t - kT_c),
\]

where different BOC subcarrier symbols are used for different values of \( k \) in a TMBOC time sequence, we assume that the signal power is split into 25%/75% between data and pilot components, and the TMBOC subcarrier is used on pilot components. Then, the MBOC signal produced by using the TMBOC method, denoted as TMBOC(6,1,4/33), can be implemented by placing the BOC(6,1) subcarrier symbol in locations 1, 5, 7, and 30 of each 33 BOC(1,1) subcarrier symbol location.

1.1.3. AltBOC Signal. AltBOC can transmit four signal components at most; its spectrum has two sidebands, and each sideband can be processed independently as a
quadrature phase shift keying (QPSK) signal. The BOC signal is a particular case of AltBOC when the four PRN spreading codes are made identical. The AltBOC subcarrier based on the BOC subcarrier is complex, so the spectrum of the signal component is not split up but only shifted to upper or lower sidebands [7]. We assume that both pilot and data components are introduced, and four PRN spreading codes are needed. The signal model of AltBOC can be expressed as

\[
S_{\text{AltBOC}} = \sum_{k=-\infty}^{\infty} \left( e_L^D + jc_L^P \right) p_{\text{AltBOC}}(t - kT_c) + \left( e_U^D + jc_U^P \right) p_{\text{AltBOC}}(t - kT_c),
\]

where \( p_{\text{AltBOC}}(t) = p_{\text{cos}}(t) + j p_{\text{sin}}(t) \) denotes the subcarrier symbol. \( e_L^D \) and \( e_U^P \) are the data codes of lower and upper sidebands, respectively. \( c_L^P \) and \( c_U^P \) are the pilot codes of lower

\[\text{Figure 2: Cosine-phased BOC subcarrier wave: (a) cosine-phased BOC subcarrier wave when } N = 2, 6, 10, \ldots, \text{ (b) cosine-phased BOC subcarrier wave when } N = 4, 8, 12, \ldots, \text{ (c) cosine-phased BOC subcarrier wave when } N = 3, 7, 11, \ldots, \text{ and (d) cosine-phased BOC subcarrier wave when } N = 5, 9, 13, \ldots.\]

\[\text{Figure 3: CBOC subcarrier waves.}\]
and upper sidebands, respectively. Furthermore, \( P_{\text{AltBOC}}(t) \) can also be expressed as

\[
P_{\text{AltBOC}}(t) = \sum_{k=\infty}^{+\infty} \mu_{\text{AltBOC},k}(t) \otimes \sum_{m=0}^{2N-1} (-1)^{[(m+1)/2]} \delta(t - mt_s/2 - kT_c) + j \sum_{k=\infty}^{+\infty} \mu_{\text{AltBOC},k}(t) \otimes \sum_{m=0}^{N-1} (-1)^m \delta(t - mt_s - kT_c).
\]

(12)

However, the signal defined in (11) may be distorted within the high-power amplifier of the satellite payload due to nonlinear amplification because it does not have a constant envelope. Thus, the constant envelope AltBOC with four codes is derived in [7] and defined as follows:

\[
s_{\text{AltBOC}} = \sum_{k=\infty}^{+\infty} (c_D^p + jc_\pi^p) p_d^*(t - kT_c)
+ (c_D^p + jc_\pi^p) p_d(t - kT_c) + (c_D^p + jc_\pi^p) p_p^*(t - kT_c)
+ (c_D^p + jc_\pi^p) p_p(t - kT_c).
\]

(13)

\[
s_c^d(t) = \begin{cases} \frac{\sqrt{2}}{4} \, \text{sign} \left[ \cos \left( 2\pi f_s t - \frac{\pi}{4} \right) \right] + \frac{1}{2} \, \text{sign} \left[ \cos \left( 2\pi f_s t \right) \right] + \frac{\sqrt{2}}{4} \, \text{sign} \left[ \cos \left( 2\pi f_s t + \frac{\pi}{4} \right) \right], & 0 \leq t \leq T_c, \\ 0, & \text{others}, \end{cases}
\]

(15)

\[
s_c^p(t) = \begin{cases} -\frac{\sqrt{2}}{4} \, \text{sign} \left[ \cos \left( 2\pi f_s t - \frac{\pi}{4} \right) \right] + \frac{1}{2} \, \text{sign} \left[ \cos \left( 2\pi f_s t \right) \right] - \frac{\sqrt{2}}{4} \, \text{sign} \left[ \cos \left( 2\pi f_s t + \frac{\pi}{4} \right) \right], & 0 \leq t \leq T_c, \\ 0, & \text{others}. \end{cases}
\]

Furthermore, \( s_c^d(t) \) and \( s_c^p(t) \) can also be expressed as

\[
s_c^d(t) = \mu_{s_c^d}(t) \otimes \sum_{m=0}^{4N-1} \left[ \frac{\sqrt{2}}{4} (-1)^{[(m+1)/4]} + \frac{1}{2} (-1)^{[(m+2)/4]} + \frac{\sqrt{2}}{4} (-1)^{[(m+3)/4]} \right] \delta(t - mt_s/4),
\]

(16)

\[
s_c^p(t) = \mu_{s_c^p}(t) \otimes \sum_{m=0}^{4N-1} \left[ \frac{\sqrt{2}}{4} (-1)^{[(m+1)/4]} + \frac{1}{2} (-1)^{[(m+2)/4]} + \frac{\sqrt{2}}{4} (-1)^{[(m+3)/4]} \right] \delta(t - mt_s/4).
\]

The top panel of Figure 4 shows the wave of \( s_c^d(t) \), and the bottom panel shows the wave of \( s_c^p(t) \). Similar to the BOC signal, the AltBOC signal is generally denoted as AltBOC \((p, q)\) with the reference frequency \( f_{\text{ref}} \) which equals to 1.023 MHz.

1.2. Autocorrelation Functions. The ACF of the BOC signal can be expressed as

\[
R(t, t + \tau) = E(s(t)^* s(t + \tau))
= E \left( \sum_{k=\infty}^{+\infty} (c_k p_t - kT_c) \sum_{l=\infty}^{+\infty} c_l^* p^*(t + \tau - lT_c) \right)
= \sum_{k=\infty}^{+\infty} \sum_{l=\infty}^{+\infty} R_{\text{BOC}}(l) E(p(t - kT_c) \cdot p^*(t + \tau - kT_c - lT_c)),
\]

(17)
where $R_c(l) = E(c_k c_{k+l}^*)$ denotes the ACF of the PRN spreading code. For an ideal spreading code, $R_c(l)$ becomes

$$R_c(l) = \begin{cases} 1, & l = 0, \\ 0, & l \neq 0. \end{cases} \tag{18}$$

Substituting (18) into (17), $R(t, t + \tau)$ can be simplified to

$$R(t, t + \tau) = \sum_{k=-\infty}^{\infty} E(p(t - kT_c) p^*(t + \tau - kT_c)). \tag{19}$$

The BOC signal is not a wide-sense stationary process but rather has the following characteristics:

$$E(s(t + T_c)) = E(s(t)), \quad R(t + T_c, t + \tau + T_c) = R(t, t + \tau). \tag{20}$$

Thus, the BOC signal is cyclostationary, and its ACF can be obtained by averaging the ACF over the interval $t \in [0, T_c]$:

$$R(\tau) = \frac{1}{T_c} \int_{0}^{T_c} R(t, t + \tau) dt. \tag{21}$$

Substituting (19) into (21), the ACF of the BOC signal can be written as

$$R(\tau) = \frac{1}{T_c} \int_{0}^{T_c} \sum_{k=-\infty}^{\infty} p(t - kT_c) p^*(t + \tau - kT_c) dt$$

$$= \frac{1}{T_c} \sum_{k=-\infty}^{\infty} \int_{-kT_c}^{(1-k)T_c} p(t) p^*(t + \tau) dt$$

$$= \frac{1}{T_c} \int_{-\infty}^{\infty} p(t) p^*(t + \tau) dt$$

$$= \frac{1}{T_c} \int_{-\infty}^{\infty} p(t) p^*(-\tau) dt. \tag{22}$$

Therefore, the ACF of the BOC signal can be obtained by the convolution of the subcarrier symbol and the conjugate of the mirror function of the subcarrier symbol.

1.2.1. BOC Signal. For the sine-phased BOC subcarrier symbol, according to Figure 1, we found the following property:

$$p_{\text{sin}}(t) = \begin{cases} -p_{\text{sin}}(T_c - t), & N \text{ is even}, \\ p_{\text{sin}}(T_c - t), & N \text{ is odd.} \end{cases} \tag{23}$$

Then, substituting (23) into (22), the ACF of the sine-phased BOC signal can be expressed as

$$R_{\text{BOCsin}}(\tau) = \frac{1}{T_c} (-1)^{N-1} p_{\text{sin}}(\tau) \otimes p_{\text{sin}}(\tau) \otimes \delta(\tau + T_c). \tag{24}$$

Substituting (3) into (24), $R_{\text{BOCsin}}(\tau)$ can be derived:

$$R_{\text{BOCsin}}(\tau) = \frac{1}{T_c} (-1)^{N-1} \mu_{\text{sin}}(\tau) \otimes \sum_{m=0}^{N-1} (-1)^m \delta(\tau - mT_c) \otimes \mu_{\text{sin}}(\tau)$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} (-1)^m \delta(\tau - mT_c) \otimes \delta(\tau + T_c)$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} (-1)^{m+n} \frac{T_c}{T} \times \Lambda_T$$

$$\cdot (\tau - (m + n + 1 - N)T), \tag{25}$$

where $\Lambda_T(t)$ is the triangle function with support $2T$, defined as

$$\Lambda_T(t) = \begin{cases} 1 - \frac{|t|}{T}, & |t| < T, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

For the cosine-phased BOC subcarrier symbol, according to Figure 2, we found the following property:

$$p_{\text{cos}}(t) = \begin{cases} p_{\text{cos}}(T_c - t), & N \text{ is even}, \\ -p_{\text{cos}}(T_c - t), & N \text{ is odd.} \end{cases} \tag{27}$$

Then, substituting (27) into (22), the ACF of the cosine-phased BOC signal can be expressed as

$$R_{\text{BOCcos}}(\tau) = \frac{1}{T_c} (-1)^{N} p_{\text{cos}}(\tau) \otimes p_{\text{cos}}(\tau) \otimes \delta(\tau + T_c). \tag{28}$$

Substituting (5) into (28), $R_{\text{BOCcos}}(\tau)$ can be derived:
The ACFs of the sine-phased and cosine-phased BOC signals when \( N = 2, 3, 4, \) and 5, which are constructed according to (25) and (29), are presented in Figure 5.

1.2.2. MBOC Signal. The CBOC subcarrier symbol, according to the definition, has the following property:

\[
P_{\text{CBOC}}(t) = -P_{\text{CBOC}}(T_c - t).
\]  

For the TMBOC signal, its ACF can be expressed as

\[
R_{\text{TMBOC}}(\tau) = \frac{1}{2} \sum_{m=0}^{11} \sum_{n=0}^{11} w_1(-1)^{w_1(n/6)+w_2(-1)^n} \delta\left(\tau - \frac{mT_c}{12}\right) \otimes \mu_{\Delta \tau /2}(\tau) \sum_{n=0}^{11} w_1(-1)^{w_1(n/6)+w_2(-1)^n} \delta\left(\tau - \frac{nT_c}{12}\right) \otimes \delta(\tau + T_c).
\]  

Substituting (25) and (29) into (34), the ACF can be derived:

\[
R_{\text{MBOC}}(\tau) = \frac{1}{2} \sum_{m=0}^{11} \sum_{n=0}^{11} w_1(-1)^{w_1(n/6)+w_2(-1)^n} \delta\left(\tau - \frac{mT_c}{12}\right) \otimes \mu_{\Delta \tau /2}(\tau) \sum_{n=0}^{11} w_1(-1)^{w_1(n/6)+w_2(-1)^n} \delta\left(\tau - \frac{nT_c}{12}\right) \otimes \delta(\tau + T_c).
\]  

Substituting (7) and (8) into (31), the ACF of the CBOC signal can be derived:

\[
R_{\text{CBOC}}(\tau) = \frac{1}{2} \sum_{m=0}^{11} \sum_{n=0}^{11} w_1(-1)^{w_1(n/6)+w_2(-1)^n} \delta\left(\tau - \frac{mT_c}{12}\right) \otimes \mu_{\Delta \tau /2}(\tau) \sum_{n=0}^{11} w_1(-1)^{w_1(n/6)+w_2(-1)^n} \delta\left(\tau - \frac{nT_c}{12}\right) \otimes \delta(\tau + T_c).
\]  

1.2.3. AltBOC Signal. The ACF of the nonconstant envelope AltBOC can be expressed as

\[
R_{\text{NCE-AltBOC}}(\tau) = 2 \int_0^{T_c} P_{\text{AltBOC}}(t) P_{\text{AltBOC}}^*(t + \tau) + P_{\text{AltBOC}}(t) P_{\text{AltBOC}}(t + \tau) dt = 4(R_{\text{BOCsin}}(\tau) + R_{\text{BOCcos}}(\tau)).
\]  

Substituting (25) and (29) into (34), the ACF can be derived as follows:

\[
R_{\text{NCE-AltBOC}}(\tau) = 4 \left( \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} (-1)^{m+n+(N-1)} \frac{T_c}{T_c} \times \Lambda_{\Delta \tau /2}(\tau - (m + n + 1 - N)t_c) + \frac{1}{2} \sum_{m=0}^{2N-1} \sum_{n=0}^{2N-1} (-1)^{m+n+1} \frac{T_c}{T_c} \times \Lambda_{\Delta \tau /2}(\tau - (m + n + 1 - 2N)t_c) \right).
\]  

For the constant envelope AltBOC signal, its ACF can be expressed as

\[
R_{\text{CE-AltBOC}}(\tau) = \frac{2}{T_c} \int_0^{T_c} (p_d(t) \delta(t + \tau) + p_d^*(t) \delta(t + \tau) + p_p(t) \delta(t + \tau) + p_p^*(t) \delta(t + \tau)) dt
\]  

\[
= 4 \int_0^{T_c} (s_d(t) \delta(t + \tau) + s_d(t - T/4) \delta(t - T/4 + \tau) + s_p(t) \delta(t + \tau) + s_p(t - T/4) \delta(t - T/4 + \tau)) dt.
\]
Figure 5: Normalized ACFs of BOC signals when $N = 2, 3, 4, \text{ and } 5$.

Figure 6: Normalized ACFs of CBOC($6, 1/11, ^{+}$), CBOC($6, 1/11, ^{-}$), and TMBOC($6, 1, 4/33$).
Let $s_{c1}(t) = s_{c2}(t)$, $s_{c2}(t) = s_{c3}(t - T_c/4)$, $s_{c1}(t) = s_{c3}(t)$, and $s_{c2}(t) = s_{c3}(t - T_c/4)$; then, the ACF of the constant envelope AltBOC signal can be expressed as

$$R_{\text{CE-AltBOC}}(\tau) = \frac{4}{T_c} \left( s_{c1}(\tau) \otimes s_{c1}(-\tau) + s_{c2}(\tau) \otimes s_{c2}(-\tau) + s_{c3}(\tau) \otimes s_{c3}(-\tau) \right)$$

According to the definition, $s_{c1}(t)$, $s_{c2}(t)$, $s_{c1}(t)$, and $s_{c2}(t)$ have the following property:

$$\begin{align*}
s_{c1}(t) &= (-1)^N s_{c1}(T_c - t), \\
s_{c2}(t) &= (-1)^{N-1} s_{c2}(T_c - t), \\
s_{c3}(t) &= (-1)^{N} s_{c3}(T_c - t), \\
s_{c4}(t) &= (-1)^{N-1} s_{c4}(T_c - t). \\
\end{align*}$$

Thus, substituting (38) into (37), the ACFs of $s_{c1}(t)$, $s_{c2}(t)$, $s_{c1}(t)$, and $s_{c2}(t)$ can be derived:

$$\begin{align*}
R_{s_{c1}}(\tau) &= (-1)^N s_{c1}(\tau) \otimes s_{c1}(-\tau) \otimes \delta(\tau + T_c), \\
R_{s_{c2}}(\tau) &= (-1)^{N-1} s_{c2}(\tau) \otimes s_{c2}(-\tau) \otimes \delta(\tau + T_c), \\
R_{s_{c3}}(\tau) &= (-1)^{N} s_{c3}(\tau) \otimes s_{c3}(-\tau) \otimes \delta(\tau + T_c), \\
R_{s_{c4}}(\tau) &= (-1)^{N-1} s_{c4}(\tau) \otimes s_{c4}(-\tau) \otimes \delta(\tau + T_c). \\
\end{align*}$$

Substituting (16) and (39) into (37), the ACF of the constant envelope AltBOC signal can be obtained:

$$R_{\text{CE-AltBOC}}(\tau) = \sum_{m=0}^{4N-1} \sum_{n=0}^{4N-1} \frac{1}{2} (-1)^{m/4 + n/4 + N-1}$$

$$+ \frac{1}{2} (-1)^{(m+2)/4 + (n+2)/4 + N}$$

$$+ (-1)^{(m+1)/4 + (n+3)/4 + N} \frac{T_c}{T_c}$$

$$\times \Lambda_{t/4}(\tau - (m + n + 1 - 4N) \frac{T_c}{T_c}).$$

The ACFs of nonconstant and constant envelope AltBOC(15,10), which are constructed according to (35) and (40), are presented in Figure 7.

1.3. Power Spectral Density. According to the Wiener–Khinchin theorem [20], the PSD of the BOC signal is the Fourier transform of its ACF:

$$G(f) = \mathcal{F}[R(\tau)].$$

1.3.1. BOC Signal. Considering that $N$ is either even or odd, the PSD of the sine-phased BOC signal can be derived as follows:

$$G_{\text{BOCsin}}(f) = \left\{ \begin{array}{ll}
\frac{1}{T_c} \left( \sin(\pi f T_c) \sin(\pi f T_c) \right)^2, & N \text{ is even}, \\
\frac{1}{T_c} \left( \sin(\pi f T_c) \sin(\pi f T_c) \right)^2, & N \text{ is odd},
\end{array} \right.$$  

Considering that $N$ is either even or odd, the PSD of the cosine-phased BOC signal can be derived as follows:

$$G_{\text{BOCcos}}(f) = \left\{ \begin{array}{ll}
\frac{1}{T_c} \left( 1 - \cos(\pi f T_c) \cos(\pi f T_c) \right)^2, & N \text{ is even}, \\
\frac{1}{T_c} \left( 1 - \cos(\pi f T_c) \cos(\pi f T_c) \right)^2, & N \text{ is odd},
\end{array} \right.$$

The PSDs of the sine-phased and cosine-phased BOC signals when $N=2$, 3, 4, and 5, which are constructed according to (42) and (43), are presented in Figure 8.

1.3.2. MBOC Signal. For the CBOC signal, its PSD can be expressed as

$$G_{\text{CBOC}}(f) = \frac{1}{T_c} \left( \sin^2(\pi f T_c/12) + \sin^2(\pi f T_c/12) \right)$$

$$+ 2\omega_7 \omega_8 \left( 1 - \cos(\pi f T_c) \sin(\pi f T_c) \cos(\pi f T_c/12) \right)$$

$$\times \frac{\sin(\pi f T_c/12)}{\pi f}.$$  

For the TMBOC signal, its PSD can be expressed as

$$G_{\text{TMBOC}}(f) = \frac{1}{T_c} \left( \frac{3\omega_7 \omega_8 \sin^2(\pi f T_c/12) \sin^2(\pi f T_c)}{\pi^2 f^2 \cos^2(\pi f T_c/12)} \right)$$

$$+ \left( 1 - \frac{3\omega_7 \omega_8 \sin^2(\pi f T_c/12) \sin^2(\pi f T_c)}{\pi^2 f^2 \cos^2(\pi f T_c/12)} \right).$$

The PSDs of CBOC(6,1,1/11,’+’), CBOC(6,1,1/11,’-’), and TMBOC(6,1,4/33), which are constructed according to (44) and (45), are presented in Figure 9.

1.3.3. AltBOC Signal. Considering that $N$ is either even or odd, the PSD of the nonconstant envelope AltBOC signal can be derived as follows:
Figure 7: Normalized ACFs of nonconstant and constant envelope AltBOC(15,10).

Figure 8: PSDs of BOC signals when $N = 2, 3, 4, and 5$. 
Considering that \( N \) is either even or odd, the PSD of the constant envelope AltBOC signal can be derived as follows:

\[
G_{\text{NCE-AltBOC}}(f) = \begin{cases} 
\frac{4}{T_c} \left( \frac{\sin^2(\pi f t_s) \sin^2(\pi f T_c)}{\pi^2 f^2 \cos^2(\pi f t_s)} + \frac{(1 - \cos(\pi f t_s))^2 \sin^2(\pi f T_c)}{\pi^2 f^2 \cos^2(\pi f t_s)} \right), & \text{if } N \text{ is even}, \\
\frac{4}{T_c} \left( \frac{\sin^2(\pi f T_c) \cos^2(\pi f t_s)}{\pi^2 f^2 \cos^2(\pi f t_s)} + \frac{(1 - \cos(\pi f t_s))^2 \cos^2(\pi f T_c)}{\pi^2 f^2 \cos^2(\pi f t_s)} \right), & \text{if } N \text{ is odd}.
\end{cases}
\]  

(46)

The PSDs of nonconstant and constant envelope AltBOC(15,10), which are constructed according to (46) and (47), are presented in Figure 10.

2. Conclusion

This paper derives explicit analytical expressions for the ACFs of the BOC, MBOC, and AltBOC modulations. By expressing the ACFs as the sum of triangle functions, it is possible to determine expressions for the PSDs additionally. The derive method uses the conversion relationship between the convolution operation and the calculation of the correlation function. The method is common and can be used to derive analytical expressions for the ACFs of other BOC-based modulations. With the knowledge of the analytical expressions for the ACFs, for a satellite navigation system, it is possible to calculate the potential code tracking accuracy quantitatively and to estimate the signal resolution under multipath propagation and interference conditions. Designers can consciously overcome difficulties when developing a discriminator for a receiver to ensure unambiguous tracking of the main peak of ACFs and minimize the probability of capture of their false peaks. Moreover, the analytical expressions for the ACFs are useful for GNSS signal simulation and performance evaluation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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